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Some more algebra on ultrafilters in metric spaces

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Abstract. We continue algebraization of the set of ultrafilters on a metric spaces initiated in [6]. In particular, we define and study metric counterparts of prime, strongly prime and right cancellable ultrafilters from the Stone-Čech compactification of a discrete group as a right topological semigroup [3]. Our approach is based on the concept of parallelity introduced in the context of balleans in [4].

1. Introduction

Let X be a discrete space, and let βX be the Stone-Čech compactification of X. We take the point of βX to be the ultrafilters on X, with the point of X identified with principal ultrafilters, so $X^* = \beta X \setminus X$ is the set of all free ultrafilters. The topology of βX can be defined by stating that the sets of the form $\bar{A} = \{p \in \beta X : A \in p\}$, where A is a subset of X, are base for the open sets. Given a filter φ on X, the set $\bar{\varphi} = \bigcap\{\bar{A} : A \in \varphi\}$ is closed in βX, and every non-empty closed subset of βX can be obtained in this way.

If S is a discrete semigroup, the semigroup multiplication has a natural extension to βS, see [3, Chapter 4]. The compact right topological semigroup has very rich algebraic structure and a plenty applications to combinatorics, topological algebra and functional analysis, see [1, 2, 3, 7, 8, 10]. To get the product pq of $p, q \in \beta S$, one can take an arbitrary

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$P \in p$, and for each $x \in P$, pick $Q_x \in q$. The $\bigcup_{x \in P} xQ_x \in pq$ and these subsets form a basis of pq.

In [6], given a metric space X, we endowed X with the discrete topology, introduced and characterized the metric counterparts in βX of minimal left ideals and the closure of the minimal ideal in βS.

In this note, we continue algebraization of βX, define and describe the metric analogues of prime, strongly prime and right cancellable ultrafilters from βG, G is a discrete group. We recall that an ultrafilter $p \in G^*$ is *prime* if $p \notin G^*G^*$, and p is strongly *prime* if $p \notin \mathrm{cl}\, G^*G^*$. An ultrafilter $p \in G^*$ is called *right cancellable* if, for any $q, r \in \beta G$, $qp = rp$ implies $q = r$.

The key observation: to detect whether $p \in G^*$ is prime or strongly prime, we do not need to know how to multiply any two individual ultrafilters but only what is the set G^*q, $q \in G^*$. Indeed, p is prime if and only if $p \notin G^*q$ for each $q \in G^*$. If G is countable then $p \in G^*$ is right cancellable if and only if $p \notin G^*p$, see [3, Theorem 8.18]. But the natural metric counterpart of G^*p in βX can be defined by means of the parallelity relation on ultrafilters introduced in [4] for the general case of balleans, and applied for algebraization of βX, X is a metric space, in [6].

2. Ball invariance and parallelity

Let (X, d) be a metric space. For any $x \in X$, $A \subseteq X$, $r \in \mathbb{R}^+$, $\mathbb{R}^+ = \{r \in \mathbb{R} : r \geqslant 0\}$, we denote

$$B(x, r) = \{y \in X : d(x, y) \leqslant r\},\ B(A, r) = \bigcup_{a \in A} B(a, r).$$

Given an ideal $\mathcal{I}$ in the Boolean algebra $\mathcal{P}_X$ of all subsets of X, ($A, B \in \mathcal{I}$, $C \subseteq A \Longrightarrow A \bigcup B \in \mathcal{I},\ C \in \mathcal{I}$), we say that $\mathcal{I}$ is *ball invariant* if, for every $A \in \mathcal{I}$ and $r \in \mathbb{R}^+$, we have $B(A, r) \in \mathcal{I}$. If $\mathcal{I}$ is ball invariant and $\mathcal{I} \neq \{\varnothing\}$ then $\mathcal{I}$ contains the ideal $\mathcal{I}_b$ of all bounded subsets of X. A subset A of X is called *bounded* if $A \subseteq B(x, r)$ for some $x \in X$ and $r \in \mathbb{R}^+$.

We say that a filter φ on X is *ball invariant* if, for every $A \in \varphi$ and $r \in \mathbb{R}^+$, there exists $C \in \varphi$ such that $B(C, r) \subseteq A$.

An ideal $\mathcal{I}$ is called proper if $\mathcal{I} \neq \mathcal{P}_X$. For a proper ideal in $\mathcal{P}_X$, we denote by $\varphi_{\mathcal{I}}$ the filter $\{X \setminus A : A \in \mathcal{I}\}$ and put $A^{\wedge} = \bar{\varphi}_{\mathcal{I}}$ so

$$A^{\wedge} = \{p \in \beta X : X \setminus A \in p\}.$$

We remind the reader that X in βX is endowed with the discrete topology and use the parallelity equivalence on βX defined in [6] by the rule: $p \parallel q$ if and only if there exists $r \in \mathbb{R}^+$ such that $B(P, r) \in q$ for each $P \in p$. A subset S of βX is called *invariant* (with respect to the parallelity equivalence) if, for all $p, q \in \beta X$, $p \in S$ and $p \parallel q$ imply $q \in S$.

