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Algorithmic computation of principal posets using Maple and Python

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Abstract. We present symbolic and numerical algorithms for a computer search in the Coxeter spectral classification problems. One of the main aims of the paper is to study finite posets I that are principal, i.e., the rational symmetric Gram matrix $G_I := \frac{1}{2}[C_I + C_I^{tr}] \in \mathbb{M}_I(\mathbb{Q})$ of I is positive semi-definite of corank one, where $C_I \in \mathbb{M}_I(\mathbb{Z})$ is the incidence matrix of I. With any such a connected poset I, we associate a simply laced Euclidean diagram $DI \in \{\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$, the Coxeter matrix $\mathrm{Cox}_I := -C_I \cdot C_I^{-tr}$, its complex Coxeter spectrum $\mathbf{specc}_I$, and a reduced Coxeter number $\check{\mathbf{c}}_I$. One of our aims is to show that the spectrum $\mathbf{specc}_I$ of any such a poset I determines the incidence matrix C_I (hence the poset I) uniquely, up to a $\mathbb{Z}$-congruence. By computer calculations, we find a complete list of principal one-peak posets I (i.e., I has a unique maximal element) of cardinality ≤ 15, together with $\mathbf{specc}_I$, $\check{\mathbf{c}}_I$, the incidence defect $\partial_I : \mathbb{Z}^I \to \mathbb{Z}$, and the Coxeter-Euclidean type DI. In case when $DI \in \{\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ and $n := |I|$ is relatively small, we show that given such a principal poset I, the incidence matrix C_I is $\mathbb{Z}$-congruent with the non-symmetric Gram matrix $\check{G}_{DI}$ of DI, $\mathbf{specc}_I = \mathbf{specc}_{DI}$ and $\check{\mathbf{c}}_I = \check{\mathbf{c}}_{DI}$. Moreover, given a pair of principal posets I and J, with $|I| = |J| \leq 15$, the matrices C_I and C_J are $\mathbb{Z}$-congruent if and only if $\mathbf{specc}_I = \mathbf{specc}_J$.

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1. Introduction

Throughout, we freely use the terminology and notation introduced in [45], [47], [50], [51]. We denote by $\mathbb{N}$ the set of non-negative integers, by $\mathbb{Z}$ the ring of integers, and by $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ the field of the rational, the real, and the complex numbers, respectively. We view $\mathbb{Z}^n$, with $n \geq 1$, as a free abelian group. By $e_1, \dots, e_n$ we denote the standard $\mathbb{Z}$-basis of the group $\mathbb{Z}^n$. Given $n \geq 1$, we denote by $\mathbb{M}_n(\mathbb{Z})$ the $\mathbb{Z}$-algebra of all square n by n matrices $A = [a_{ij}]$, with $a_{ij} \in \mathbb{Z}$, and by $E \in \mathbb{M}_n(\mathbb{Z})$ the identity matrix. Given a finite set J, we denote by $\mathbb{M}_J(\mathbb{Z})$ the $\mathbb{Z}$-algebra of all square J by J matrices. The group

$$\mathrm{Gl}(n, \mathbb{Z}) := \{A \in \mathbb{M}_n(\mathbb{Z}),\ \det A \in \{-1, 1\}\} \subseteq \mathbb{M}_n(\mathbb{Z})$$

is called the (integral) **general linear group**. We say that two square rational matrices $A, A' \in \mathbb{M}_n(\mathbb{Q})$ are $\mathbb{Z}$-congruent (and denote by $A \sim_{\mathbb{Z}} A'$) if there exists a $\mathbb{Z}$-invertible matrix $B \in \mathrm{Gl}(n, \mathbb{Z})$ such that $A' = B^{tr} \cdot A \cdot B$.

By a poset $J \equiv (J, \preceq)$ we mean a finite partially ordered set J with respect to a partial order relation $\preceq$. Following [43], J is called a **one-peak poset** if it has a unique maximal element $*$. Given a poset J, with $m = |J|$, we denote by

$$C_J = [c_{ij}] \in \mathbb{M}_J(\mathbb{Z}) \equiv \mathbb{M}_m(\mathbb{Z}) \tag{1.1}$$

the **incidence matrix** of J, with $c_{ij} = 1$, for all $i \preceq j$, and $c_{ij} = 0$ otherwise. The rational matrix

$$G_J := \tfrac{1}{2}[C_J + C_J^{tr}] \in \mathbb{M}_J(\mathbb{Q}) \equiv \mathbb{M}_m(\mathbb{Q}) \tag{1.2}$$

is called the symmetric Gram matrix of J. Following [50] and [51], we call the symmetric matrix $Ad_J := C_J + C_J^{tr} - 2 \cdot E$ the **adjacency matrix** of J, and

$$P_J(t) = \det(t \cdot E - Ad_J) \in \mathbb{Z}[t], \tag{1.3}$$

the **characteristic polynomial** of the poset J. We say that the poset J is **$\mathbb{Z}$-bilinear equivalent** to a poset J' (and we write $J \approx_{\mathbb{Z}} J'$) if $C_J \sim_{\mathbb{Z}} C_{J'}$.

We define J to be non-negative (resp. positive) if the rational symmetric Gram matrix G_J (1.2) is positive semi-definite (resp. positive definite). If J is connected and the symmetric Gram matrix G_J is positive semi-definite of rank $|J| - 1$, we call J a **principal poset**, see [47, Definition 2.1] and [51, Section 3]. In other words, J is principal if and only if the quadratic form $q_J(x) = x \cdot C_J \cdot x^{tr}$ is non-negative and the subgroup $\mathrm{Ker}\, q_J := \{v \in \mathbb{Z}^J;\ q_J(v) = 0\}$ of $\mathbb{Z}^J$ is infinite cyclic.

Our study is inspired by important applications of the quadratic forms and edge-bipartite graphs in constructing linear algebra invariants that measure a geometric complexity of Nazarova-Roiter matrix problems over a field K and in the study of module categories and their derived categories, see the monographs [1], [11], [43], [53], and the articles [2]-[9], [12]-[24], [27], [29]-[34], [39]-[42], [45]-[52], and [54], [55]. In particular, our study is inspired by a well-known result of Drozd [10], by the Coxeter spectral analysis of loop-free edge-bipartite graphs developed in [50], and the Coxeter spectral classification technique of finite posets introduced in [44], [45], and [51], see also [26], [28], [36]-[38], and [56].

In the present paper we are mainly interested in the class of non-negative posets J; in particular, in the class of principal posets. We study them by applying our recent results on the Coxeter spectral classification of loop-free edge-bipartite graphs defined in [50] (see also [51]) as follows.

An **edge-bipartite graph** (bigraph, for short), with $n \geq 2$ vertices, is a pair $\Delta = (\Delta_0, \Delta_1 = \Delta_1^- \sqcup \Delta_1^+)$, where $\Delta_0 = \{a_1, \dots, a_n\}$ is a set of vertices and Δ_1 is a finite set of edges such that $|\Delta_1^-(a_i, a_j)| \cdot |\Delta_1^+(a_i, a_j)| = 0$, for all $a_i \neq a_j \in \Delta_0$. Edges in $\Delta_1^-(a_i, a_j)$ and $\Delta_1^+(a_i, a_j)$ are visualized as continuous a_i —— a_j, and dotted ones a_i - - - - a_j, respectively. We say that Δ is loop-free if $\Delta_1(a_i, a_i)$ is empty, for all $a_i \in \Delta_0$. We denote by $\mathcal{UB}igr_n$ the set of all connected loop-free edge-bipartite graphs, with $n \geq 2$ vertices.

We view any finite graph $\Delta = (\Delta_0, \Delta_1)$ as an edge-bipartite graph by setting $\Delta_1^-(a_i, a_j) = \Delta_1(a_i, a_j)$ and $\Delta_1^+(a_i, a_j) = \emptyset$, for $a_i, a_j \in \Delta_0$.

A non-symmetric **Gram matrix** of $\Delta \in \mathcal{UB}igr_n$ is the matrix

$$\check{G}_\Delta = \begin{bmatrix} 1 & d_{12}^\Delta & \dots & d_{1n}^\Delta \\ 0 & 1 & \dots & d_{2n}^\Delta \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \in \mathbb{M}_n(\mathbb{Z}),$$

where $d_{ij}^\Delta = -|\Delta_1^-(a_i, a_j)|$, if there is an edge a_i —— a_j and $i \leq j$, $d_{ij}^\Delta = |\Delta_1^+(a_i, a_j)|$, if there is an edge a_i- - - - a_j and $i \leq j$. We set $d_{ij}^\Delta = 0$, if the set $\Delta_1(a_i, a_j)$ is empty or $j < i$. The rational matrix

$$G_\Delta := \frac{1}{2}[\check{G}_\Delta + \check{G}_\Delta^{tr}] \in \mathbb{M}_n(\mathbb{Q})$$

is called the **symmetric Gram matrix** of Δ. The **Gram quadratic form** of $\Delta \in \mathcal{UB}igr_n$ is defined by the formula

$$q_\Delta(x) = q_\Delta(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2 + \sum_{i<j} d_{ij}^\Delta x_i x_j = x \cdot \check{G}_\Delta \cdot x^{tr} = x \cdot G_\Delta \cdot x^{tr}.$$

We call $\Delta \in \mathcal{UB}igr_n$, with $n \geq 2$ numbered vertices, **positive** (resp. **non-negative of corank s** ≥ 1), if its symmetric Gram matrix $G_\Delta \in \mathbb{M}_n(\mathbb{Q})$ is positive definite (resp. positive semi-definite of rank $n - s$). Moreover, we call $\Delta \in \mathcal{UB}igr_n$ **principal** if the matrix G_Δ is positive semi-definite of rank $n - 1$, see [50]. Note that a non-negative loop-free bigraph Δ is of corank $s \geq 1$ if and only if the kernel

$$\operatorname{Ker} q_\Delta := \{v \in \mathbb{Z}^n;\ q_\Delta(v) = v \cdot \check{G}_\Delta \cdot v^{tr} = 0\} \subseteq \mathbb{Z}^n$$

of $q_\Delta : \mathbb{Z}^n \to \mathbb{Z}$ is a free subgroup of $\mathbb{Z}^n$ of $\mathbb{Z}$-rank s. Obviously, Δ is principal if and only if Δ is non-negative loop-free and $\operatorname{Ker} q_\Delta = \mathbb{Z} \cdot \mathbf{h}$, with $\mathbf{h} \neq 0$.

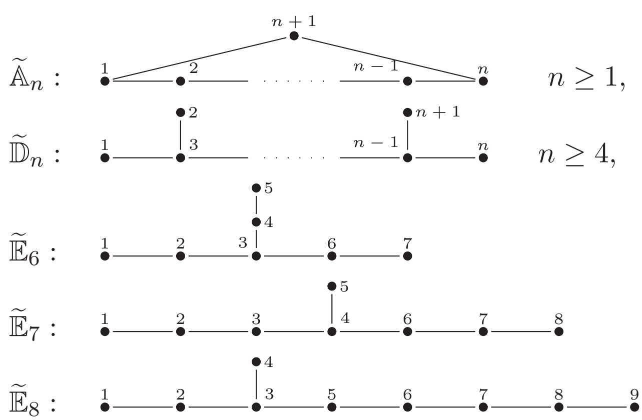
The matrix $Ad_\Delta := \check{G}_\Delta + \check{G}_\Delta^{tr} - 2 \cdot E$ is called the **symmetric adjacency matrix** of a loop-free edge-bipartite graph $\Delta \in \mathcal{UB}igr_n$, and the **spectrum** of Δ is the set $\mathbf{spec}_\Delta \subset \mathbb{R}$ of n real roots of the polynomial

$$P_\Delta(t) = \det(t \cdot E - Ad_\Delta) \in \mathbb{Z}[t],$$

called the **characteristic polynomial** of the edge-bipartite graph Δ.

Following [50], with any principal poset J, we have associated in [51] a loop-free edge-bipartite principal graph Δ_J, and a simply laced Euclidean diagram DJ, that is, one of the graphs presented in the following table.

TABLE 1.1. Simply laced Euclidean diagrams



We recall that DJ is the simply laced Euclidean diagram $\widetilde{D}\Delta_J$ obtained from Δ_J by applying the inflation algorithm $\Delta_J \mapsto \widetilde{D}\Delta_J$ presented in [28, Algorithm 5.4] and [50, Algorithm 3.1] (see also [52]). Consequently, we have the passage

$$J \mapsto \Delta_J \mapsto DJ := \widetilde{D}\Delta_J.$$

We study the Coxeter spectral properties of any principal poset J by means of the Coxeter spectral properties of the associated simply laced Euclidean diagram $DJ \in \{\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$.

Following [45], with any poset J, we associate the **Coxeter matrix**

$$\mathrm{Cox}_J := -C_J \cdot C_J^{-tr},$$

with $\det \mathrm{Cox}_J = (-1)^m$, where $m = |J|$ and $C_J^{-tr} = (C_J^{tr})^{-1}$. The **Coxeter spectrum** $\mathbf{specc}_J \subseteq \mathbb{C}$ of J is defined to be the set of all $m = |J|$ complex roots of the **Coxeter polynomial**

$$\mathrm{cox}_J(t) = \det(t \cdot E - \mathrm{Cox}_J) \in \mathbb{Z}[t],$$

the **Coxeter number** $\mathbf{c}_J \geq 2$ is the minimal integer such that $\mathrm{Cox}_J^{\mathbf{c}_J} = E$, and the **Coxeter transformation** of J is the group automorphism

$$\Phi_J : \mathbb{Z}^J \to \mathbb{Z}^J, \quad \Phi_J(v) := v \mapsto v \cdot \mathrm{Cox}_J,$$

see [45] and [51] for details. If J is non-negative, the Coxeter spectrum $\mathbf{specc}_J$ lies on the unit circle $\mathcal{S}^1 = \{z \in \mathbb{C};\ |z| = 1\}$, consists of roots of unity, and $1 \notin \mathbf{specc}_J$ if and only if J is positive, see [50, Lemma 2.1] and [51]. In this case we have associated with J (see [47] and [51]) a reduced Coxeter number $\check{\mathbf{c}}_J$ and the **incidence defect homomorphism** $\widetilde{\partial}_J : \mathbb{Z}^J \to \mathrm{Ker}\, q_J \subseteq \mathbb{Z}^J$ such that

$$\Phi_J^{\check{\mathbf{c}}_J}(v) = v + \widetilde{\partial}_J(v), \text{ for all } v \in \mathbb{Z}^J,$$

where $\mathrm{Ker}\, q_J := \{v \in \mathbb{Z}^J;\ q_J(v) = 0\}$ is the kernel of the incidence quadratic form $q_J : \mathbb{Z}^J \to \mathbb{Z}$ defined by the formula

$$q_J(x) = \sum_{j \in J} x_j^2 + \sum_{i \prec j} x_i x_j = x \cdot C_J \cdot x^{tr}.$$

Since J is assumed to be non-negative, the quadratic form q_J is non-negative and $\mathrm{Ker}\, q_J$ is a subgroup of $\mathbb{Z}^J$, see [47]. If, in addition, the poset J is principal, the kernel $\mathrm{Ker}\, q_J$ is an infinite cyclic subgroup of

$\mathbb{Z}^J$ of the form $\operatorname{Ker} q_J = \mathbb{Z} \cdot \mathbf{h}_J$, where $\mathbf{h}_J$ is a non-zero vector in $\operatorname{Ker} q_J$ uniquely determined by J, up to multiplication by -1. In this case, $\widetilde{\partial}_J(v) = \partial_J(v) \cdot \mathbf{h}_J$, where $\partial_J : \mathbb{Z}^J \to \mathbb{Z}$ is a group homomorphism, called the **incidence defect** of J, see [51].

