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Rigid, quasi-rigid and matrix rings with $(\overline{\sigma}, 0)$-multiplication

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Abstract. Let R be a ring with an endomorphism σ. We introduce $(\overline{\sigma}, 0)$-multiplication which is a generalization of the simple 0- multiplication. It is proved that for arbitrary positive integers $m \leq n$ and $n \geq 2$, $R[x;\sigma]$ is a reduced ring if and only if $S_{n,m}(R)$ is a ring with $(\overline{\sigma}, 0)$-multiplication.

1. Introduction

Throughout this paper, we will assume that R is an associative ring with non-zero identity, σ is an endomorphism of the ring R and the polynomial ring over R is denoted by $R[x]$ with x its indeterminate.

In [6], the authors introduced and studied the notion of simple 0-multiplication. A subring S of the full matrix ring $\mathbb{M}_n(R)$ of $n \times n$ matrices over R is with simple 0- multiplication if for arbitrary $(a_{ij}), (b_{ij}) \in S$ then $(a_{ij})(b_{ij}) = 0$ implies that $a_{il}b_{lj} = 0$, for arbitrary $1 \leq i, j, l \leq n$. This definition is not meaningless because of the [4, Lemma 1.2]. Let R be a domain (commutative or not) and $R[x]$ is its polynomial ring. Let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ be elements of $R[x]$. It is easy to see that if $f(x)g(x) = 0$, then $a_i b_j = 0$ for every i and j since $f(x) = 0$ or $g(x) = 0$. Armendariz [1] noted that the above result can be extended the class of reduced rings, i.e., if it has no non-zero nilpotent elements. A ring R is said to have the *Armendariz property* or is an *Armendariz ring* if whenever polynomials

$$f(x) = a_0 + a_1 x + \cdots + a_m x^m, \quad g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x]$$

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satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each i, j. In [6, Theorem 2.1], the authors show that many matrix rings with simple 0-multiplication are Armendariz rings.

Recall that an endomorphism σ of a ring R is said to be *rigid* if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is *σ-rigid* if there exists a rigid endomorphism σ of R. Note that σ-rigid rings are reduced rings,i.e., the rings contains no nonzero nilpotent elements.

An ideal I of a ring R is said to be a *σ-ideal* if I is invariant under the endomorphism σ of the ring R, i.e., $\sigma(I) \subseteq I$. Now, let σ be an automorphism of the ring R and I be a σ-ideal of R. I is called a *quasi σ-rigid ideal* if $aR\sigma(a) \subseteq I$, then $a \in I$ for any $a \in R$ [3]. If the zero ideal $\{0\}$ of R is a quasi σ-rigid ideal, then R is said to be a *quasi σ-rigid ring* [3]. In Section 2, we obtain some ring extensions of quasi σ-rigid rings. We prove that; the class of quasi σ-rigid rings is closed under taking finite direct products (see Corollary 2.4).

We denote RG the group ring of a group G over a ring R and, for cyclic group order n, write C_n. We also prove that; if RG is quasi $\overline{\sigma}$-rigid, then R is a quasi σ-rigid ring (see Theorem 2.6), and R is quasi σ-rigid if and only if RC_2 is quasi σ-rigid where R is a ring with $2^{-1} \in R$ (see Corollary 2.8).

Let R be a quasi σ- rigid ring with $\sigma : R \to R$ endomorphism. In Example 2.1, it is shown that $\mathbb{M}_2(R)$ is not a quasi σ-rigid ring. Again, in Example 2.12, we proved that

$$S_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} | a, a_{ij} \in R \right\}$$

is not a quasi σ-rigid ring however R is a quasi σ-rigid ring. Naturally, these examples are starting points of our study. In this article, we introduce and study subrings with $(\overline{\sigma}, 0)$-multiplication of matrix rings which is a generalization of the simple 0-multiplication. They are related to $\overline{\sigma}$-skew Armendariz rings. Applying them, we obtain the following result in Section 3. Let σ be an endomorphism of a ring R. For arbitrary positive integers $m \leq n$ and $n \geq 2$, the following conditions are equivalent.