Proposition 1. *For a proper ideal $\mathcal{I}$ in βX, the following statements are equivalent:*
(i) *$\mathcal{I}$ is ball invariant;*
(ii) *$\varphi_{\mathcal{I}}$ is ball invariant;*
(iii) *$\mathcal{I}^{\wedge}$ is invariant.*

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is evident. To prove (ii) $\Longrightarrow$ (iii), let $p \in \mathcal{I}^{\wedge}$ and $q \parallel p$. We choose $r \geqslant 0$ such that $B(P, r) \in q$ for each $P \in p$. Given an arbitrary $Y \in \varphi_{\mathcal{I}}$, we choose $Z \in \varphi_{\mathcal{I}}$ such that $B(Z, r) \subseteq Y$. Then $Z \in p$ and $B(Z, r) \in q$ so $Y \in q$ and $q \in \mathcal{I}^{\wedge}$.

To see that (iii) $\Longrightarrow$ (i), we assume the contrary and choose $Y \in \varphi_{\mathcal{I}}$ and $r \geqslant 0$ such that $B(Z, r) \setminus Y \neq \varnothing$ for each $r \geqslant 0$. Then we take $q \in \beta X$ such that $B(Z, r) \setminus Y \in q$ for each $Z \in \varphi_{\mathcal{I}}$. By [6, Lemma 2.1], there exists $p \in \varphi_{\mathcal{I}}$ such that $q \parallel p$. Since $q \notin \mathcal{I}^{\wedge}$, we get a contradiction. $\square$

In what follows, we suppose that every metric space X under consideration is unbounded, put

$$X^{\sharp} = \{p \in \beta X : \textit{ every member } P \in p \textit{ is unbounded in } X\}$$

and note that $X^{\sharp}$ is a closed invariant subset of βX.

We say that a subset A of X is
- *large* if $X = B(A, r)$ for some $r \geqslant 0$;
- *small* if $L \setminus A$ is large for every large subset L;
- *thick* if, for every $r \geqslant 0$, there exists $a \in A$ such that $B(a, r) \subseteq A$;
- *prethick* if $B(A, r)$ is thick for some $r \geqslant 0$.

The family Sm_X of all small subsets of X is an ideal in $\mathcal{P}_X$, and a subset A is small if and only if A is not prethick [7, Theorems 11.1 and 11.2].

Proposition 2. *For every metric space X, the ideal Sm_X is ball invariant and*

$$\mathrm{Sm}_X^{\wedge} = \mathrm{cl}\Big\{\bigcup\{K : K \textit{ is a minimal non-empty closed invariant subset of } X^{\sharp}\}\Big\}.$$

Proof. The second statement is the dual form of Theorem 3.2 from [6]. Assume that A is small but $B(A,r)$ is not small for some $r \geqslant 0$. Then $B(A,r)$ is prethick so there is $m \geqslant 0$ such that $B(B(A,r),m)$ is thick. It follows that A is prethick and we get a contradiction. $\square$

For every metric space X, by [6, Corollary 3.1], the set of all minimal non-empty closed invariant subset of X has cardinality $2^{2^{\operatorname{asden} X}}$, where $\operatorname{asden} X = \min\{|Y| : Y \text{ is a large subset of } X\}$. Applying Proposition 1, we get $2^{2^{\operatorname{asden} X}}$ maximal proper ball invariant ideals in $\mathcal{P}_X$.

Proposition 3. *Let $\mathcal{I}$ be a ball invariant ideal in $\mathcal{P}_X$ such that $\mathcal{I} \neq \mathcal{I}_b$, $\mathcal{I}_b$ is the ideal of all bounded subsets of X. Then there exists a ball invariant ideal $\mathcal{J}$ such that $\mathcal{I}_b \subset \mathcal{J} \subset \mathcal{I}$.*

Proof. We take an unbounded subset $A \in \mathcal{I}$ and choose a sequence $(a_n)_{n\in\omega}$ in A such that $B(a_n, n) \bigcap B(a_m, m) = \varnothing$ for all distinct $n, m \in \omega$. We put $A_0 = \{a_{2n} : n \in \omega\}$, $A_1 = \{a_{2n+1} : n \in \omega\}$ and denote by $\mathcal{J}$ the smallest ball invariant ideal such that $A_0 \in \mathcal{I}$. Then $Y \in \mathcal{I}$ if and only if $Z \subseteq B(Y_0, m)$ for some $m \in \omega$. By the choice of $(a_n)_{n\in\omega}$, $Y \setminus B(Y_0, m) \neq \varnothing$ for each $m \in \omega$, so $Y_1 \notin \mathcal{J}$ but $Y_1 \in \mathcal{I}$. $\square$

3. Prime and strongly prime ultrafilters

For each $q \in X^\sharp$, we denote $q^= = \{r \in X^\sharp : r \parallel q\}$ and say that $p \in X^\sharp$ is *divisible* if there exists $q \in X^\sharp$ such that $\bar{P} \bigcap q^=$ is infinite for each $P \in p$. An ultrafilter $p \in X^\sharp$ is called *prime* if p is not divisible, and *strongly prime* if p is not in the closure of the set of all divisible ultrafilters.