The following characterisation of principal posets obtained in [51, Proposition 9] is of importance.

Theorem 1.4. *Assume that J is a connected poset with $m = |J| \geq 2$ and let $G_J \in \mathbb{M}_J(\mathbb{Q})$ be the symmetric incidence Gram matrix of J* (1.2). *The following four conditions are equivalent.*

(a) *The poset J is principal.*

(b) *The incidence symmetric Gram matrix G_J is positive semidefinite of rank $m-1$.*

(c) *The incidence quadratic form $q_J : \mathbb{Z}^J \to \mathbb{Z}$ of J is non-negative and* $\operatorname{Ker} q_J = \mathbb{Z} \cdot \mathbf{h}$, *for some non-zero vector $\mathbf{h} \in \mathbb{Z}^J$.*

(d) *There exists a simply laced Euclidean diagram*

$$DJ \in \{\widetilde{\mathbb{A}}_s,\ s \geq 3,\ \widetilde{\mathbb{D}}_n,\ n \geq 4,\ \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$$

(*uniquely determined by J*) *such that the incidence symmetric Gram matrix G_J is $\mathbb{Z}$-congruent to the symmetric Gram matrix $G_{DJ} \in \mathbb{M}_{DJ}(\mathbb{Q})$ of the Euclidean diagram DJ, that is, there exists a $\mathbb{Z}$-invertible matrix* $B \in \mathrm{Gl}(m, \mathbb{Z})$ *such that $G_{DJ} = B^{tr} \cdot G_J \cdot B$.*

Proof. Apply [51, Proposition 3.2] and a characterisation of principal loop-free edge-bipartite graphs given in [50]. □

One of the aims of the Coxeter spectral analysis of finite posets is to study the following problem.

Problem 1.5. When the Coxeter spectrum $\mathbf{specc}_J$ of a poset J determines the incidence matrix C_J (hence the poset J) uniquely, up to a $\mathbb{Z}$-congruence.

In connection with Problem 1.5, and a problem studied by Horn and Sergeichuck in [25], we also consider the problem if for any $\mathbb{Z}$-invertible matrix $A \in \mathbb{M}_n(\mathbb{Z})$, there exists $B \in \mathbb{M}_n(\mathbb{Z})$ such that $A^{tr} = B^{tr} \cdot A \cdot B$ and $B^2 = E$ (the identity matrix).

We would like to note that the following problem remains unsolved.

Problem 1.6. Show that $DJ \neq \widetilde{\mathbb{A}}_{m-1}$, if J is a one-peak connected principal poset, with $m = |J| \geq 5$.

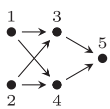
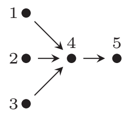
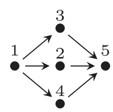
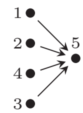
A partial solution of Problems 1.5 and 1.6 is given in the following three theorems proved in Sections 2 and 3.

Theorem 1.7. (a) *Assume that I is a poset of the shape* $\widetilde{\mathcal{D}}^{(1)}_{m,s,p}, \widetilde{\mathcal{D}}^{(2)}_{m,s}, \widetilde{\mathcal{D}}^{(3)}_{m,s,p}, \widetilde{\mathcal{D}}^{(4)}_{m,s,p}, \widetilde{\mathcal{D}}^{(5)}_{m,s,p,r}, \widetilde{\mathcal{D}}^{(6)}_{m,s,p,r}$ *or* $\widetilde{\mathcal{D}}^{(7)}_{m,s}$ *listed in Table* 1.11, *and* $|I| = m \geq 5$. *Then I is principal and the associated Euclidean graph DI of I is the diagram* $\widetilde{\mathbb{D}}_m$.

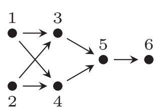
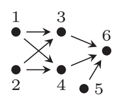
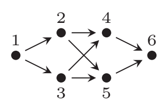
(b) *If I is any of the one-peak posets listed in Tables* 4.1 *and* 4.2 *then I is principal and the associated Euclidean graph DI of I is the diagram* $\widetilde{\mathbb{E}}_{m-1}$, *where* $m = |I|$.

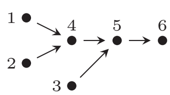
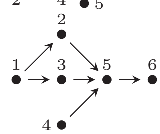
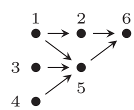
Theorem 1.8. *Assume that I is a one-peak principal poset with* $m := |I| \leq 15$ *and DI is its associated Euclidean diagram.*

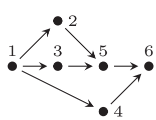
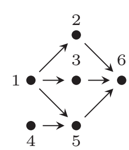
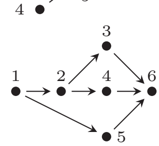
(a) $m \geq 5$ *and DI is not the diagram* $\widetilde{\mathbb{A}}_{m-1}$. *In particular, for* $m = 5$, *we have* $DI = \widetilde{\mathbb{D}}_4$ *and I is one of the four posets:*

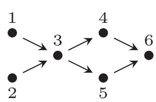
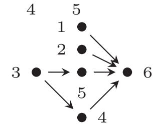
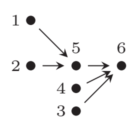
$\widetilde{\mathcal{D}}^{(1)}_{4,0,5}$: 
$\widetilde{\mathcal{D}}^{(2)}_{4,0}$: 
$\widetilde{\mathcal{D}}^{(4)}_{4,1,4}$: 
$\widetilde{\mathcal{D}}^{(6)}_{4,0,4,2}$: 

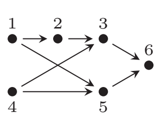
(b) *If* $m \geq 6$ *and* $DI = \widetilde{\mathbb{D}}_{m-1}$ *then I is one of the* 2.115 *posets of the shapes presented in Table* 1.11. *In particular, for* $m = 6$, *we have* $DI = \widetilde{\mathbb{D}}_5$ *and I is one of the* 13 *posets:*

$\widetilde{\mathcal{D}}^{(1)}_{5,0,5}$: 
$\widetilde{\mathcal{D}}^{(1)}_{5,0,6}$: 
$\widetilde{\mathcal{D}}^{(1)}_{5,1,6}$: 

$\widetilde{\mathcal{D}}^{(2)}_{5,0}$: 
$\widetilde{\mathcal{D}}^{(2)}_{5,1}$: 
$\widetilde{\mathcal{D}}^{(3)}_{5,2,2}$: 

$\widetilde{\mathcal{D}}^{(4)}_{5,1,4}$: 
$\widetilde{\mathcal{D}}^{(4)}_{5,1,5}$: 
$\widetilde{\mathcal{D}}^{(4)}_{5,2,5}$: 

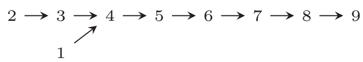
$\widetilde{\mathcal{D}}^{(5)}_{5,0,3,6}$: 
$\widetilde{\mathcal{D}}^{(6)}_{5,0,5,3}$: 
$\widetilde{\mathcal{D}}^{(6)}_{5,0,5,2}$: 

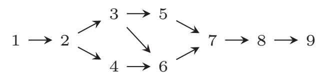
$\widetilde{\mathcal{D}}^{(7)}_{5,3}$: 

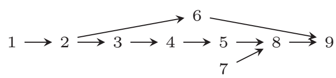
(c) *If* $m = 7$ *and* $DI = \widetilde{\mathbb{E}}_6$ *then* I *is one of the* 31 *posets presented in Table* 4.1.

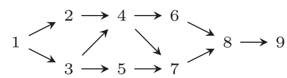
(d) *If* $m = 8$ *and* $DI = \widetilde{\mathbb{E}}_7$ *then* I *is one of the* 132 *posets presented in Table* 4.2.

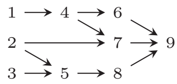
(e) *If* $m = 9$ *and* $DI = \widetilde{\mathbb{E}}_8$ *then* I *is one of the posets* $\mathcal{J}_1^{\widetilde{E}8}, \ldots, \mathcal{J}_{422}^{\widetilde{E}8}$ *listed in* [21]*. In particular, we have*

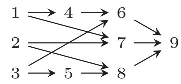
$\mathcal{J}_1^{\widetilde{E}8}$:


$\mathcal{J}_{31}^{\widetilde{E}8}$:


$\mathcal{J}_{48}^{\widetilde{E}8}$:


$\mathcal{J}_{147}^{\widetilde{E}8}$:


$\mathcal{J}_{412}^{\widetilde{E}8}$:


$\mathcal{J}_{422}^{\widetilde{E}8}$:


(f) *The total number of principal posets* J (*not necessarily one-peak ones*)*, with* $m = |J| \leq 15$*, equals* 158.448 *and the number* $\# J$ *of such posets* J *of the Coxeter-Euclidean type* $DJ \in \{\widetilde{\mathbb{D}}_{m-1}, \widetilde{\mathbb{E}}_{m-1}\}$ *is listed in the following tables.*

$n = \lvert J \rvert$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
DJ	$\widetilde{\mathbb{D}}_4$	$\widetilde{\mathbb{D}}_5$	$\widetilde{\mathbb{D}}_6$ / $\widetilde{\mathbb{E}}_6$	$\widetilde{\mathbb{D}}_7$ / $\widetilde{\mathbb{E}}_7$	$\widetilde{\mathbb{D}}_8$ / $\widetilde{\mathbb{E}}_8$	$\widetilde{\mathbb{D}}_9$
$\# J$	8	30	92 / 84	227 / 470	577 / 2 102	1.357

$n = \lvert J \rvert$	$n = 11$	$n = 12$	$n = 13$	$n = 14$	$n = 15$
DJ	$\widetilde{\mathbb{D}}_{10}$	$\widetilde{\mathbb{D}}_{11}$	$\widetilde{\mathbb{D}}_{12}$	$\widetilde{\mathbb{D}}_{13}$	$\widetilde{\mathbb{D}}_{14}$
$\# J$	3.217	7.371	16.897	38.069	85.561

Theorem 1.9. *Assume that* I *is a one-peak principal poset with* $5 \leq m := |I| \leq 15$, DI *is its associated Euclidean diagram,* $\check{G}_{DI}$ *is the non-symmetric Gram matrix of the graph* DI*, and* $\Phi_I : \mathbb{Z}^I \to \mathbb{Z}^I$ *is the incidence Coxeter transformation of* I*. Denote by*

$$\mathcal{R}_I := \mathcal{R}_{q_I} = \{v \in \mathbb{Z}^I;\ q_I(v) = 1\}$$

the set of roots of the incidence quadratic form $q_I : \mathbb{Z}^I \to \mathbb{Z}$.

(a) *There exists a* $\mathbb{Z}$*-invertible matrix* $B \in \mathbb{M}_m(\mathbb{Z})$ *such that* $\check{G}_{DI} = B^{tr} \cdot C_I \cdot B$,

(b) *The Coxeter number* $\mathbf{c}_I$ *of* I *is infinite, the incidence defect homomorphism* $\partial_I : \mathbb{Z}^I \to \mathbb{Z}$ *is non-zero and the set* $\partial_I^0\mathcal{R}_I \cup \operatorname{Ker} q_I$ *admits a* Φ_I*-mesh translation quiver* $\Gamma(\partial_I^0\mathcal{R}_I \cup \operatorname{Ker} q_I, \Phi_I)$ *of a sand-glass tube shape* (*in the sense of* [46]-[47]), *where*

$$\partial_I^0\mathcal{R}_I = \{v \in \mathbb{Z}^I;\ q_I(v) = 1 \text{ and } \partial_I(v) = 0\}.$$

(c) *There exists a* $\mathbb{Z}$*-invertible matrix* $C \in \mathbb{M}_m(\mathbb{Z})$ *such that* $C^2 = E$ *and* $C_I^{tr} = C^{tr} \cdot C_I \cdot C$.

The proofs of Theorems 1.7-1.9 are given in Sections 2 and 3.

We finish this section by a result that relates the Coxeter spectrum $\mathbf{specc}_I$ with the usual spectrum $\mathbf{spec}_I$ of a poset I, compare with [50, Proposition 2.4(c)].

Theorem 1.10. *Assume that* $I = \{1, \dots, m\}$ *is an arbitrary poset and* $\overline{P}_I(t) := \det(t \cdot E - \overline{A}d_I)$ *is the characteristic polynomial of the Euler adjacency matrix* $\overline{A}d_I := \overline{C}_I^{tr} + \overline{C}_I - 2E$ *of* I, *where* $\overline{C}_I := C_I^{-1}$ *is the Euler matrix of* I, *see* [45].

(a) *If the Hasse quiver of* I (*see* [43]) *is a tree then*

$$\operatorname{cox}_I(t^2) = t^m \cdot \overline{P}_I(t + \tfrac{1}{t}).$$

(b) *Assume that* I *is a one-peak poset and* $m := |I| \le 15$. *If* I *is positive or* I *is principal then* $\operatorname{cox}_I(t^2) = t^m \cdot \overline{P}_I(t + \frac{1}{t})$ *if and only if the Hasse quiver of* I *is a tree.*

Proof. Since $\det C_I = 1$, the Euler matrix $\overline{C}_I := C_I^{-1}$ lies in $\mathbb{M}_m(\mathbb{Z})$ and $C_I^{tr} = C_I^{tr} \cdot \overline{C}_I \cdot C_I$.

(a) Assume that the Hasse quiver of I is a tree. Without loss of generality, we may assume that the points of $I = \{a_1, \dots, a_m\}$ are numbered in such a way that $a_i \preceq a_j$ implies $i \le j$ in the natural order. Let $\overline{\Delta}_I$ be the Euler edge-bipartite graph associated with I, see [51, (33)].