(i) $R[x; \sigma]$ is a reduced ring.

(ii) $S_{n,m}(R)$ is a ring with $(\overline{\sigma}, 0)$-multiplication.

(iii) $S_{n,m}(R)$ is a $\overline{\sigma}$-skew Armendariz ring (see Theorem 3.3).
See Example 4 for the definition of the ring $S_{n,m}(R)$.

2. Extensions of quasi σ-rigid rings

The following example shows that the class of quasi σ-rigid rings is not closed under taking subrings.

Example 1. Let R be a quasi σ-rigid ring with $\sigma : R \to R$ endomorphism defined by $\sigma(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. We take the nonzero element $a = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$. Since

$$a\mathbb{M}_2(R)\sigma(a) = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 0,$$

$\mathbb{M}_2(R)$ is not a quasi σ-rigid ring.

This example is one of the starting point of our study. First, we prove that the finite direct product of quasi σ-rigid rings is again a quasi σ-rigid ring.

Proposition 1. *Let σ_1 and σ_2 be automorphisms of rings R_1 and R_2, respectively. Assume that I_1 is a quasi σ_1-rigid ideal of R_1 and I_2 is a quasi σ_2-rigid ideal of R_2. Then $I_1 \times I_2$ is a quasi σ-rigid ideal of $R_1 \times R_2$, where σ is an automorphism of $R_1 \times R_2$ such that $\sigma(a, b) = (\sigma_1(a), \sigma_2(b))$ for any $a \in R_1$ and $b \in R_2$.*

Proof. We assume that $(a, b)R_1 \times R_2\sigma(a, b) \subseteq I_1 \times I_2$, equivalently,

$$(a, b)R_1 \times R_2(\sigma_1(a), \sigma_2(b)) \subseteq I_1 \times I_2.$$

Then we have $(aR_1\sigma_1(a), 0) \subseteq I_1 \times I_2$ and $(0, bR_2\sigma_2(b)) \subseteq I_1 \times I_2$. Thus we obtain that $aR_1\sigma_1(a) \subseteq I_1$ and $bR_2\sigma_2(b) \subseteq I_2$. Since I_1 is a quasi σ_1-rigid ideal of R_1 and I_2 is a quasi σ_2-rigid ideal of R_2, we have $a \in I_1$ and $b \in I_2$. Hence, $(a, b) \in I_1 \times I_2$. □

Theorem 1. *Assume that each σ_i, $1 \leq i \leq n$, is an automorphism of rings R_i. Then the finite direct product of quasi σ_i-rigid ideals I_i of R_i, $1 \leq i \leq n$, is a quasi σ-rigid ideal, where σ is an automorphism of $\prod_{i=1}^{n} R_i$ such that $\sigma(a_1, a_2, \cdots, a_n) = (\sigma_1(a_1), \sigma_2(a_2), \cdots, \sigma_n(a_n))$ for any $a_i \in R_i$.*

As a parallel result to Theorem 1, we have the following corollaries for quasi σ-rigid rings.

Corollary 1. *Assume that each σ_i, $1 \leq i \leq n$, is an automorphism of rings R_i. Then the finite direct product of quasi σ_i-rigid rings R_i,*

$1 \leq i \leq n$, is a quasi σ-rigid ring, where σ is an automorphism of $\prod_{i=1}^{n} R_i$ such that $\sigma(a_1, a_2, \cdots, a_n) = (\sigma_1(a_1), \sigma_2(a_2), \cdots, \sigma_n(a_n))$ for any $a_i \in R_i$.