A subset A of X is called *sparse* if $\bar{A} \bigcap q^=$ is finite for each $q \in X^\sharp$. We denote by Sp_X the family of all sparse subsets of X and observe that Sp_X is an ideal in $\mathcal{P}_X$.

Proposition 4. *An ultrafilter $p \in X^\sharp$ is strongly prime if and only if there exists $A \in Sp_X$ such that $A \in p$ so $Sp_X^\wedge = \operatorname{cl} \mathcal{D}$, where $\mathcal{D}$ is the set of all divisible ultrafilters.*

Proof. Assume that each member $P \in p$ is not sparse and choose $q \in X^\sharp$ such that $\bar{P} \bigcap q^=$ is infinite. We take an arbitrary limit point r of the set $\bar{P} \bigcap q^=$. Then $P \in r$ and r is divisible so $p \in \operatorname{cl} \mathcal{D}$ and p is not strongly prime.

On the other hand, if A is sparse and $A \in p$ then $\bar{A} \bigcap \mathcal{D} = \varnothing$ and $p \notin \operatorname{cl} \mathcal{D}$. $\square$

A subset A of X is called *thin* if, for every $r \geqslant 0$, there exists a bounded subset V of X such that $B(a,r) \bigcap A = \{a\}$ for each $a \in A \setminus V$.

Proposition 5. *If $p \in X^\sharp$ and some member $P \in p$ is thin then p is strongly prime.*

Proof. By [6, Theorem 4.3], P is thin if and only if $|\bar{P} \bigcap q^=| \leqslant 1$ for each $q \in X^\sharp$ so we can apply Proposition 4. □

Since every unbounded subset of X contains some unbounded thin subset, we conclude that the set of all strongly prime ultrafilters is dense in $X^\sharp$.

Is the ideal Sp_X ball invariant? In Proposition 7, we give a negative example. In Proposition 6, we describe a class of metric spaces for which the answer is positive.

A metric space X is called *uniformly locally finite* if, for every $r \geqslant 0$, there exists $m \in \mathbb{N}$ such that $|B(x,r)| \leqslant m$ for each $x \in X$.

Proposition 6. *If a metric space X is uniformly locally finite then the ideal Sp_X is ball invariant.*

Proof. By [5, Theorem 1], there exists a countable group G of permutations of X such that

(1) for each $r \geqslant 0$, there exists a finite subset F of G such that $B(x,r) \subseteq F(x)$ for each $x \in X$, where $F(x) = \{g(x) : g \in F\}$;

(2) for every finite subset F of G, there exists $r \geqslant 0$ such that $F(x) \subseteq B(x,r)$ for each $x \in X$.

It follows that, for $p, q \in X^\sharp$, $p \parallel q$ if and only if there exists $g \in G$ such that $q = g(p)$, where $g(p) = \{g(P) : P \in p\}$.

Now let A be a sparse subset of X and $r \geqslant 0$. We choose F satisfying (2) so $B(A,r) \subseteq F(A)$, where $F(A) = \bigcup_{g \in F} g(A)$. We take an arbitrary $q \in X^\sharp$. Since A is sparse, $q^= \bigcap \bar{A}$ is finite. Then $q^= \bigcap \overline{B(A,r)} \subseteq \bigcup_{g \in F}(q^= \bigcap \overline{g(A)})$. Since $|q^= \bigcap \overline{g(A)}| = |(g^{-1}q)^= \bigcap A|$ and A is sparse, $|q^= \bigcap \overline{B(A,r)}|$ is finite and $B(A,r)$ is sparse. □

Proposition 7. *Let $\mathbb{Q}$ be the set of rational numbers endowed with the metric $d(x,y) = |x-y|$. The ideal $Sp_{\mathbb{Q}}$ is not ball invariant.*

Proof. We put $A = \{2^n : n \in \mathbb{N}\}$. By Proposition 5, A is sparse. We take an arbitrary free ultrafilter $q \in \bar{A}$. Then $B(A,1) \in x + q$ for each $x \in [0,1]$. Since $x + q \parallel q$, $q^= \bigcap \overline{B(A,1)}$ is infinite so $B(A,1)$ is not sparse. □

We say that an ultrafilter $p \in X^\sharp$ is *discrete* if each $q \in p^=$ is an isolated point in the set $p^=$. In view of [3, Theorem 8.18], a discrete ultrafilter can be considered as a counterpart of a right cancellable ultrafilter. Clearly, if each $q \in p^=$ is prime then p is discrete.