By the definition of $\overline{\Delta}_I$, the non-symmetric Gram matrix of the edge-bipartite graph $\overline{\Delta}_I$ coincides with the Euler matrix $\overline{C}_I$. Hence, $\operatorname{Cox}_{\overline{\Delta}_I} = \overline{\operatorname{C}}\mathrm{ox}_I = -C_I^{-tr} \cdot C_I$ and $\operatorname{cox}_{\overline{\Delta}_I}(t) = \det(t \cdot E - \overline{\operatorname{C}}\mathrm{ox}_I) = \operatorname{cox}_I(t)$, see [50] and [51, Corollary 6]. Since the Hasse quiver of I is a tree then, by [44, Proposition 2.12], the Euler matrix $\overline{C}_I := C_I^{-1}$ of I has the form

$$\overline{C}_I := C_I^{-1} = \begin{bmatrix} 1 & c_{12}^{-} & \dots & c_{1m}^{-} \\ 0 & 1 & \dots & c_{2m}^{-} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \in \mathbb{M}_m(\mathbb{Z}),$$

with

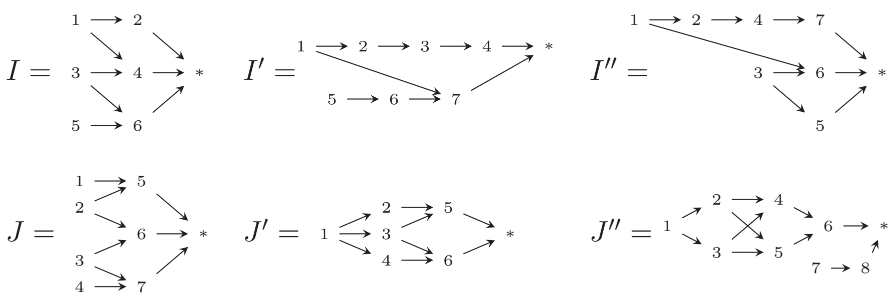
- $c^-_{11} = \ldots = c^-_{mm} = 1$,
- $c^-_{ab} = -1$, if there is an arrow $a \to b$ in the Hasse quiver Q_I of I,
- $c^-_{ab} = 0$, if $a \not\prec b$ or there is a path $a \prec j_1 \prec \ldots \prec j_{s-1} \prec j_s = b$ of length $s \geq 2$ in the Hasse quiver Q_I.

It follows from [51, (33)] that the Euler edge-bipartite graph $\overline{\Delta}_I$ is a tree and, by [50, Proposition 2.4(c)], we get

$$\mathrm{cox}_I(t^2) = \mathrm{cox}_{\overline{\Delta}_I}(t^2) = t^m \cdot \overline{P}_{\overline{\Delta}_I}(t + \tfrac{1}{t}) = t^m \cdot \overline{P}_I(t + \tfrac{1}{t}).$$

(b) Assume that I is a one-peak poset, $m := |I| \leq 15$, and I is positive or I is principal. If the Hasse quiver Q_I of I is a tree, the statement (a) applies. To prove the converse implication, assume to the contrary that the Hasse quiver Q_I of I is not a tree. If I is positive, I is one of the non-tree shape posets described in [18, Theorem 5.2(e)], see also [19]. If I is principal, I is one of the non-tree shape poset described in Theorem 1.8. By a case by case inspection of the posets from our lists, a straightforward computer calculations show that $\mathrm{cox}_I(t^2) \neq t^m \cdot \overline{P}_I(t + \frac{1}{t})$.

For example, if I, I', I'', and J, J', J'' are the positive and principal posets



then we have

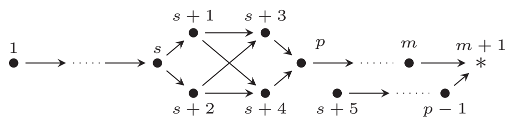
$\mathcal{L}$	$\mathrm{cox}_{\mathcal{L}}(t^2)$	$t^m \cdot \overline{P}_{\mathcal{L}}(t + \frac{1}{t})$
I	$t^{14} + t^{12} - t^8 - t^6$ $+t^2 + 1$	$t^{14} - 3t^{12} - 8t^{11} - 13t^{10} - 18t^9 - 21t^8 - 22t^7 - 21t^6 - 18t^5$ $-13t^4 - 8t^3 - 3t^2 + 1$
I'	$t^{16} + t^{14} + t^2 + 1$	$t^{16} - t^{14} - 2t^{13} - 3t^{12} - 6t^{11} - 7t^{10} - 8t^9 - 8t^8 - 8t^7 - 7t^6$ $-6t^5 - 3t^4 - 2t^3 - t^2 + 1$
I''	$t^{16} + t^{14} - t^{10} - t^8$ $-t^6 + t^2 + 1$	$t^{16} - 3t^{14} - 6t^{13} - 10t^{12} - 14t^{11} - 16t^{10} - 18t^9 - 19t^8 - 18t^7$ $-16t^6 - 14t^5 - 10t^4 - 6t^3 - 3t^2 + 1$

J	$t^{14}+t^{12}-2t^{8}-2t^{6}$ $+t^{2}+1$	$t^{14}-5t^{12}-12t^{11}-20t^{10}-28t^{9}-36t^{8}-40t^{7}-36t^{6}-28t^{5}$ $-20t^{4}-12t^{3}-5t^{2}+1$
J'	$t^{16}+t^{14}-t^{10}-2t^{8}$ $-t^{6}+t^{2}+1$	$t^{16}-3t^{14}-8t^{13}-14t^{12}-20t^{11}-25t^{10}-28t^{9}-30t^{8}-28t^{7}$ $-25t^{6}-20t^{5}-14t^{4}-8t^{3}-3t^{2}+1$
J''	$t^{18}+t^{16}-t^{14}-t^{12}$ $-t^{6}-t^{4}+t^{2}+1$	$t^{18}-7t^{16}-24t^{15}-53t^{14}-88t^{13}-121t^{12}-144t^{11}-156t^{10}$ $-160t^{9}-156t^{8}-144t^{7}-121t^{6}-88t^{5}-53t^{4}-24t^{3}-7t^{2}+1$

This finishes the proof of Theorem 1.10. □

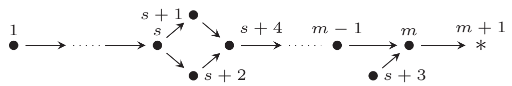
TABLE 1.11. Seven infinite series of principal one-peak posets of Coxeter-Euclidean type $\widetilde{\mathbb{D}}_m$

$\widetilde{\mathcal{D}}^{(1)}_{m,s,p}$:



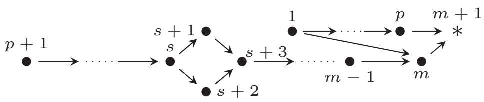
$0 \leq s, s+5 \leq p \leq m+1$, $4 \leq m$

$\widetilde{\mathcal{D}}^{(2)}_{m,s}$:



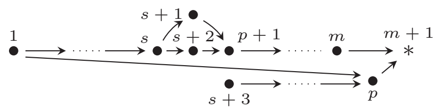
$0 \leq s, 4 \leq m, s+4 \leq m$

$\widetilde{\mathcal{D}}^{(3)}_{m,s,p}$:



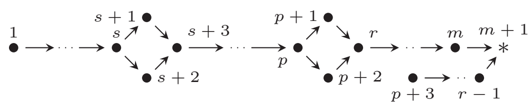
$1 < p \leq s \leq m-3$, $5 \leq m$, delete $\bullet_s$ if $s = p$

$\widetilde{\mathcal{D}}^{(4)}_{m,s,p}$:



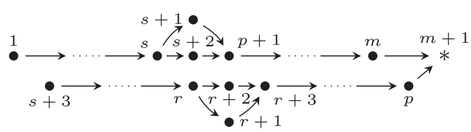
$1 \leq s, s+3 \leq p \leq m$, $4 \leq m$

$\widetilde{\mathcal{D}}^{(5)}_{m,s,p,r}$:



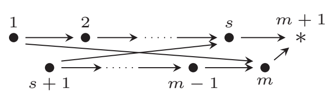
$0 \leq s, s+3 \leq p$, $6 \leq p+3 \leq r \leq m+1$, delete $\bullet_{p+3}$ if $p+3 = r$

$\widetilde{\mathcal{D}}^{(6)}_{m,s,p,r}$:



$0 \leq s, s+2 \leq r$, $4 \leq r+2 \leq p \leq m$, delete $\bullet_r$ if $r = s+2$

$\widetilde{\mathcal{D}}^{(7)}_{m,s}$:



$5 \leq m, 1 < s < m-2$

Convention 1.12. In Table 1.11, the following convention is used: we delete the vertex $\bullet_a$ if $a = 0$.

Remark 1.13. Let I be a one-peak principal poset I such that $|I| \leq 15$. It follows from Theorem 1.9 that the Coxeter polynomial $\mathrm{cox}_I(t) \in \mathbb{Z}[t]$ coincides with the Coxeter polynomial $\mathrm{cox}_{DI}(t)$ of the simply laced Euclidean diagram $DI \in \{\widetilde{\mathbb{D}}_n,\ n \geq 4,\ \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ associated with I, where

$$\mathrm{cox}_{DI}(t) = \begin{cases} t^{n+1}+t^n - t^{n-1} - t^{n-2} - t^3 - t^2+t+1, & \text{if } DI = \widetilde{\mathbb{D}}_n, \\ t^7 + t^6 - 2t^4 - 2t^3 + t + 1, & \text{if } DI = \widetilde{\mathbb{E}}_6, \\ t^8 + t^7 - t^5 - 2t^4 - t^3 + t + 1, & \text{if } DI = \widetilde{\mathbb{E}}_7, \\ t^9 + t^8 - t^6 - t^5 - t^4 - t^3 + t + 1, & \text{if } DI = \widetilde{\mathbb{E}}_8, \end{cases} \tag{1.13}$$

see [33] and [45]. In particular, for $n = 4$ and $n = 5$, we have

$\mathrm{cox}_{\widetilde{\mathbb{D}}_4}(t) = t^5 + t^4 - 2t^3 - 2t^2 + t + 1,$

$\mathrm{cox}_{\widetilde{\mathbb{D}}_5}(t) = t^6 + t^5 - t^4 - 2t^3 - t^2 + t + 1.$

2. Proof of Theorems 1.7 and 1.8

In this section we present the proof of Theorems 1.7 and 1.8 that are main results of the paper. First, we note that every poset $J = (\{1, \ldots, m\}, \preceq)$ is $\mathbb{Z}$-bilinear equivalent to a poset J' with vertices numbered in such a way that $i \preceq j$ implies $i \leq j$ in a natural order and the matrix $C_{J'}$ is upper triangular. Therefore, without loss of generality, we can assume that the matrix $C_J \in \mathbb{M}_J(\mathbb{Z})$ of any poset J has an upper triangular form.

Following [51], we identify a poset $J = (J, \preceq)$ with its acyclic edge-bipartite graph Δ_J (usually viewed as a signed graph in the sense of [56]) without continuous edges, and with the dotted edges $\bullet_i$- - - -$\bullet_j$, for all $i \prec j$. We recall from [51] that Δ_J is uniquely determined by its non-symmetric Gram matrix $\check{G}_{\Delta_J} \equiv C_J$.

One of our main tools is the second step $\Delta \mapsto \Delta' := t^-_{ab}\Delta$ of the inflation algorithm [28, Algorithm 5.4] and [50, Algorithm 3.1] (see also [52]) that associates to any loop-free principal edge-bipartite graph Δ, with a fixed dotted edge $\bullet_a$- - - -$\bullet_b$ in $\Delta_1(a, b)$, $a \neq b$, the loop-free principal edge-bipartite graph $\Delta' := t^-_{ab}\Delta$ as follows:

- we set $\Delta_0 = \Delta'_0$ and replace the dotted edge $\bullet_a$- - - -$\bullet_b$ in Δ by a continuous one $\bullet_a$———$\bullet_b$ in Δ',
- given $c \neq a$ in $\Delta_0 = \Delta'_0$ such that $\Delta_1(a, c) \neq \emptyset$, we define $\Delta'_1(b, c)$ to be the set with exactly $d^{\Delta'}_{bc}$ dotted edges if $d^{\Delta'}_{bc} > 0$, and exactly $-d^{\Delta'}_{bc}$ continuous edges if $d^{\Delta'}_{bc} \leq 0$, where $d^{\Delta'}_{bc} := d^{\Delta}_{bc} - d^{\Delta}_{ac} \cdot d^{\Delta'}_{ab}$,
- each of the remaining edges of Δ_1 becomes an edge in Δ'_1, i.e., we set $\Delta'^+_1(a', b') = \Delta^+_1(a', b')$ and $\Delta'^-_1(a', b') = \Delta^-_1(a', b')$, if $(a', b') \neq (a, b)$ or $(a', b') \neq (b, c)$.

The main idea of the inflation algorithm is to reduce the number of dotted edges in Δ. It is shown in [50] that $\check{G}_{\Delta'} = \nabla((T_{ab}^{-})^{tr} \cdot \check{G}_{\Delta} \cdot T_{ab}^{-})$, where

$$T_{ab}^{-} = [t_{ij}], \text{ with } t_{ij} = \begin{cases} 1, & \text{if } i = j \text{ or } (i,j) \neq (a,b), \\ -1, & \text{if } (i,j) = (a,b), \\ 0, & \text{otherwise,} \end{cases}$$

and the operation $C \mapsto \nabla(C) = [c_{ij}^{\triangledown}]$ (introduced in [47, (4.5)]) associates to any square matrix $C = [c_{ij}] \in \mathbb{M}_m(\mathbb{R})$ the upper triangular matrix

$$\nabla(C) = [c_{ij}^{\triangledown}] = \begin{bmatrix} c_{11}^{\triangledown} & c_{12}^{\triangledown} & \dots & c_{1m}^{\triangledown} \\ 0 & c_{22}^{\triangledown} & \dots & c_{1m}^{\triangledown} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{mm}^{\triangledown} \end{bmatrix} \in \mathbb{M}_m(\mathbb{R})$$

by setting $c_{ij}^{\triangledown} = c_{ij} + c_{ji}$, for $i < j$, $c_{ij}^{\triangledown} = 0$, for $i > j$, and $c_{jj}^{\triangledown} = c_{jj}$, for $j = 1, \dots, m$.

Note that the inflation matrix $T_{ab}^{-} = [t_{ij}] \in \mathbb{M}_m(\mathbb{Z})$ is the identity matrix with an element t_{ab} changed to -1.