Lemma 1. *Let R be a subring of a ring S such that both share the same identity. Suppose that S is a free left R-module with a basis G such that $1 \in G$ and $ag = ga$ for all $a \in R$ ring. Let σ be an endomorphism of R. Assume that the epimorphism $\overline{\sigma} : S \to S$ defined by $\overline{\sigma}(\sum_{g \in G} r_g g) = \sum_{g \in G} \sigma(r_g) g$ extends σ. If S is a quasi $\overline{\sigma}$-rigid ring, then R is a quasi σ-rigid ring.*

Proof. Suppose that $rR\sigma(r) = 0$ for $r \in R$. Then, by hypothesis, we can obtain that $r \sum_{g \in G} Rg\overline{\sigma}(r) = 0$. Hence $r = 0$, since S is a quasi $\overline{\sigma}$-rigid ring. □

Theorem 2. *Let R be a ring and G be a group. If the group ring RG is quasi $\overline{\sigma}$-rigid, then R is a quasi σ-rigid ring.*

Proof. Since $S = RG = \oplus_{g \in G} Rg$ is a free left R-module with a basis G satisfying the assumptions of Lemma 1, the proof of theorem is clear. □

Example 2. Let R be a ring. Note that if G is a semigroup or C_2, then clearly the epimorphism $\overline{\sigma} : S \to S$ defined by $\overline{\sigma}(\sum_{g \in G} r_g g) = \sum_{g \in G} \sigma(r_g) g$ extends σ. If the semigroup ring RG or RC_2 is quasi $\overline{\sigma}$-rigid, then R is a quasi σ-rigid ring by Theorem 2.

Corollary 2. *Let R be a ring with $2^{-1} \in R$. Then R is quasi σ-rigid if and only if RC_2 is quasi σ-rigid.*

Proof. If $2^{-1} \in R$, then the mapping $RC_2 \to R \times R$ which is given by $a + bg \to (a + b, a - b)$, is a ring isomorphism. The rest is clear from Example 2.7 and Corollary 1. □

Let σ be an epimorphism of a ring R. Then $\overline{\sigma} : R[x] \to R[x]$, defined by $\overline{\sigma}(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} \sigma(a_i) x^i$, is an epimorphism of the polynomial ring $R[x]$, and $\overline{\sigma}$ extends to σ.

Corollary 3. *R is a quasi σ-rigid ring if and only if $R[x]$ is a quasi $\overline{\sigma}$-rigid ring.*

Since, for an automorphism σ of R, every σ-rigid ring is a quasi σ-rigid ring, Corollary 1 holds for quasi σ-rigid rings.

Now we investigate a sufficient condition for Corollary 1.

Proposition 2. *Assume that σ is an automorphism of a ring R and e is a central idempotent of R. If R is a quasi σ-rigid ring then eR is also a quasi σ-rigid ring.*

Proof. For $ea \in eR$, we assume that $ea(eR)\sigma(ea) = 0$, equivalently,

$$0 = ea(eR)\sigma(ea) = eaeR\sigma(ea) = (ea)R\sigma(ea).$$

Since R is a quasi σ-rigid ring, we have $ea = 0$. □

The following example show that the condition e is a central idempotent of R" in Proposition 2 is necessary.

Example 3. Let F be a field with $\mathrm{char}(F) \neq 2$. It is easy to see that the ring $R = \mathbb{M}_2(F)$ with an endomorphism $\sigma(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ is a quasi σ-rigid ring. Consider the idempotent element $e = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ of R. Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

the idempotent e is not central. Let $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Now it is easy to see that $ea \neq 0$ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\sigma(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}) = 0$.

Example 3 shows that for a quasi σ-rigid ring R, $\mathbb{M}_n(R)$ or the full upper triangular matrix ring $\mathbb{T}_n(R)$ is not necessarily quasi $\overline{\sigma}$-rigid.