Proposition 8. *There exist two ultrafilters $p, q \in \mathbb{Q}^\sharp$ such that $p \parallel q$, p is isolated in $p^=$ but q is not isolated in $p^=$.*

Proof. For each $n \in \mathbb{N}$, we put $A_n = \bigcup_{m \geqslant n}[2^{m, 2^m+1}]$ and take a maximal filter q such that $A_n \in q$, $n \in \mathbb{N}$ and each member $A \in q$ is somewhere dense, i.e. the closure of A in $\mathbb{Q}$ has non-empty interior. It is easy to see that q is an ultrafilter and q has a basis consisting of subsets without isolated points. We consider the mapping $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ defined by $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the nearest from the left integer to x. The set $\{f(\mathcal{U}) : \mathcal{U} \in q\}$ is a basis for some uniquely determined ultrafilter p such that $\{2^n : n \in \mathbb{N}\} \in p$. Clearly $p \parallel q$ and, by Proposition 5, p is isolated in $p^=$.

We show that q is not isolated in $p^=$. We take an arbitrary $Q \in q$ such that Q has no isolated points, $f(Q) \subseteq \{2^n : n \in \mathbb{N}\}$ and choose an arbitrary mapping $h : f(Q) \longrightarrow Q$ such that $h(2^n) \in [2^n, 2^n + 1]$ for each $2^n \in f(Q)$. We denote by q_h the ultrafilter with the basis $\{h(\mathcal{V}) : \mathcal{V} \in p\}$. Then $Q \in q_h$ and $q_h \parallel p$. Since Q has no isolated points, we have countably many ways to choose h and get countably many distinct ultrafilters from $p^= \bigcap \bar{Q}$. $\square$

A subset A of X is called *disparse* if $\bar{A} \bigcap p^=$ is discrete for each $p \in X^\sharp$. The family dSp_X of all disparse subsets of X is an ideal in $\mathcal{P}_X$ and we get the following evident

Proposition 9. *For every metric space X, $\mathrm{dSp}_X^\wedge$ is the of all ultrafilters $p \in X^\sharp$ such that $p^=$ has no isolated points.*

Proposition 10. *For every $p \in X^\sharp$, the set $p^=$ is nowhere dense in $X^\sharp$.*

Proof. We take an arbitrary $A \in p$ and coming back to the proof of Proposition 3, consider the subsets A_0, A_1 of A. If $A_0 \in q$, $A_1 \in r$ then q and r are not parallel. Then either $\overline{A_0} \bigcap p^=$ or $\bar{A}_1 \bigcap p^= = \varnothing$. $\square$

4. Ballean context

Following [7, 8], we say that a *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets and, for every $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$ and $\alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radii.*

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we set

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, \ B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

A ball structure $\mathcal{B} = (XP, B)$ is called a *ballean* if

- for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \ B^*(x, \beta) \subseteq B(x, \beta');$$

- for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma);$$

- for any $x, y \in X$, there is $\alpha \in P$ such that $y \in B(x, \alpha)$.

A ballean $\mathcal{B}$ on X can also be determined in terms of entourages of the diagonal $\triangle_X$ in $X \times X$ (in this case it is called a *coarse structure* [9]) and can be considered as an asymptotic counterpart of a uniform topological space.

Every metric space (X, d) defines the ballean $(X, \mathbb{R}^+, B_d)$, where $B_d(x, r) = \{y \in X : d(x, y) \leqslant r\}$. For criterion of metrizability of balleans see [8, Theorem 2.1.1].

We observe that all definitions in this paper do not use the metric on X directly but only balls so can be literally rewritten for any ballean in place of metric space. Moreover, a routine verification ensures that Propositions 1, 2, 4, 5, 9 remain true for any balleans.

Let G be a group with the identity e. We denote by $\mathcal{F}_G$ the family of all finite subsets of G containing e and get the group ballean $\mathcal{B}(G) = (G, \mathcal{F}_G, B)$, where $B(g, F) = Fg$ for all $g \in G$, $F \in \mathcal{F}_G$. We note that $G^\sharp = G^*$ and, for $p, q \in G^*$, $p \parallel q$ if and only if $q = gp$ for some $g \in G$. Hence, $p^= = Gp$, $\mathrm{cl}\, p^= = (\beta G)p$ and the minimal non-empty closed invariant subsets in $G^\sharp$ are precisely the minimal left ideals of the semigroup βG. The ballean and semigroup notions of divisible, prime and strongly prime ultrafilters coincide.

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