Proof of Theorem 1.7. (a) Assume that

$$I \in \{\widetilde{\mathcal{D}}_{m,s,p}^{(1)}, \widetilde{\mathcal{D}}_{m,s}^{(2)}, \widetilde{\mathcal{D}}_{m,s,p}^{(3)}, \widetilde{\mathcal{D}}_{m,s,p}^{(4)}, \widetilde{\mathcal{D}}_{m,s,p,r}^{(5)}, \widetilde{\mathcal{D}}_{m,s,p,r}^{(6)}, \widetilde{\mathcal{D}}_{m,s}^{(7)}\}$$

with $|I| = m + 1 \geq 5$, as listed in Table 1.11. Our aim is to construct a matrix $B_I \in \mathbb{M}_{m+1}(\mathbb{Z})$ such that

$$(\check{G}_{\widetilde{\mathbb{D}}_m}^{tr} + \check{G}_{\widetilde{\mathbb{D}}_m}) = B_I^{tr} \cdot (C_I^{tr} + C_I) \cdot B_I.$$

Since $\det C_I = 1$, the matrix C_I is $\mathbb{Z}$-invertible and $C_I^{-1} = C_I^{-1} \cdot C_I^{tr} \cdot C_I^{-tr}$. Note that, for $B_I'' := C_I^{-tr}$, we have $(C_I^{-tr} + C_I^{-1}) = B_I''^{tr} \cdot (C_I^{tr} + C_I) \cdot B_I''$. Therefore, it is sufficient to construct a $\mathbb{Z}$-invertible matrix $B_I' \in \mathrm{Gl}(m+1, \mathbb{Z})$ such that

$$B_I'^{tr} \cdot (C_I^{-tr} + C_I^{-1}) \cdot B_I' = (\check{G}_{\widetilde{\mathbb{D}}_m}^{tr} + \check{G}_{\widetilde{\mathbb{D}}_m}),$$

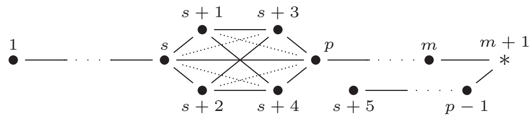
because then, for $B_I := B_I'' \cdot B_I' \in \mathbb{M}_{m+1}(\mathbb{Z})$, we have

$$\begin{aligned} (\check{G}_{\widetilde{\mathbb{D}}_m}^{tr} + \check{G}_{\widetilde{\mathbb{D}}_m}) &= B_I'^{tr} \cdot (C_I^{-tr} + C_I^{-1}) \cdot B_I' \\ &= B_I'^{tr} \cdot (B_I''^{tr} \cdot (C_I^{tr} + C_I) \cdot B_I'') \cdot B_I' \\ &= (B_I'' \cdot B_I')^{tr} \cdot (C_I^{tr} + C_I) \cdot (B_I'' \cdot B_I') \\ &= B_I^{tr} \cdot (C_I^{tr} + C_I) \cdot B_I. \end{aligned}$$

We construct the matrix $B'_I \in \mathbb{M}_{m+1}(\mathbb{Z})$ in six cases considered below, in the form of the product of inflation matrices $T^-_{ab} = [t'_{ij}] \in \mathbb{M}_{m+1}(\mathbb{Z})$, by applying the inflation algorithm procedure.

First, we note that our assumptions imply that the matrix C_I^{-1} is upper triangular and coincides with the non-symmetric Gram matrix $\check{G}_{\overline{\Delta}_I}$ that defines the Euler acyclic edge-bipartite graph $\overline{\Delta}_I$ of I, see [51].

Case 1° Assume that $I = \widetilde{\mathcal{D}}^{(1)}_{m,s,p}$. The Euler bigraph $\overline{\Delta}_I$ of I, with $\check{G}_{\overline{\Delta}_I} = C_I^{-1}$, has the shape

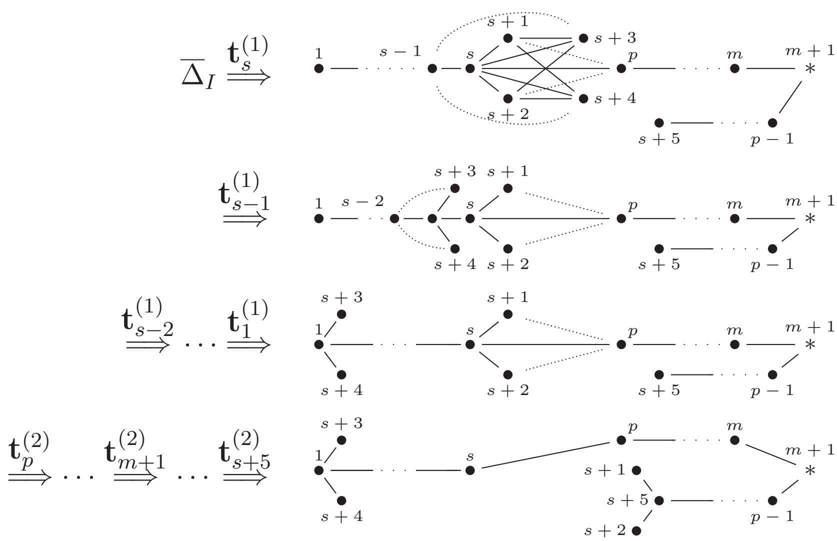
$\overline{\Delta}_I =$ 

To simplify the notation, we set

$\mathbf{t}^{(1)}_j := \mathbf{t}^-_{j\,s+3} \circ \mathbf{t}^-_{j\,s+4}$, $\quad \mathbf{t}^{(2)}_j := \mathbf{t}^-_{j\,s+1} \circ \mathbf{t}^-_{j\,s+2}$,

$T^{(1)}_j := T^-_{j\,s+3} \cdot T^-_{j\,s+4}$ and $T^{(2)}_j := T^-_{j\,s+1} \cdot T^-_{j\,s+2}$.

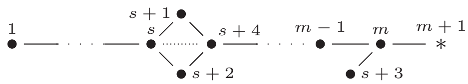
The passage $\overline{\Delta}_I \mapsto D\overline{\Delta}_I$ and the construction of the matrix $B'_I \in \mathbb{M}_{m+1}(\mathbb{Z})$ can be illustrated as follows.



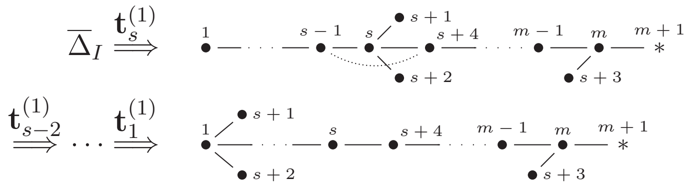
Summing up, $D\overline{\Delta}_I = \widetilde{\mathbb{D}}_m$ and we define $B'_I \in \mathrm{Gl}(m+1,\mathbb{Z})$ to be the matrix

$$B'_I := T^{(2)}_{s+5} \cdots T^{(2)}_{p-1} \cdot T^{(2)}_{m+1} \cdot T^{(2)}_{m} \cdots T^{(2)}_{p} \cdot T^{(1)}_{1} \cdots T^{(1)}_{s-1} \cdot T^{(1)}_{s}$$

Case 2° Assume that $I = \widetilde{\mathcal{D}}^{(2)}_{m,s}$. The Euler bigraph $\overline{\Delta}_I$ of I, with $\check{G}_{\overline{\Delta}_I} = C_I^{-1}$, has the shape

$\overline{\Delta}_I =$ 

We set $\mathbf{t}^{(1)}_j = \mathbf{t}^{-}_{j\,s+4}$ and $T^{(1)}_j = T^{-}_{j\,s+4}$. The passage $\overline{\Delta}_I \mapsto D\overline{\Delta}_I$ and the construction of the matrix $B'_I \in \mathbb{M}_{m+1}(\mathbb{Z})$ can be illustrated as follows:



We have $D\overline{\Delta}_I = \widetilde{\mathbb{D}}_m$ and we define B'_I to be the matrix

$$B'_I := T^{(1)}_1 \cdot T^{(1)}_2 \cdots T^{(1)}_{s-1} \cdot T^{(1)}_s$$

Case 3° Assume that I is one of the posets $I_1 = \widetilde{\mathcal{D}}^{(3)}_{m,s,p}$ and $I_2 = \widetilde{\mathcal{D}}^{(4)}_{m,s,p}$. Then $D\overline{\Delta}_I = \widetilde{\mathbb{D}}_m$ and B'_I is defined to be the matrix

$$\begin{aligned} B'_{I_1} :=& T^{(3)}_{p-1} \cdots T^{(3)}_2 \cdot T^{(3)}_1 \cdot T^{(1)}_{p+1} \cdot T^{(1)}_{p+2} \cdots T^{(1)}_{s-1} \cdot T^{(1)}_s, \text{ if } I = I_1 \text{ and}\\ B'_{I_2} :=& T^{(2)}_{s+3} \cdots T^{(2)}_{p-1} \cdot T^{(2)}_p \cdot T^{(2)}_1 \cdots T^{(2)}_s \cdot T^{(3)}_{m-1} \cdots T^{(3)}_2 \cdot T^{(3)}_1, \text{ if } I = I_2, \end{aligned}$$

where $T^{(1)}_j = T^{-}_{j\,s+3}$, $T^{(2)}_j = T^{-}_{j\,p+1}$ and $T^{(3)}_j = T^{-}_{j\,m+1}$.

Case 4° If $I = \widetilde{\mathcal{D}}^{(5)}_{m,s,p,r}$, then $D\overline{\Delta}_I = \widetilde{\mathbb{D}}_m$ and B'_I is defined to be the matrix

$$B'_I := T^{(2)}_{p+3} \cdots T^{(2)}_{r-1} \cdot T^{(2)}_{m+1} \cdot T^{(2)}_m \cdots T^{(2)}_{r+1} \cdot T^{(2)}_r \cdot T^{(1)}_1 \cdots T^{(1)}_{s-1} \cdot T^{(1)}_s,$$

where $T^{(1)}_j = T^{-}_{j\,s+3}$ and $T^{(2)}_j = T^{-}_{j\,r}$.

Case 5° If $I = \widetilde{\mathcal{D}}^{(6)}_{m,s,p,r}$, then $D\overline{\Delta}_I = \widetilde{\mathbb{D}}_m$ and B'_I is defined to be the matrix

$$B'_I := T^{(2)}_{s+3} \cdots T^{(2)}_{r-1} \cdot T^{(2)}_r \cdot T^{(1)}_1 \cdots T^{(1)}_{s-1} \cdot T^{(1)}_s,$$

where $T^{(1)}_j = T^{-}_{j\,p+1}$ and $T^{(2)}_j = T^{-}_{j\,r+3}$.

Case 6° If $I = \widetilde{\mathcal{D}}^{(7)}_{m,s}$, then $D\overline{\Delta}_I = \widetilde{\mathbb{D}}_m$ and B'_I is defined to be the matrix

$$B'_I := T^{(1)}_{s-2} \cdots T^{(1)}_2 \cdot T^{(1)}_1 \cdot T^{(2)}_{s+3} \cdots T^{(2)}_m \cdot T^{(2)}_{m+1} \cdot T^{-}_{m+1\,1},$$

where $T^{(1)}_j = T^{-}_{j\,s}$ and $T^{(2)}_j = T^{-}_{j\,s+1}$.

To finish the proof of (a), we recall from [50] that the Euclidean diagrams are principal and the existence of a $\mathbb{Z}$-equivalence of a finite poset I with a Euclidean diagram $\widetilde{\mathbb{D}}_m$ implies that I is principal.

(b) The proof is a computational one and we proceed similarly as in the proof of (a). As earlier, we identify the poset I with its acyclic edge-bipartite graph Δ_I and apply the inflation algorithm to Δ_I. In this way we obtain a matrix B_I defining the $\mathbb{Z}$-equivalence of the poset I with the Euclidean diagram $\widetilde{\mathbb{E}}_m$, where $m = |I| - 1$. Hence I is principal and the proof is complete. □

Proof of Theorem 1.8. Assume that $m \leq 15$ and I is a principal one-peak poset. We compute a complete list of such posets I, with $|I| = m \leq 15$, and their Coxeter-Euclidean types DI by applying the inflation algorithm [50] and Algorithm 3.1 described in Section 3. In particular, the computations show that:

(a) there is no such a poset I that $DI = \widetilde{\mathbb{A}}_{m-1}$,

(b) if $DI = \widetilde{\mathbb{D}}_{m-1}$, then I is one of the posets listed in Theorem 1.7,

(c) if $7 \leq m \leq 9$ and $DI = \widetilde{\mathbb{E}}_{m-1}$, then I is one of the posets listed in (c), (d), and (e) of Theorem 1.8. □

3. Algorithms

In this section, we outline a description of computational algorithms we use in the proofs of Theorems 1.8 and 1.9. First, we discuss an algorithm used in computation of all principal posets with at most 15 elements, compare with [37] and [38].

Algorithm 3.1. Input: An integer $1 \leq n \leq 15$.

Output: Finite sets $\mathbf{princ}[1], \ldots, \mathbf{princ}[n]$ of all connected non-negative posets of corank 1 encoded in the form of their incidence matrices. That is, the set $\mathbf{princ}[i]$ contains the principal posets with i vertices.

Step 1° Initialize the set $\mathbf{princ}[1]$ with the matrix $\begin{bmatrix} 1 \end{bmatrix} \in \mathbb{M}_1(\mathbb{Z})$.

Step 2° For every m from 2 to n:

Step 2.1° Initialize an empty list $\mathbf{candidate_m}$.

Step 2.2° For every poset $J \in \mathbf{princ}[m-1]$, generate a list of all possible extensions of J to a poset with m vertices. In other words, generate a list $W_J \ni w$ of all vectors $w = [w_2, \ldots, w_m] \in \{0,1\}^{m-1}$ such that the matrix

$$C_{J_w} = \left[\begin{array}{c|c} 1 & w \\ \hline 0 & C_J \end{array}\right] = [c_{ij}] \in \mathbb{M}_m(\mathbb{Z})$$

is an incidence matrix of a poset J_w (matrix with the following transitivity property: $c_{ij} = 1$ and $c_{js} = 1$ implies $c_{is} = 1$, for $1 \le i, j, s \le m$).

Step 2.3° For every poset $J \in \mathbf{princ}[m-1]$ and for every vector $w \in W_J$, construct the matrix $C_{J_w} \in \mathbb{M}_m(\mathbb{Z})$ and add the poset J_w to the list $\mathbf{candidate_m}$ if the symmetric matrix $C_{J_w} + C_{J_w}^{tr}$ is non-negative of corank at most 1, (by checking, for example, if all diagonal minors of the matrix are non-negative - the extended Sylvester criterion, see [16]).

Step 2.4° Construct the set $\mathbf{princ}[m]$ by selecting non isomorphic posets from the list $\mathbf{candidate_m}$ (using the Hasse digraph representation in order to test poset isomorphism).

Step 3° Remove from the sets $\mathbf{princ}[1], \ldots, \mathbf{princ}[n]$ the matrices of posets J_w that are not connected (using a graph search algorithm, such as breadth-first search – BFS) or have the symmetric matrix $C_{J_w} + C_{J_w}^{tr}$ positive definite (for example, by checking if all principal minors of $C_{J_w} + C_{J_w}^{tr}$ are positive - Sylvester criterion).