Example 4. Let R be a ring. We consider the following subrings of $\mathbb{T}_n(R)$ for any $n \geq 2$.
(1)

$$\begin{aligned} R_n &= RI_n + \textstyle\sum_{i=1}^{n}\sum_{k=i+1}^{n} RE_{ij} \\ &= \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R \right\}, \end{aligned}$$

where E_{ij} is the matrix units for all i, j and I_n is the identity matrix.
(2)

$$T(R,n) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} : a_i \in R \right\}.$$

(3) Let $m \leq n$ be positive integers. Let $S_{n,m}(R)$ be the set of all $n \times n$ matrices (a_{ij}) with entiries in a ring R such that

(a) for $i > j$, $a_{ij} = 0$,

(b) for $i \leq j$, $a_{ij} = a_{kl}$ when $i - k = j - l$ and either $1 \leq i, j, k, l \leq m$ or $m \leq i, j, k, l \leq n$.

Clearly, $S_{n,1}(R) = S_{n,n}(R) = T(R, n)$.

Let σ be an endomorphism of a ring R, then $\overline{\sigma} : \mathbb{M}_n(R) \to \mathbb{M}_n(R)$, defined by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$, is an also endomorphism of $\mathbb{M}_n(R)$ and $\overline{\sigma}$ extends to σ. Now assume that R is a quasi σ-rigid ring. It is easy to see that R_n, $T(R, n)$ and $S_{n,m}(R)$ are not quasi $\overline{\sigma}$-rigid rings for $n \geq 2$. For instance, we consider the ring:

$$S_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} | a, a_{ij} \in R \right\}.$$

Although R is a quasi σ-rigid ring, S_4 is not a quasi σ-rigid ring.

Let $a \in S_4$ such that $a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$. Since

$$\begin{aligned} aS_4\overline{\sigma}(a) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} S_4\overline{\sigma} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} S_4 \begin{pmatrix} 0 & \sigma(1) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0, \end{aligned}$$

S_4 is not a quasi σ-rigid ring.

3. On σ-skew Armendariz and $(\overline{\sigma}, 0)$-multiplication rings

In Corollary 3, we proved that R is a quasi σ-rigid ring if and only if $R[x]$ is a quasi $\overline{\sigma}$-rigid ring.

Theorem 3. *Assume that σ is a monomorphism of a ring R and $\sigma(1) = 1$, where 1 denotes the identity of R. Then the factor ring $R[x]/(x^2)$ is $\overline{\sigma}$-skew Armendariz if and only if R is a σ-rigid ring, where (x^2) is an ideal of $R[x]$ generated by x^2.*

Proof. (:⇒) Assume that $R[x]/(x^2)$ is a $\overline{\sigma}$-skew Armendariz ring. Let $r \in R$ with $\sigma(r)r = 0$. Then

$$(\sigma(r) - \overline{x}y)(r + \overline{x}y) = \sigma(r)r + (\sigma(r)\overline{x} - \overline{x}\sigma(r))y - \sigma(1)\overline{x}^2y^2 = \overline{0},$$

because $\sigma(r)\overline{x} = \overline{x}\sigma(r)$ in $(R[x]/(x^2))[y;\overline{\sigma}]$, where $\overline{x} = x + (x^2) \in R[x]/(x^2)$. Since $R[x]/(x^2)$ is $\overline{\sigma}$-skew Armendariz, we can obtain that $\sigma(r)\overline{x} = \overline{0}$ so $\sigma(r) = 0$. The injectivity of σ implies $r = 0$, and so R is σ-rigid.

(⇐:) Assume that $R[x;\sigma]$ is reduced. Let $\overline{h} = h + (x^2) \in R[x]/(x^2)$. Suppose that $\overline{p}.\overline{q} = \overline{0}$ in $(R[x]/(x^2))[y;\overline{\sigma}]$, where $\overline{p} = \overline{f}_0 + \overline{f}_1y + ... + \overline{f}_my^m$ and $\overline{q} = \overline{g}_0 + \overline{g}_1y + ... + \overline{g}_ny^n$. Let $\overline{f}_i = a_{i_0} + a_{i_1}\overline{x}$, $\overline{g}_j = b_{j_0} + b_{j_1}\overline{x}$ for each $0 \leq i \leq m$, and $0 \leq j \leq n$, where $\overline{x}^2 = \overline{0}$. Note that $\overline{x}y = y\overline{x}$ since $\alpha(1) = 1$, $a\overline{x} = \overline{x}a$ for any $a \in R$. Thus $\overline{p} = h_0 + h_1\overline{x}$ and $\overline{q} = k_0 + k_1\overline{x}$, where $h_0 = \sum_{i=0}^m a_{i_0}y^i$, $h_1 = \sum_{i=0}^m a_{i_1}y^i$, $k_0 = \sum_{j=0}^n b_{j_0}y^j$, $k_1 = \sum_{j=0}^n b_{j_1}y^j$ in $R[y]$. Since $\overline{p}.\overline{q} = \overline{0}$ and $\overline{x}^2 = \overline{0}$, we have

$$\overline{0} = \overline{p}.\overline{q} = \overline{0} = h_0k_0 + (h_0k_1 + h_1k_0)\overline{x} + h_1k_1\overline{x}^2 = h_0k_0 + (h_0k_1 + h_1k_0)\overline{x}.$$

We get $h_0k_0 = 0$ and $h_0k_1 + h_1k_0 = 0$ in $R[y;\sigma]$. Since $R[y;\sigma]$ is reduced, $k_0h_0 = 0$ and so $0 = k_0(k_0k_1 + h_1k_0)h_1 = (k_0h_1)^2$. Thus $k_0h_1 = 0$, $h_1k_0 = 0$ and $h_0k_1 = 0$. Moreover, R is σ-skew Armendariz by [8, Corollary 4]. Thus $a_{0_i}\sigma^i(b_{0_j}) = 0$, $a_{0_i}\sigma^i(b_{1_j}) = 0$ and $a_{1_i}\sigma^i(b_{0_j}) = 0$ for all $0 \leq i \leq m$, $0 \leq j \leq n$. Hence $f_i\overline{\sigma}^i(\overline{g}_j) = \overline{0}$ for all $0 \leq i \leq m$, $0 \leq j \leq n$. Therefore $R[x]/(x^2)$ is σ-skew Armendariz. □

In [6], the author defined and studied Armendariz and simple 0-multiplication rings. In other words, a subring S of the ring $\mathbb{M}_n(R)$ of $n \times n$ matrices over R is with simple 0-multiplication if for arbitrary $(a_{ij}), (b_{ij}) \in S$ then $(a_{ij})(b_{ij}) = 0$ implies that for arbitrary $1 \leq i, j, l \leq n$, $a_{il}b_{lj} = 0$.

Let σ be an endomorphism of a ring R. As we mentioned before, $\overline{\sigma} : \mathbb{M}_n(R) \to \mathbb{M}_n(R)$, defined by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$, is an also endomorphism of $\mathbb{M}_n(R)$ and $\overline{\sigma}$ extends to σ. We shall say that a subring S of the ring $\mathbb{M}_n(R)$ of $n \times n$ matrices over R is with $(\overline{\sigma}, 0)$-multiplication if for arbitrary $(a_{ij}), (b_{ij}) \in S$, $(a_{ij})\overline{\sigma}((b_{ij})) = 0$ implies that for arbitrary $1 \leq i, j, l \leq n$ $a_{il}\sigma(b_{lj}) = 0$.

Let σ be an epimorphism of a ring R. We know that $\overline{\sigma} : R[x] \to R[x]$, defined by $\overline{\sigma}(\sum_{i=0}^n a_ix^i) = \sum_{i=0}^n \sigma(a_i)x^i$, is an also epimorphism of the polynomial ring $R[x]$, and $\overline{\sigma}$ extends to σ. So, the map

$$\mathbb{M}_n(R)[x] \longrightarrow \mathbb{M}_n(R)[x],$$

defined by

$$\sum_{i=0}^{m} A_i x^i \mapsto \sum_{i=0}^{m} \overline{\sigma}(A_i) x^i$$

is an endomorphism of the matrix ring $\mathbb{M}_n(R)[x]$ and clearly this map extends σ. By the same notation of authors in [6], ϕ denotes the canonical isomorphism of $\mathbb{M}_n(R)[x]$ onto $\mathbb{M}_n(R[x])$. It is given by