Step 4° Return the sets $\mathbf{princ}[1], \ldots, \mathbf{princ}[n]$ as a result.

Remark 3.2. (a) Note that posets J, with $|J| \le 3$, need not to be checked, because any such a poset has the shape $(\circ)$, $(\circ\ \circ)$, $(\circ \to \circ)$, $(\circ\ \circ\ \circ)$, $(\circ\ \circ \!\to\! \circ)$, $(\circ \!\to\! \circ \!\to\! \circ)$, $(\circ \!\to\! \circ \!\leftarrow\! \circ)$ or $(\circ \!\leftarrow\! \circ \!\to\! \circ)$ and is positive.

(b) Note that Step 2.3° and Step 2.4° can be done simultaneously by adding to the set $\mathbf{princ}[m]$ only these non-negative posets that have Hasse digraphs not isomorphic to the posets that are already in the set $\mathbf{princ}[m]$.

(c) In our implementation of the algorithm we use the **igraph** package (`http://igraph.sourceforge.net/`) to test digraph isomorphism in step 2.4°.

(d) A simple check of the equivalence of the degrees of vertices to detect non-isomorphic digraphs before the usage of more advance algorithm in the step 2.4° gave us a considerable speed up.

(e) It is easy to efficiently implement the algorithm in a parallel environment (for example, Step 3° can be executed in parallel without need of any synchronisation).

In the proof of Theorem 1.9 we determine the $\mathbb{Z}$-congruence $C_I \sim_{\mathbb{Z}} \check{G}_{DI}$, for any principal poset I, $|I| \leq 15$, of the Coxeter-Euclidean type

$$DI \in \{\widetilde{\mathbb{D}}_m,\ m \geq 4,\ \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}.$$

We do it by constructing a matrix $B \in \mathbb{M}_n(\mathbb{Z})$ with $n = |I|$, such that $C_I = B^{tr} \cdot \check{G}_{DI} \cdot B$. We essentially use the following theorem and Procedure 3.6.

Theorem 3.3. (a) *Let D be one of the Euclidean diagrams $\widetilde{\mathbb{D}}_m, m \geq 4$, $\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$, with vertices numbered as in Introduction. Let $\mathcal{R}_D$ be the set of roots of the Euler quadratic form q_D of D. Then there exists a unique Φ_D-mesh translation quiver $\Gamma(\mathcal{R}_D, \Phi_D)$ of Φ_D-orbits of the set $\mathcal{R}_D$ (called a Φ_D-mesh geometry of roots of D) such that $\Gamma(\mathcal{R}_D, \Phi_D)$ admits a principal Coxeter Φ_Δ-orbit configuration $\Gamma_D^{\check{G}_D}$ of simple roots of the form presented below* (*see also* [47] *and* [48, Table 4.7]).

(b) *If I is a connected principal poset, with $n \geq 5$ elements, of the Coxeter-Euclidean type DI and $B \in \mathrm{Gl}(n, \mathbb{Z})$ is such a matrix that $\check{G}_{DI} = B \cdot \check{G}_I \cdot B^{tr}$, then the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{\ \Phi_D\ } & \mathbb{Z}^n \\ h\downarrow\simeq & & h\downarrow\simeq \\ \mathbb{Z}^n & \xrightarrow{\ \Phi_I\ } & \mathbb{Z}^n \end{array} \tag{3.4}$$

where $h = h_B$ is the group isomorphism defined by $h_B(x) = x \cdot B$.

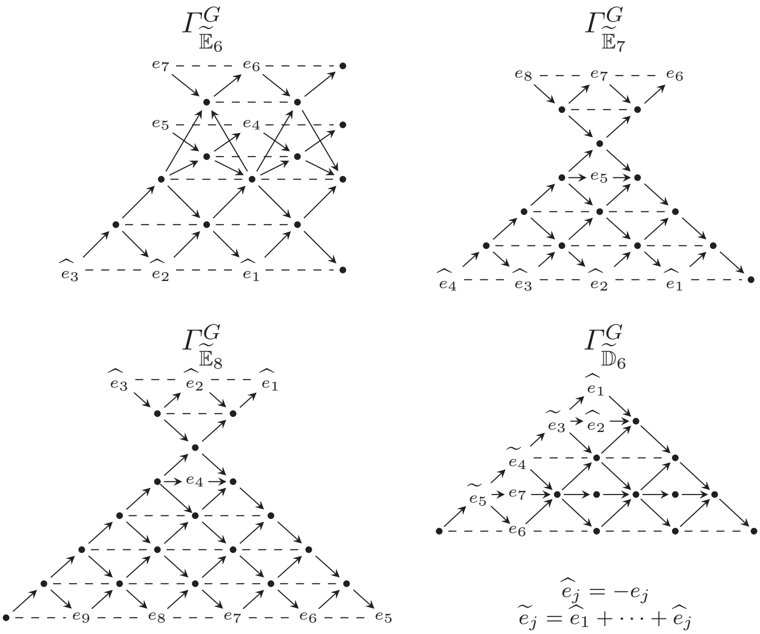
If $\mathcal{R}_I := \mathcal{R}_{q_I}$ is the set of roots of the unit quadratic form $q_I : \mathbb{Z}^I \to \mathbb{Z}$ and $\Gamma(\mathcal{R}_{DI}, \Phi_{DI})$ is Φ_{DI}-mesh geometry of roots of DI (with principal Coxeter Φ_{DI}-orbit configuration $\Gamma_{DI}^{\check{G}_{DI}}$), then the group isomorphism $h = h_B$ carries it to the Φ_I-mesh geometry $\Gamma(\mathcal{R}_I, \Phi_I)$ (with a principal Coxeter Φ_I-orbit configuration $\Gamma_I^{C_I}$), induces the mesh translation quiver isomorphism

$$h : \Gamma(\mathcal{R}_{DI}, \Phi_{DI}) \xrightarrow{\simeq} \Gamma(\mathcal{R}_I, \Phi_I)$$

and the quiver isomorphism $h : \Gamma_{DI}^{\check{G}_{DI}} \xrightarrow{\simeq} \Gamma_I^{C_I}$. Moreover, the matrix B has the form

$$B = \begin{bmatrix} h(e_1) \\ h(e_2) \\ \vdots \\ h(e_n) \end{bmatrix}. \tag{3.5}$$

Proof. Apply [46, Section 5], [46, Theorems 4.7 and Proposition 4.8] and their proofs, see also [47], [48, Lemma 4.3], and [50]. □



Procedure 3.6. Assume that I is a poset of the Coxeter-Euclidean type $DI \in \{\widetilde{\mathbb{D}}_{n-1}, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$, with $n = |I|$.

To construct a matrix B that defines a $\mathbb{Z}$-equivalence $C_I \sim_{\mathbb{Z}} \check{G}_{DI}$, we proceed as follows.

1° First, by applying the mesh toroidal algorithm described in [46, Proposition 4.5] and [47] (see also [48, Lemma 4.6 and Algorithm 4.8.2]), we construct the Φ_I-mesh translation quiver $\Gamma(\mathcal{R}_I, \Phi_I)$, with a principal Coxeter Φ_I-orbit configuration $\Gamma_I^{C_I}$ of simple roots of q_I, together with a mesh quiver isomorphism $h : \Gamma_{DI}^{\check{G}_{DI}} \to \Gamma_I^{C_I}$.

2° Next, we define the matrix B by setting

$$B = \begin{bmatrix} h(e_1) \\ h(e_2) \\ \vdots \\ h(e_n) \end{bmatrix},$$

as in (3.5) and [48, Lemma 4.3].

3° Finally, we check the matrix equality $\check{G}_{DI} = B \cdot C_I \cdot B^{tr}$. □

Remark 3.7. Procedure 3.6 has implementations in Maple and Python, with an assistance of programming and graphics in Java.

Algorithm 3.8. Input: A non-negative poset I, with n elements, encoded in the form of incidence matrix $C_I \in \mathbb{M}_n(\mathbb{Z})$.

Output: Reduced Coxeter number $\check{\mathbf{c}}_I \in \mathbb{Z}$.

Step 1° Initialize the symbolic vector $v := [v_1, \dots, v_n]$.

Step 2° Calculate the Coxeter matrix $\mathrm{Cox}_I := -C_I \cdot C_I^{-tr}$.

Step 3° For $r = 1, 2, 3, \dots$

Step 3.1° Calculate $w := v \cdot \mathrm{Cox}_I^r - v$.

Step 3.2° If $q_I(w) := w \cdot C_I \cdot w^{tr}$ equals zero then stop the calculations and return $\check{\mathbf{c}}_I$ as a result.

Remark 3.9. By [51, Theorem 18], Algorithm 3.8 returns $\check{\mathbf{c}}_I$ in a finite number of steps.

We use the following algorithm computing the incidence defect of any principal poset.

Algorithm 3.10. Input: A principal poset I, with n elements, encoded in the form of incidence matrix $C_I \in \mathbb{M}_n(\mathbb{Z})$.

Output: The incidence defect $\partial_I : \mathbb{Z}^n \to \mathbb{Z}$.

Step 1° Initialize the symbolic vector $v := [v_1, \dots, v_n]$ and calculate the Coxeter matrix $\mathrm{Cox}_I := -C_I \cdot C_I^{-tr}$.

Step 2° Using Algorithm 3.8, compute the reduced Coxeter number $\check{\mathbf{c}}_I \in \mathbb{Z}$ and calculate the vector $w = [w_1, \dots, w_n] := v \cdot \mathrm{Cox}_I^{\check{\mathbf{c}}} - v$.

Step 3° Compute the vector $\mathbf{h} \in \mathbb{Z}^n$ such that $\mathrm{Ker}\, q_I = \mathbb{Z} \cdot \mathbf{h}$ by solving in $\mathbb{Z}$ the system of $\mathbb{Z}$-linear equations $(C_I + C_I^{tr}) \cdot v^{tr} = 0$ (for example, using the procedure **isolve** of the **LinearAlgebra** package in MAPLE).

Step 4° Solve the symbolic system of linear equations $\lambda \cdot \mathbf{h}_I = w$ (in $\mathbb{Z}$), for the unknown λ, and return computed λ as a result.

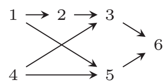
Outline of proof of Theorem 1.9. Assume that I is a one-peak principal poset with $m := |I| \leq 15$. By Theorem 1.8, I is one of the four five elements posets listed in 1.8(a), one of the 2.115 posets with $6 \leq m \leq 15$ of the shape listed in Table 1.11, one of the 31 posets listed in Table 4.1, one of the 132 posets listed in Table 4.2, or one of the 422 posets listed in [21]. By applying Algorithms 3.8 and 3.10, for each of the posets I from this finite list, we calculate its reduced Coxeter number $\check{\mathbf{c}}_I$ and the incidence defect $\partial_I : \mathbb{Z}^I \to \mathbb{Z}$. In particular, we show that ∂_I is non-zero. It follows from [51, Theorem 1.18 (c)] that the Coxeter number $\mathbf{c}_I$ of I is infinite, see also [47, Corollary 4.15 (c)] and [49, Proposition 3.12].

(a) By applying Procedure 3.6 as in [17, Section 7], for each of the posets I from the finite list described above, we construct a $\mathbb{Z}$-invertible matrix B_I such that $\check{G}_{DI} = B_I \cdot C_I \cdot B_I^{tr}$. By setting $B := B_I^{tr}$ we get the required equality $\check{G}_{DI} = B^{tr} \cdot C_I \cdot B$.

The proof is long and a computational one. Here we do not present a complete proof, but we illustrate its idea on examples of several posets. The proof for the remaining posets of the list is analogous.

First we illustrate an application of Procedure 3.6 for the posets $\widetilde{\mathcal{D}}_{5,3}^{(7)}$ and $\mathcal{J}_{29}^{\widetilde{E}6}$ presented in Theorem 1.8 (b) and Table 4.1, respectively.

1° If we enumerate the elements of the poset $J := \widetilde{\mathcal{D}}_{5,3}^{(7)}$ from Theorem 1.8 (b) as follows

$J =$ 

then the incidence matrix $C_J \in \mathbb{M}_6(\mathbb{Z})$, the Coxeter matrix and the incidence quadratic form $q_J : \mathbb{Z}^6 \to \mathbb{Z}$ are given as follows

$$C_J = \begin{bmatrix} 1&1&1&0&1&1\\ 0&1&1&0&0&1\\ 0&0&1&0&0&1\\ 0&0&1&1&1&1\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{bmatrix}, \quad \mathrm{Cox}_J = \begin{bmatrix} 0&0&0&1&0&-1\\ 0&0&0&0&1&-1\\ -1&1&0&0&1&-1\\ 0&1&0&0&0&-1\\ 0&0&1&0&0&-1\\ -1&0&1&-1&1&-1 \end{bmatrix},$$

$$\begin{aligned} q_J(x) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_1x_2 + x_3(x_1 + x_2 + x_4) \\ &\quad +(x_1 + x_4)x_5 + (x_1 + x_2 + x_3 + x_4 + x_5)x_6 \\ &= \left(x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 + \tfrac{1}{2}x_5 + \tfrac{1}{2}x_6\right)^2 + \tfrac{3}{4}\left(x_2 + \tfrac{1}{3}x_3 - \tfrac{1}{3}x_5 + \tfrac{1}{3}x_6\right)^2 \\ &\quad + \tfrac{2}{3}\left(x_3 + \tfrac{3}{4}x_4 - \tfrac{1}{4}x_5 + \tfrac{1}{4}x_6\right)^2 + \tfrac{5}{8}\left(x_4 + x_5 + \tfrac{3}{5}x_6\right)^2 + \tfrac{2}{5}x_6^2. \end{aligned}$$

It follows that $q_J : \mathbb{Z}^6 \to \mathbb{Z}$ is non-negative and $\operatorname{Ker} q_J = \mathbb{Z} \cdot \mathbf{h}_J$, where $\mathbf{h}_J = [1, 0, -1, 1, -1, 0]$; hence J is principal. By Theorem 1.7, the poset J is of Euclidean type $DJ = \widetilde{\mathbb{D}}_5$. A routine calculations show that

- $\mathrm{cox}_J(t) = t^6 + t^5 - t^4 - 2t^3 - t^2 + t + 1 = \mathrm{cox}_{\widetilde{\mathbb{D}}_5}(t)$,
- $\check{\mathbf{c}}_J = 6$, $\mathbf{c}_J = \infty$,
- $\widetilde{\partial}_J(v) = 2(v_1 + v_2 + v_3 + v_4 + v_5) \cdot \mathbf{h}_J$, $\partial_J(v) = 2(v_1 + v_2 + v_3 + v_4 + v_5)$,

and the set $\mathcal{R}_J$ of roots of q_J has the decomposition

$$\mathcal{R}_J = \mathcal{R}_J^{red} + \mathbb{Z} \cdot \mathbf{h}_J,$$

where the finite set

$$\mathcal{R}_J^{red} = \{v \in \mathbb{Z}^6;\, v_1 = 0 \text{ and } q_J(v) = 1\}$$