$$\phi(\overline{\sigma}(A_0) + \overline{\sigma}(A_1)x + ... + \overline{\sigma}(A_m)x^m) = (f_{ij}(x)),$$

where

$$f_{ij}(x) = (\sigma(a_{ij}^{(0)}) + \sigma(a_{ij}^{(1)})x + ... + \sigma(a_{ij}^{(m)})x^m)$$

and $\sigma(a_{ij}^{(k)})$ denotes the (i,j) entry of $\overline{\sigma}(A_k)$. In fact follows E_{ij} will denote the usual matrix unit.

Theorem 4. *Let σ be an endomorphism of a ring R, and R be a σ-skew Armendariz ring.*

(1) *If S is a subring of $\mathbb{M}_n(R)$ with $(\overline{\sigma}, 0)$-multiplication and, for some $A_i, B_i \in S$ $0 \leq i \leq 1$, $(A_0 + A_1x)(B_0 + B_1x) = 0$ then $A_t\overline{\sigma^t}(B_u) = 0$ for $0 \leq t, u \leq 1$.*

(2) *If, for a subring S of $\mathbb{M}_n(R)$, $\phi(S[x])$ is a subring of $\mathbb{M}_n(R[x])$ with $(\overline{\sigma}, 0)$-multiplication, then S is an $\overline{\sigma}$-skew Armendariz ring.*

Proof. (1) Assume that S is a subring of $\mathbb{M}_n(R)$ with $(\overline{\sigma}, 0)$-multiplication and, for some $A_i, B_i \in S$, $0 \leq i \leq 1$. Then

$$\begin{aligned} 0 &= (A_0 + A_1x)(B_0 + B_1x) \\ &= A_0B_0 + A_0B_1x + A_1xB_0 + A_1xB_1x \\ &= A_0B_0 + [A_0B_1 + A_1\overline{\sigma}(B_0)]x + A_1\overline{\sigma}(B_1)x^2 \\ &= a_{il}^{(0)}b_{lj}^{(0)} + (a_{il}^{(0)}b_{lj}^{(1)} + a_{il}^{(1)}\sigma(b_{lj}^{(0)}))x + a_{il}^{(1)}\sigma(b_{lj}^{(1)})x^2. \end{aligned}$$

Now, we set $p = \sum_{t=0}^{1} a_{il}^{(t)} x^t$ and $q = \sum_{u=0}^{1} b_{lj}^{(u)} x^u$. Then $pq = 0$ and $a_{il}^{(t)}\sigma^t(b_{lj}^{(u)}) = 0$, since R is a σ-skew Armendariz ring.

(2) We prove only when $n = 2$. Other cases can be proved by the same method. Suppose that

$$\begin{aligned} p(x) &= \overline{\sigma}(A_0) + \overline{\sigma}(A_1)x + ... + \overline{\sigma}(A_m)x^m \in S[x; \overline{\sigma}] \\ q(x) &= \overline{\sigma}(B_0) + \overline{\sigma}(B_1)x + ... + \overline{\sigma}(B_m)x^m \in S[x; \overline{\sigma}] \end{aligned}$$

such that $p(x)q(x) = 0$, where

$$\overline{\sigma}(A_i) = \begin{pmatrix} \sigma(a_{11}^{(i)}) & \sigma(a_{12}^{(i)}) \\ \\ \sigma(a_{21}^{(i)}) & \sigma(a_{22}^{(i)}) \end{pmatrix} \text{ and } \overline{\sigma}(B_j) = \begin{pmatrix} \sigma(b_{11}^{(j)}) & \sigma(b_{12}^{(j)}) \\ \\ \sigma(b_{21}^{(j)}) & \sigma(b_{22}^{(j)}) \end{pmatrix}$$

for $0 \leq i \leq m$ and $0 \leq j \leq m$. We claim that $\overline{\sigma}(A_i)\overline{\sigma}^i(\overline{\sigma}(B_j)) = 0$ for $0 \leq i \leq m$ and $0 \leq j \leq m$. Then $p(x)q(x) = 0$ implies that

$$(\sigma(a_{il}^{(0)}) + ... + \sigma(a_{il}^{(m)})x^m)\sigma((b_{lj}^{(0)}) + ... + (b_{lj}^{(m)})x^m) = 0$$

since $\phi(S[x])$ is $(\overline{\sigma}, 0)$-multiplication. Now we can obtain that $\sigma(a_{il}^{(t)})\sigma^t(\sigma(b_{lj}^{(u)})) = 0$ for all $0 \leq i, j, u, t \leq m$ since R is σ-skew Armendariz. □

Now we return one of the important examples in the paper, the ring $S_{n,m}(R)$ that is not a (quasi) $\overline{\sigma}$-rigid rings for $n \geq 2$ by Example 2.12. We consider our ring S_4. Note that if R is an σ-rigid ring, then $\sigma(e) = e$ for $e^2 = e \in R$. Let $p = e_{12} + (e_{12} - e_{13})x$ and $q = e_{34} + (e_{24} + e_{34})x \in S_4[x; \overline{\sigma}]$, where e_{ij}'s are the matrix units in S_4. Then $pq = 0$, but $(e_{12} - e_{13})\overline{\sigma}(e_{34}) \neq 0$. Thus S_4 is not $\overline{\sigma}$-skew Armendariz. Similarly, for the case of $n \geq 5$, we have the same result.

Theorem 5. *Let σ be an endomorphism of a ring R. For arbitrary positive integers $m \leq n$ and $n \geq 2$, the following conditions are equivalent.*

(1) *$R[x; \sigma]$ is a reduced ring.*

(2) *$S_{n,m}(R)$ is a ring with $(\overline{\sigma}, 0)$-multiplication.*

(3) *$S_{n,m}(R)$ is an $\overline{\sigma}$-skew Armendariz ring.*

Proof. To prove, we completely follow the proof of [6, Theorem 2.3].
(1) ⇒ (2) We will proceed by induction on n. Suppose that $n \geq 2$ and the result holds for smaller integers. Let $A = (a_{ij})$, $\overline{A} = (\overline{a}_{ij}) \in S_{n,m}(R)$ and $A\overline{\sigma}(\overline{A}) = 0$.

Note that the matrices obtained from A and $\overline{A}$ by deleting their first rows and columns belong to $S_{n-1,m-1}(R)$ when $m > 1$, and $S_{n-1,n-1}(R)$ when $m = 1$. The product of obtained matrices is equal to 0. So, applying the induction assumption, we get that $a_{ij}\sigma^i(\overline{a}_{jl}) = 0$ for $i \geq 2$ and all j, l. Similarly, by deleting the last rows and columns, we get that $a_{ij}\sigma^i(\overline{a}_{jl}) = 0$ for $l \leq n - 1$ and all i, j. Moreover,

$$a_{11}\sigma(\overline{a}_{1n}) + a_{12}\sigma(\overline{a}_{2n}) + \cdots + a_{1n}\sigma(\overline{a}_{nn}) = 0. \quad (1)$$

It is left to prove that $a_{1j}\sigma(\overline{a}_{jn}) = 0$ for $1 \leq j \leq n$. Let $1 \leq j < k \leq n$.

If $k \leq m$, then $a_{ij} = a_{k-j-1,k}$ and $k-j+1 \geq 2$, so from the induction conclusion, we get that $a_{1j}\sigma(\overline{a}_{kn}) = a_{k-j-1,k}\sigma(\overline{a}_{kn}) = 0$.

Similarly, we get that if $m \leq j$ (which is possible only when $m < n$), then $a_{1j}\sigma(\overline{a}_{kn}) = a_{1j}\sigma(\overline{a}_{j,j-k+n}) = 0$.