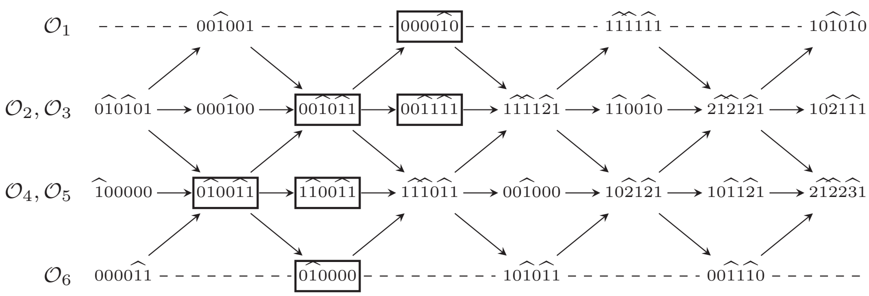
is a reducer of $\mathcal{R}_J$ in the sense of [47]. One shows that

$$|\mathcal{R}_J^{red}| = 40,\ |\partial_J^0\mathcal{R}_J^{red}| = 10,\ |\partial_J^-\mathcal{R}_J^{red}| = |\partial_J^+\mathcal{R}_J^{red}| = 15,$$

where $\partial_J^0\mathcal{R}_J^{red}$, $\partial_J^-\mathcal{R}_J^{red}$, and $\partial_J^+\mathcal{R}_J^{red}$ is the subset of $\mathcal{R}_J^{red}$ consisting of the zero-defect vectors, negative-defect vectors, and positive-defect vectors, respectively. It is easy to see that the negative-defect part $\partial_J^-\mathcal{R}_J$ of $\mathcal{R}_J$ is the set

$$\partial_J^-\mathcal{R}_J = \partial_J^-\mathcal{R}_J^{red} + \mathbb{Z}\cdot\mathbf{h}_J$$

and admits the following Φ_J-translation quiver structure $\Gamma(\partial_J^-\mathcal{R}_J, \Phi_J)$:

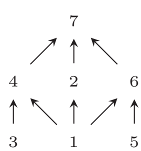


Hence, by applying Procedure 3.6 and using the vectors in $\Gamma(\partial_J^-\mathcal{R}_J, \Phi_J)$ marked by the framed boxes, we obtain the $\mathbb{Z}$-invertible matrix

$$B_J = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

such that $\check{G}_{DJ} = \check{G}_{\widetilde{\mathbb{D}}_5} = B_J \cdot \check{G}_J \cdot B_J^{tr}$.

2° If we enumerate the points of the poset $I := \mathcal{J}_{29}^{\widetilde{E}6}$ from Table 4.1 as follows

$I =$  , then the incidence matrix $C_I \in \mathbb{M}_7(\mathbb{Z})$, the Coxeter matrix and the incidence quadratic form $q_I : \mathbb{Z}^7 \to \mathbb{Z}$ have the forms

$$C_I = \begin{bmatrix} 1&1&0&1&0&1&1\\ 0&1&0&0&0&0&1\\ 0&0&1&1&0&0&1\\ 0&0&0&1&0&0&1\\ 0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1 \end{bmatrix}, \quad \mathrm{Cox}_I = \begin{bmatrix} 0&0&1&0&1&0&-1\\ -1&0&0&1&0&1&-1\\ -1&1&0&0&0&1&-1\\ -1&1&1&0&0&1&-1\\ -1&1&0&1&0&0&-1\\ -1&1&0&1&1&0&-1\\ -2&1&0&1&0&1&-1 \end{bmatrix},$$

$$\begin{aligned} q_I(x) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_1x_2 + (x_1 + x_3)x_4 \\ &\quad + (x_1 + x_5)x_6 + (x_1 + x_2 + x_3 + x_4 + x_5 + x_6)x_7 \\ &= \left(x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_4 + \tfrac{1}{2}x_6 + \tfrac{1}{2}x_7\right)^2 + \tfrac{3}{4}\left(x_2 - \tfrac{1}{3}x_4 - \tfrac{1}{3}x_6 + \tfrac{1}{3}x_7\right)^2 \\ &\quad + \left(x_3 + \tfrac{1}{2}x_4 + \tfrac{1}{2}x_7\right)^2 + \tfrac{5}{12}\left(x_4 - \tfrac{4}{5}x_6 + \tfrac{1}{5}x_7\right)^2 \\ &\quad + \left(x_5 + \tfrac{1}{2}x_6 + \tfrac{1}{2}x_7\right)^2 + \tfrac{3}{20}\left(x_6 + x_7\right)^2. \end{aligned}$$

It follows that $q_I : \mathbb{Z}^7 \to \mathbb{Z}$ is non-negative and $\operatorname{Ker} q_I = \mathbb{Z} \cdot \mathbf{h}_I$, where $\mathbf{h}_I = [1, -1, 0, -1, 0, -1, 1]$; hence I is principal. By theorem 1.7 the poset I is of Euclidean type $DI = \widetilde{\mathbb{E}}_6$. A routine calculations show that

- $\mathrm{cox}_I(t) = t^7 + t^6 - 2t^4 - 2t^3 + t + 1 = \mathrm{cox}_{\widetilde{\mathbb{E}}_6}(t)$,
- $\check{\mathbf{c}}_I = 6$, $\mathbf{c}_I = \infty$,
- $\widetilde{\partial}_I(v) = (v_1 - v_7) \cdot \mathbf{h}_I$, $\partial_I(v) = v_1 - v_7$,

and the set $\mathcal{R}_I$ of roots of q_I has the decomposition

$$\mathcal{R}_I = \mathcal{R}_I^{red} + \mathbb{Z} \cdot \mathbf{h}_I,$$

where the finite set

$$\mathcal{R}_I^{red} = \{v \in \mathbb{Z}^7;\ v_1 = 0 \text{ and } q_J(v) = 1\}$$

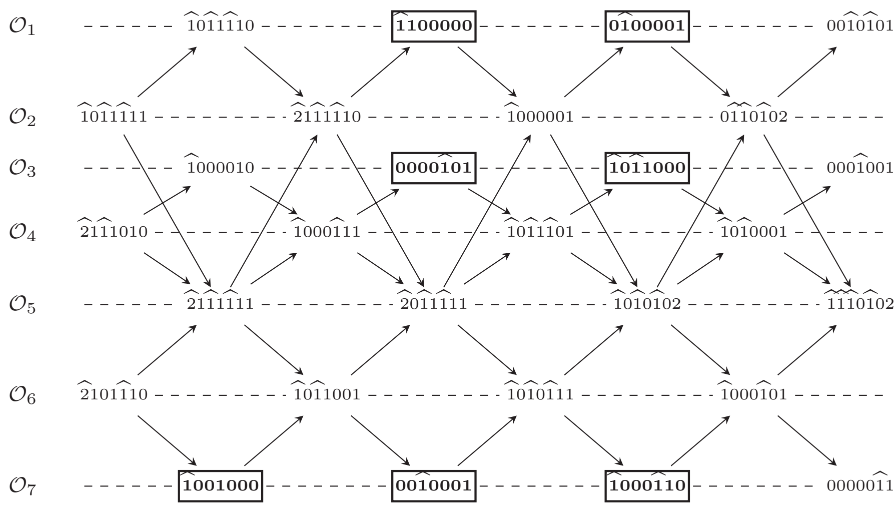
is a reducer of $\mathcal{R}_I$ in the sense of [47]. One shows that

$$|\mathcal{R}_I^{red}| = 72,\ |\partial_I^0 \mathcal{R}_I^{red}| = 14,\ |\partial_I^- \mathcal{R}_I^{red}| = |\partial_I^+ \mathcal{R}_I^{red}| = 29,$$

where $\partial_I^0 \mathcal{R}_I^{red}$, $\partial_I^- \mathcal{R}_I^{red}$, and $\partial_I^+ \mathcal{R}_I^{red}$ is the subset of $\mathcal{R}_I^{red}$ consisting of the zero-defect vectors, negative-defect vectors, and positive-defect vectors, respectively. It is easy to see that the negative-defect part $\partial_I^- \mathcal{R}_I$ of $\mathcal{R}_I$ is the set

$$\partial_I^- \mathcal{R}_I = \partial_I^- \mathcal{R}_I^{red} + \mathbb{Z} \cdot \mathbf{h}_I$$

and admits the following Φ_I-translation quiver structure $\Gamma(\partial_I^- \mathcal{R}_I, \Phi_I)$:



Hence, by applying Procedure 3.6 and using the vectors in $\Gamma(\partial_I^- \mathcal{R}_I, \Phi_I)$ distinguished by the framed boxes, we obtain the $\mathbb{Z}$-invertible matrix

$$B_I = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

such that $\check{G}_{DI} = \check{G}_{\widetilde{\mathbb{E}}_6} = B_I \cdot \check{G}_I \cdot B_I^{tr}$.

Now we collect a final effect of the computational Procedure 3.13 (presented below) applied to principal posets of type $\widetilde{\mathbb{E}}_8$. For example, consider the principal posets

$I_1 := \mathcal{J}_1^{\widetilde{E8}}$, $I_2 := \mathcal{J}_{31}^{\widetilde{E8}}$, $I_3 := \mathcal{J}_{48}^{\widetilde{E8}}$, $I_4 := \mathcal{J}_{147}^{\widetilde{E8}}$, $I_5 := \mathcal{J}_{412}^{\widetilde{E8}}$, $I_6 := \mathcal{J}_{422}^{\widetilde{E8}}$ from Theorem 1.8 (e). Using Procedure 3.13, computer calculations yield the following $\mathbb{Z}$-invertible matrices $B_1, \ldots, B_6 \in \mathbb{M}_9(\mathbb{Z})$:

$$B_1 = \begin{bmatrix} 5 & 3 & 4 & -2 & -1 & -2 & -2 & -1 & -2 \\ 5 & 4 & 3 & -1 & -2 & -2 & -1 & -2 & -2 \\ 6 & 3 & 4 & -2 & -2 & -1 & -2 & -2 & -2 \\ -8 & -5 & -5 & 2 & 3 & 2 & 3 & 2 & 3 \\ -3 & -1 & -2 & 1 & 1 & 1 & 0 & 1 & 1 \\ -2 & -2 & -2 & 1 & 1 & 0 & 1 & 1 & 1 \\ -3 & -2 & -1 & 1 & 0 & 1 & 1 & 1 & 1 \\ -2 & -1 & -2 & 0 & 1 & 1 & 1 & 1 & 1 \\ -3 & -2 & -2 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 1 & 0 & -2 & -2 & -1 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 & -2 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -2 & -1 & -1 & 0 & 1 & 1 \\ -2 & -1 & 1 & 2 & 2 & 2 & -1 & -1 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 1 & -1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 2 & 2 & -1 & -1 & -2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & -2 & -2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 & -2 & -2 & 1 & 0 \\ -1 & -2 & -2 & 0 & 1 & 3 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 & 2 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 2 & 2 & 2 & 1 & -1 & -1 & -2 & -1 & -1 \\ 1 & 3 & 1 & 1 & -1 & -1 & -2 & -1 & -1 \\ 1 & 3 & 2 & 2 & -1 & -1 & -3 & -2 & 0 \\ -2 & -4 & -2 & -2 & 1 & 1 & 4 & 2 & 1 \\ 0 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 \\ -1 & -2 & -1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & -1 & 0 & 1 & 1 & 1 & 0 \\ -1 & -2 & -1 & -1 & 1 & 1 & 2 & 0 & 0 \end{bmatrix}, \quad B_6 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & -1 \\ 1 & 2 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -2 & -1 & -1 & -1 & 1 & 1 & 1 & 2 \\ -1 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix},$$

for the poset I_1, I_2, I_3, I_4, I_5, and I_6, respectively. One checks that $\check{G}_{DI} = \check{G}_{\widetilde{\mathbb{E}}_8} = B_j \cdot \check{G}_{I_j} \cdot B_j^{tr}$, for $j = 1, \ldots, 6$.

(b) By (a) and [46, Theorems 4.7 and 4.8], for each of the posets I from the finite list described above, there is a $\mathbb{Z}$-invertible matrix B such that $\check{G}_{DI} = B \cdot \check{G}_I \cdot B^{tr}$ and, by [46, Proposition 4.8], we have the commutative diagram (3.4), with $D = DI$ and $h = h_B$. Hence, in view of our remarks made in the first part of proof, (b) is a consequence of the results in [46, Section 5].

(c) Given a principal poset I such that $m := |I| \leq 15$, fix a matrix $B_I \in \mathrm{Gl}(m, \mathbb{Z})$ such that $\check{G}_{DI} = B_I^{tr} \cdot C_I \cdot B_I$, as in (a). By applying the technique used in [17, Section 7], one can construct a matrix $C \in \mathrm{Gl}(m, \mathbb{Z})$ such that $\check{G}_{DI}^{tr} = C^{tr} \cdot \check{G}_{DI} \cdot C$ and $C^2 = E$. Then

$$\check{G}_{DI}^{tr} = B_I^{tr} \cdot C_I^{tr} \cdot B_I \text{ and } \check{G}_{DI}^{tr} = C^{tr} \cdot \check{G}_{DI} \cdot C = C^{tr} \cdot B_I^{tr} \cdot C_I \cdot B_I \cdot C.$$

Hence we get

$$C_I^{tr} = B_I^{-tr} \cdot C^{tr} \cdot B_I^{tr} \cdot C_I \cdot B_I \cdot C \cdot B_I^{-1} = \overline{C}^{tr} \cdot C_I \cdot \overline{C},$$

where $\overline{C} = B_I \cdot C \cdot B_I^{-1}$. It follows that $\overline{C} \in \mathrm{Gl}(m, \mathbb{Z})$ and $\overline{C}^2 = E$. This finishes the proof of (c) and of Theorem 1.9. □

Corollary 3.11. *Assume that I is a principal one-peak poset such that $m = |I| \leq 15$. Then there exists a Φ_I-mesh translation quiver of roots $\Gamma(\mathcal{R}_I, \Phi_I)$ satisfying the conditions stated in Theorem* 3.3 (b)*.*

Proof. Let DI be the Coxeter-Euclidean type of I. By Theorem 1.9 (a), there exists a matrix $B \in \mathrm{Gl}(m, \mathbb{Z})$ such that $\check{G}_{DI} = B \cdot \check{G}_I \cdot B^{tr}$ and the diagram (3.4) is commutative, where $h = h_B$ is the group isomorphism defined by $h_B(x) = x \cdot B$. Therefore the corollary is a consequence of Theorem 1.9 (b). □

We recall from [50] that a $\mathbb{Z}$-invertible matrix $B \in \mathbb{M}_n(\mathbb{Z})$ defines a $\mathbb{Z}$-congruence $\Delta \approx_{\mathbb{Z}} \Delta'$ between edge-bipartite graphs Δ, Δ' in $\mathcal{UB}igr_n$ if the equality $\check{G}_{\Delta'} = B \cdot \check{G}_{\Delta} \cdot B^{tr}$ holds.

Now we outline an alternative, more general heuristic algorithm constructing a $\mathbb{Z}$-invertible matrix $B \in \mathbb{M}_n(\mathbb{Z})$ defining the $\mathbb{Z}$-congruence $\check{G}_{\Delta} \sim_{\mathbb{Z}} \check{G}_{\Delta'}$ between the non-symmetric Gram matrices $\check{G}_{\Delta}, \check{G}_{\Delta'} \in \mathbb{M}_n(\mathbb{Z})$ of non-negative edge-bipartite graphs Δ and Δ', that is, satisfying the equality $\check{G}_{\Delta'} = B \cdot \check{G}_{\Delta} \cdot B^{tr}$. Its idea uses the following observations made in [45, Proposition 2.8].

Lemma 3.12. *Assume that $\Delta \approx_{\mathbb{Z}} \Delta'$ are loop-free edge-bipartite graphs in $\mathcal{UB}igr_n$ and $B \in \mathbb{M}_n(\mathbb{Z})$ is a $\mathbb{Z}$-invertible matrix satisfying the equality $\check{G}_{\Delta'} = B \cdot \check{G}_{\Delta} \cdot B^{tr}$, that is, B defines the $\mathbb{Z}$-congruence $\Delta \approx_{\mathbb{Z}} \Delta'$.*

(a) $\mathrm{Cox}_{\Delta'} = B \cdot \mathrm{Cox}_{\Delta} \cdot B^{-1}$ *and* $\mathrm{cox}_{\Delta'}(t) = \mathrm{cox}_{\Delta}(t)$.

(b) *Each of the rows w of the matrix B is a root of the unit form $q_{\Delta} : \mathbb{Z}^n \to \mathbb{Z}$ of Δ, that is, we have $q_{\Delta}(w) = 1$.*

(c) *Each of the rows of the matrix B^{-1} is a root of the unit form $q_{\Delta'} : \mathbb{Z}^n \to \mathbb{Z}$ of Δ'.*

Proof. (a) Apply [45, Proposition 2.8].

(b) Denote by $w^{(1)}, \ldots, w^{(n)}$ the rows of the matrix B. Then B has the form $B = [w^{(1)}, \ldots, w^{(n)}]^{tr}$ and, given $j \in \{1, \ldots, n\}$, we have $e_j \cdot B = w^{(j)}$ and

$$
\begin{aligned}
1 = q_{\Delta'}(e_j) &= e_j \cdot \check{G}_{\Delta'} \cdot e_j^{tr} \\
&= e_j \cdot (B \cdot \check{G}_{\Delta} \cdot B^{tr}) \cdot e_j^{tr} = (e_j \cdot B) \cdot \check{G}_{\Delta} \cdot (e_j \cdot B)^{tr} \\
&= w^{(j)} \cdot \check{G}_{\Delta} \cdot {w^{(j)}}^{tr} = q_{\Delta}(w^{(j)}).
\end{aligned}
$$

(c) The equality $\check{G}_{\Delta'} = B \cdot \check{G}_{\Delta} \cdot B^{tr}$ yields $\check{G}_{\Delta} = B^{-1} \cdot \check{G}_{\Delta'} \cdot B^{-tr}$ and (b) applies. This finishes the proof. □

It follows from Lemma 3.12, that in the situation we are interested in, the unknown matrix $B \in \mathbb{M}_n(\mathbb{Z})$ defining a $\mathbb{Z}$-congruence $\Delta \approx_{\mathbb{Z}} \Delta'$ (that is, the equality $\check{G}_{\Delta'} = B \cdot \check{G}_{\Delta} \cdot B^{tr}$ holds) satisfies the equation $\mathrm{Cox}_{\Delta'} \cdot B - B \cdot \mathrm{Cox}_{\Delta} = 0$. Moreover, the rows $w^{(1)}, \ldots, w^{(n)}$ of the matrix $B = [w^{(1)}, \ldots, w^{(n)}]^{tr}$ are roots of the quadratic form $q_{\Delta}(v) = v \cdot \check{G}_{\Delta} \cdot v^{tr}$.

Using these two observations we obtain the following heuristic procedure used already in its "positive version" [18, Algorithm 7.5] for positive edge-bipatite graphs.

Procedure 3.13. Input: The non-symmetric Gram matrices $\check{G}_\Delta, \check{G}_{\Delta'} \in \mathbb{M}_n(\mathbb{Z})$ of a pair of non-negative loop-free edge-bipartite graphs Δ and Δ' such that $\mathrm{cox}_{\Delta'}(t) = \mathrm{cox}_\Delta(t)$.

Output: A $\mathbb{Z}$-invertible matrix $B \in \mathbb{M}_n(\mathbb{Z})$ such that $\check{G}_{\Delta'} = B \cdot \check{G}_\Delta \cdot B^{tr}$, or error, if the matrix B has not been found.

Step 1° Compute the Coxeter matrices $\mathrm{Cox}_\Delta := -\check{G}_\Delta \cdot \check{G}_\Delta^{-tr}$ and $\mathrm{Cox}_{\Delta'} := -\check{G}_{\Delta'} \cdot \check{G}_{\Delta'}^{-tr}$.

Step 2° By applying [47, Algorithm 3.9] compute a finite root reducer $\mathcal{R}_\Delta^{red} \subseteq \mathcal{R}_\Delta = \{v \in \mathbb{Z}^n;\ q_\Delta(v) = 1\} \subseteq \mathbb{Z}^n$, that is, a finite set of roots of the non-negative quadratic form $q_\Delta \colon \mathbb{Z}^n \longrightarrow \mathbb{Z}$, $q_\Delta(v) = v \cdot \check{G}_\Delta \cdot v^{tr}$, such that $\mathcal{R}_\Delta = \mathcal{R}_\Delta^{red} + \mathrm{Ker}_{q_\Delta}$.

Step 3° Construct an $n \times n$ square matrix $B = [b_{ij}] = [w^{(1)}, \ldots, w^{(n)}]^{tr}$, with n^2 symbolic variables b_{ij}, $i, j \in \{1, \ldots, n\}$, and compute the matrix

$$\widetilde{B} := [\widetilde{b}_{ij}] = \mathrm{Cox}_{\Delta'} \cdot B - B \cdot \mathrm{Cox}_\Delta.$$

Solve the system

$$\widetilde{b}_{ij} = 0, \text{ for } i, j \in \{1, \ldots, n\}$$

of n^2 linear equations and update the matrix $B = [w^{(1)}, \ldots, w^{(n)}]^{tr}$ with calculated values.

Step 4° Find a row $w^{(k)}$ in the matrix B that contains any of the variables b_{ij} and proceed to *Step* 5°. If there is no such a row then proceed to *Step* 6°.

Step 5° For every root $w \in \mathcal{R}_\Delta^{red}$, replace the row $w^{(k)}$ in the matrix B by the vector w, update the matrix B accordingly and proceed recursively with Step 4°.

Step 6° If $\det B = \pm 1$ and $\check{G}_{\Delta'} = B \cdot \check{G}_\Delta \cdot B^{tr}$, stop with B as a result. Otherwise continue the search.

Step 7° If the search is completed and no matrix B has been found return the error.

Remark 3.14. Our heuristic Procedure 3.13 is a backtracking algorithm that incrementally checks all possible $\mathbb{Z}$-invertible matrices $B = [w^{(1)}, \ldots, w^{(n)}]^{tr}$, with $w^{(1)}, \ldots, w^{(n)}$ lying in the finite set

$$\mathcal{R}_\Delta^{red} \cup [\mathcal{R}_\Delta^{red} + \mathbf{h}] \cup [\mathcal{R}_\Delta^{red} - \mathbf{h}],$$

where $\mathbf{h} \in \mathbb{Z}^n$ is a generator of $\mathrm{Ker}\, q_\Delta$, if we assume that Δ and Δ' are principal edge-bipartite graphs.

Although we have not proved yet any theoretical result that would guarantee the existence of a $\mathbb{Z}$-invertible matrix B of such a form and satisfying $\check{G}_{\Delta'} = B \cdot \check{G}_{\Delta} \cdot B^{tr}$ (assuming that there exists a $\mathbb{Z}$-congruence $\Delta \approx_{\mathbb{Z}} \Delta'$), in our experience Procedure 3.13 finds a required matrix in less than a minute, if $n \leq 15$. For instance, each of the matrices $B_1, \ldots, B_6$ listed in the outline of proof of Theorem 1.9 has been computed in several seconds.

The following corollary announced in [51, Corollary 11] contains a partial solution of Problem 1.5.

Corollary 3.15. *Assume that I and J are principal one-peak posets, DI and DJ are their Coxeter-Eulidean types, $m = |I| = |J|$ and $2 \leq m \leq 15$. Let $\Gamma(\mathcal{R}_I, \Phi_I)$ and $\Gamma(\mathcal{R}_J, \Phi_J)$ be the Φ_J-mesh translation quivers of roots (see Corollary 3.11). The following conditions are equivalent.*

(a) $DI \cong DJ$.

(b) $\mathbf{specc}_I = \mathbf{specc}_J$.

(c) $C_I \approx_{\mathbb{Z}} C_J$ (*i.e., the incidence matrices C_I and C_J of I and J are $\mathbb{Z}$-congruent*).

(d) *There exists a group isomorphism $h : \mathbb{Z}^J \to \mathbb{Z}^I$ that induces the mesh translation quiver isomorphism $h : \Gamma(\mathcal{R}_J, \Phi_J) \xrightarrow{\simeq} \Gamma(\mathcal{R}_I, \Phi_I)$.*

Proof. (a)⇔(c) By Theorem 1.9 (a), we have $C_I \approx_{\mathbb{Z}} \check{G}_{DI}$ and $C_J \approx_{\mathbb{Z}} \check{G}_{DJ}$. Therefore $C_I \approx_{\mathbb{Z}} C_J$ if and only if $DI \cong DJ$.

(c)⇒(b) Note that the equality $C_I = B^{tr} \cdot C_J \cdot B$, with a matrix $B \in \mathrm{Gl}(m, \mathbb{Z})$ implies that $\mathrm{Cox}_I = B^{tr} \cdot \mathrm{Cox}_J \cdot B^{-tr}$, see Lemma 3.12 (a). Hence, $\mathrm{cox}_I(t) = \mathrm{cox}_J(t)$ and $\mathbf{specc}_I = \mathbf{specc}_J$.

(c)⇒(d) Note that the equality $C_I = B \cdot C_J \cdot B^{tr}$, with a matrix $B \in \mathrm{Gl}(m, \mathbb{Z})$ implies the commutativity of the diagram (3.4), with D and J interchanged. It follows that the group isomorphism $h : \mathbb{Z}^J \to \mathbb{Z}^I$ in (3.4) induces the mesh translation quiver isomorphism

$$h : \Gamma(\mathcal{R}_J, \Phi_J) \xrightarrow{\simeq} \Gamma(\mathcal{R}_I, \Phi_I)$$

see [47], [48, Lemma 4.3], and [50].

(b)⇒(c) Assume that $\mathbf{specc}_I = \mathbf{specc}_J$, that is, $\mathrm{cox}_I(t) = \mathrm{cox}_J(t)$. Since, by Theorem 1.9 (a), $\mathbf{specc}_{DI} = \mathbf{specc}_I = \mathbf{specc}_J = \mathbf{specc}_{DJ}$, the simple analysis of possible Coxeter polynomials (1.13) proves that $DI \cong DJ$. Hence, by applying Theorem 1.9 (a) again, we conclude that $C_I \approx_{\mathbb{Z}} C_J$.

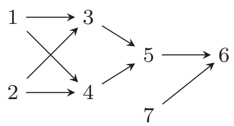
(d)⇒(a) Assume that there exists a mesh translation quiver isomorphism $h : \Gamma(\mathcal{R}_J, \Phi_J) \xrightarrow{\simeq} \Gamma(\mathcal{R}_I, \Phi_I)$ induced by a group isomorphism $h : \mathbb{Z}^J \to \mathbb{Z}^I$. By Theorem 1.9 (a), we have the isomorphisms

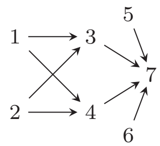
$$\Gamma(\mathcal{R}_{DJ},\Phi_{DJ})\cong\Gamma(\mathcal{R}_{J},\Phi_{J})\cong\Gamma(\mathcal{R}_{I},\Phi_{I})\cong\Gamma(\mathcal{R}_{DI},\Phi_{DI}).$$

Hence, in view of [46, Corollary 5.7], we get the graph isomorphism $DI \cong DJ$. □

The following example shows that the implication (b)⇒(c) in Corollary 3.15 does not hold for an arbitrary pair of non-negative posets I and J. We present such a pair that I is principal and J is non-negative of corank two.

Example 3.16. Consider the following pair of one-peak posets I and J, with $m = 7$ vertices.

$I =$ 

$J =$ 

One easily checks that

(a) each of the posets I, J is non-negative, I is principal, J is not principal, and

$\mathrm{cox}_I(t) = \mathrm{cox}_J(t) = t^7 + t^6 - t^5 - t^4 - t^3 - t^2 + t + 1 = \mathrm{cox}_{\widetilde{\mathbb{D}}_6}(t)$,

(b) $\mathbf{specc}_I = \mathbf{specc}_J = \mathbf{specc}_{\widetilde{\mathbb{D}}_6}$,

(c) $\mathbf{c}_I = \infty$, $\check{\mathbf{c}}_I = \check{\mathbf{c}}_J = \mathbf{c}_J = 4$,

(d) $\mathrm{Ker}\, q_I = \mathbb{Z}\cdot[1,1,-1,-1,0,0,0]$,

(e) $\mathrm{Ker}\, q_J = \mathbb{Z}\cdot[1,1,-1,-1,0,0,0] \oplus \mathbb{Z}\cdot[1,1,0,0,1,1,-2]$,

(f) $\widetilde{\partial}_I\colon \mathbb{Z}^7 \longrightarrow \mathrm{Ker}\, q_I$, $\widetilde{\partial}_I(v) = (v_1+v_2+v_3+v_4)\cdot[1,1,-1,-1,0,0,0]$,

(g) $\widetilde{\partial}_J\colon \mathbb{Z}^7 \longrightarrow \mathrm{Ker}\, q_J$ is zero.

The matrices C_I and C_J are not $\mathbb{Z}$-congruent, because the poset I is principal and the poset J is not.