If $j \leq m < k$, then $a_{1j}\sigma(\overline{a}_{kn}) = a_{m-j+1,m}\sigma(\overline{a}_{m,n+m-k}) = 0$ (because $j < k$ implies that either $m-j+1 \geq 2$ or $n+m-k \leq n-1$).

Multiplying (1) on the left by a_{11} and the foregoing, we get that $a_{11}^2\sigma(\overline{a}_{1n}) = 0$. Hence, $a_{11}\sigma(\overline{a}_{1n}) = 0$. Similarly, multiplying (1) (in which now $a_{11}\sigma(\overline{a}_{1n}) = 0$) on the left by a_{12}, we get that $a_{12}\sigma(\overline{a}_{2n}) = 0$. Continuing in this way, we get $a_{1j}\sigma(\overline{a}_{jn}) = 0$ for all $j \leq m-1$. These results and (1) gives the result when $m = n$.

If $m < n$, then, same as above, multiplying (1) on the right by $\overline{a}_{nn}$, $\overline{a}_{n-1,n}, \ldots, \overline{a}_{m+1,n}$ applying the foregoing relations , we get (successively) that

$$\begin{aligned} a_{1n}\sigma(\overline{a}_{nn}) &= a_{1,n-1}\sigma(\overline{a}_{n-1,n}) \\ &= \cdots \\ &= a_{1,m+1}\sigma(\overline{a}_{m+1,n}) \\ &= 0. \end{aligned}$$

Now (1) implies also that $a_{1m}\sigma(\overline{a}_{mn}) = 0$ and we are done.

(2) $\Rightarrow$ (3) Note that $\phi((S_{n,m}(R))[x]) = S_{n,m}(R[x])$. Now the rest follows from Theorem 4.

(3) $\Rightarrow$ (1) Clearly, $S_{n,m}(R)$ contains a subring isomorphic to $S_2(R)$. Hence (3) implies that $S_2(R)$ is a $\overline{\sigma}$-skew Armendariz ring. Then $R[x;\sigma]$ is a reduced ring. □

Theorem 6. *Let σ be an endomorphism of a ring R with $\sigma(1) = 1$. For arbitrary integers $1 < m < n$, if T is a subring of $\mathbb{T}_n(R)$, which properly contains $S_{n,m}(R)$, then there are $A_0, A_1, B_0, B_1 \in T$ such that $(A_0 + A_1x)(B_0 + B_1x) = 0$ and $A_1\overline{\sigma}(B_0) \neq 0$. In particular, T is not a $\overline{\sigma}$-skew Armendariz ring.*

Proof. We proceed by induction on n. Let us observe first that if T is an $\overline{\sigma}$-skew Armendariz subring of the ring $\mathbb{T}_n(R)$, then by deleting in every matrix from T the first(last) row and column, we get a $\overline{\sigma}$-skew Armendariz subring of the ring $\mathbb{T}_{n-1}(R)$.

To start the induction, assume that $n = 3$ and $m = 3$. Applying the above observation, it suffices to show that no subring of $\mathbb{T}_2(R)$, which properly contains $S_2(R)$, is $\overline{\sigma}$-skew Armendariz. It is clear that every such subring S contains the matrices $A = aE_{11}$, $B = -aE_{22}$, for some $0 \neq a \in R$ and $C = E_{12}$.

Let $\sigma(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$, $p = A + Cx$, $q = B + Cx \in R[x;\sigma]$. We have $(A+Cx)(B+Cx) = 0$ but $C\sigma((B)) \neq 0$, so S is not a $\overline{\sigma}$-skew Armendariz ring.

Now, the rest of the proof is similar to the proof of [6, Theorem 2.4]. □

Corollary 4. *For arbitrary integers* $1 < m < n$ *and every ring* R *with an endomorphism* σ*, no subring of* $\mathbb{T}_n(R)$*, which properly contains* $S_{n,m}(R)$*, is with* $(\overline{\sigma}, 0)$*-multiplication.*

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