4. Tables of one-peak principal posets of Coxeter-Euclidean types $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, and $\widetilde{\mathbb{E}}_8$

In this section we present two tables containing all one-peak principal posets of Coxeter-Euclidean types $\widetilde{\mathbb{E}}_6$ and $\widetilde{\mathbb{E}}_7$, respectively. A corresponding table containing all one-peak principal posets $\mathcal{J}_1^{\widetilde{E}8},\ldots,\mathcal{J}_{422}^{\widetilde{E}8}$ of the Coxeter-Euclidean type $\widetilde{\mathbb{E}}_8$ can be found in [21].

Table 4.1. Principal one-peak posets of Coxeter-Euclidean type $\widetilde{\mathbb{E}}_6$

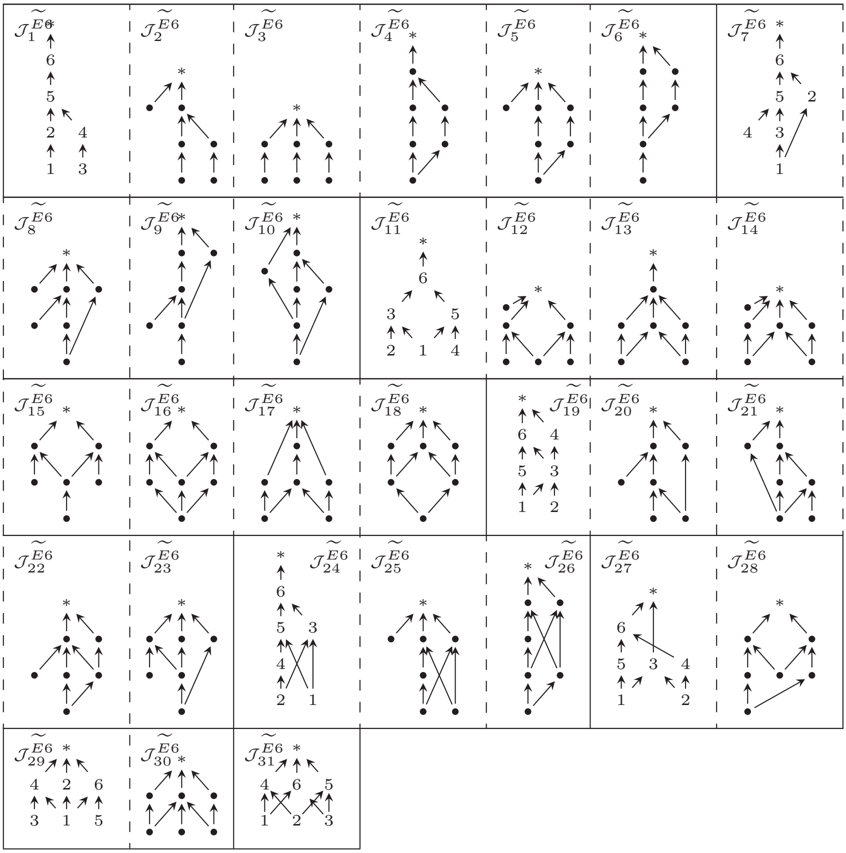
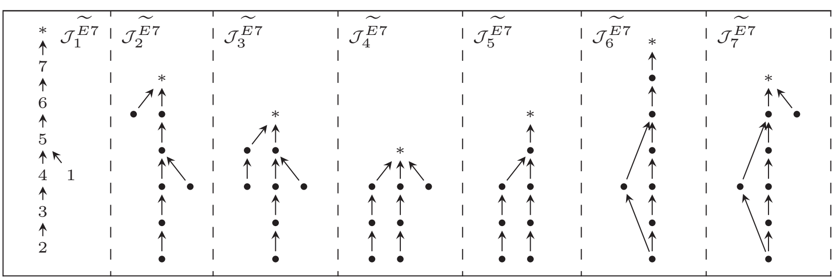
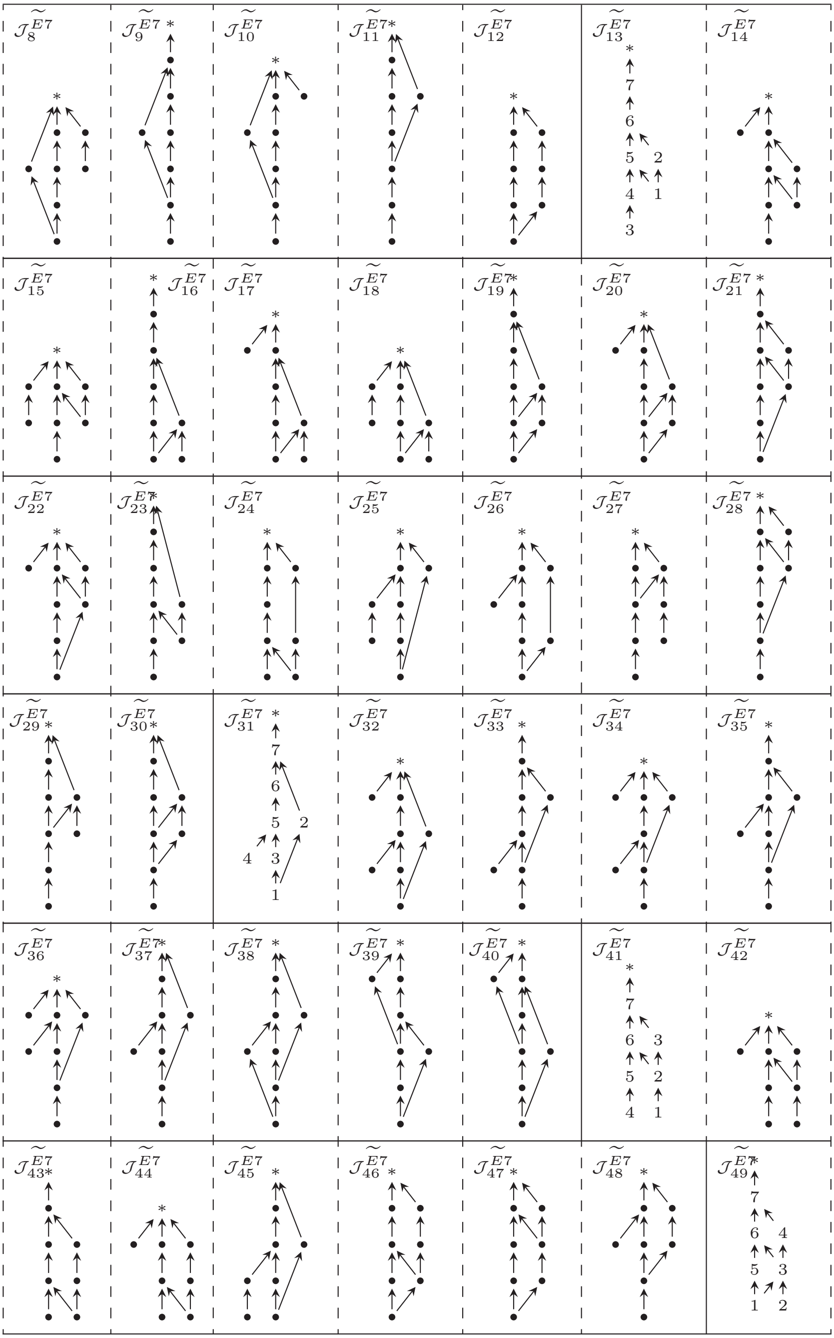


Table 4.2. Principal one-peak posets of Coxeter-Euclidean type $\widetilde{\mathbb{E}}_7$




$\widetilde{\mathcal{J}}^{E7}_{8}$
$\widetilde{\mathcal{J}}^{E7}_{9}$
$\widetilde{\mathcal{J}}^{E7}_{10}$
$\widetilde{\mathcal{J}}^{E7}_{11}$
$\widetilde{\mathcal{J}}^{E7}_{12}$
$\widetilde{\mathcal{J}}^{E7}_{13}$
$\widetilde{\mathcal{J}}^{E7}_{14}$
$\widetilde{\mathcal{J}}^{E7}_{15}$
$\widetilde{\mathcal{J}}^{E7}_{16}$
$\widetilde{\mathcal{J}}^{E7}_{17}$
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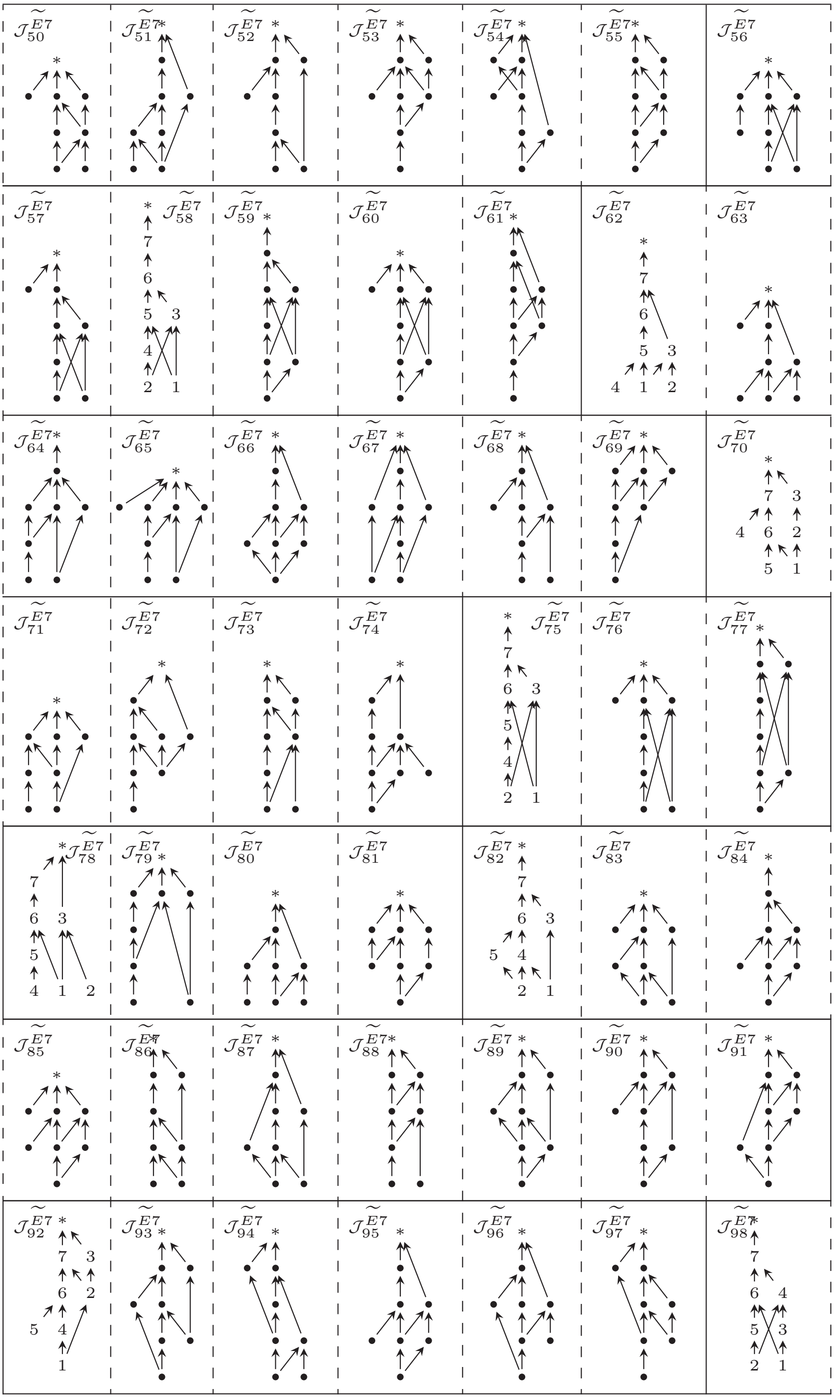
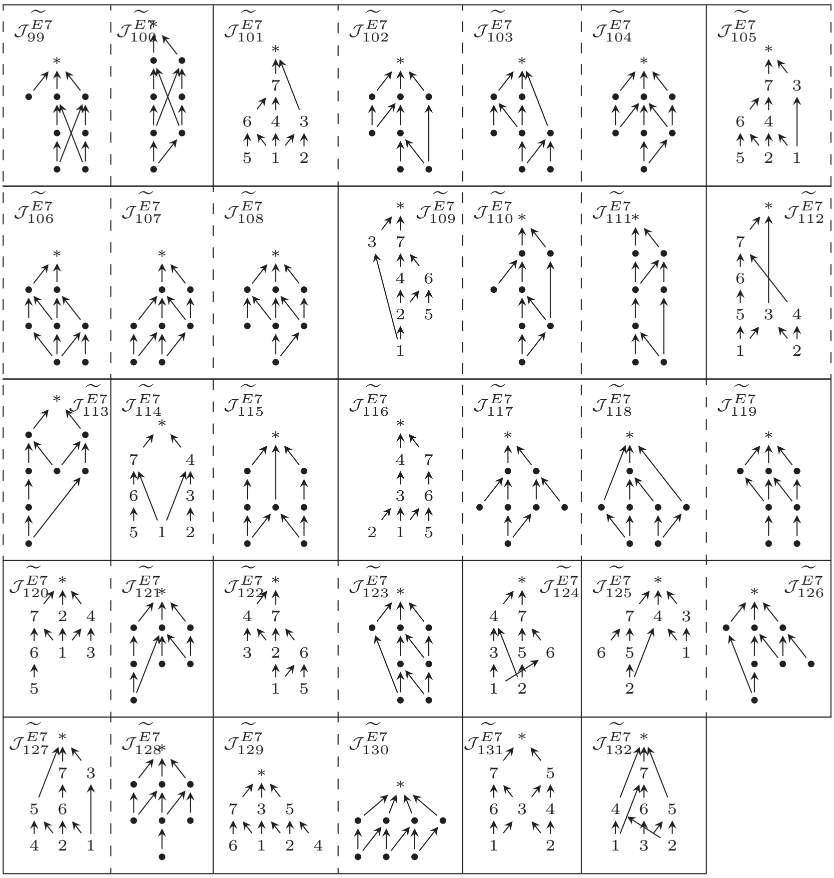
$\widetilde{\mathcal{J}}^{E7}_{43}$
$\widetilde{\mathcal{J}}^{E7}_{44}$
$\widetilde{\mathcal{J}}^{E7}_{45}$
$\widetilde{\mathcal{J}}^{E7}_{46}$
$\widetilde{\mathcal{J}}^{E7}_{47}$
$\widetilde{\mathcal{J}}^{E7}_{48}$
$\widetilde{\mathcal{J}}^{E7}_{49}$


$\widetilde{\mathcal{J}}^{E7}_{50}$ … $\widetilde{\mathcal{J}}^{E7}_{98}$



Remark 4.3. The computational technique introduced in this paper is applied and developed in [22] for non-negative posets of corank two.

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