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RESEARCH ARTICLE

Weak equivalence of representations of Kleinian 4-group

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Abstract. We give a classification of representations of the Kleinian 4-group up to weak equivalence.

1. Introduction

A classification of representation of the Kleinian 4-group G was given by Nazarova [4]. According to [3], it can be applied to the description of Chernikov 2-groups which are extensions of G with a direct sum of quasi-cyclic 2-groups. To do it, one has to consider the classes of *weak equivalence* of such representations. We recall that two $\mathbb{Z}_2G$-modules M, N are said to be *weakly equivalent* if there is an automorphism σ of the group G and an isomorphism of $\mathbb{Z}_2$-modules $f : M \to N$ such that $f(gv) = \sigma(g)f(v)$ for all $g \in G$, $v \in M$. The aim of this paper is to give a classification of indecomposable integral 2-adique representations of the Kleinian group up to weak equivalence. To do it, we use the technique proposed in [2, 7] which gives a description of representations more adapted to deal with automorphisms of the group G. We also use the technique of Auslander–Reiten quivers from [6].[1]

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[1] A preliminary version of this paper was published in [5].

2. Relation to quiver

So let $G = \langle a, b \mid a^2 = b^2 = 1,\ ab = ba\rangle$ be the Kleinian 4-group, $\mathbb{Z}_2G$ be its group ring over the ring $\mathbb{Z}_2$ of 2-adic integers. A 2-adic representation of G is given by a $\mathbb{Z}_2G$*-lattice*, i.e. $\mathbb{Z}_2G$-module M which is finitely generated and free as $\mathbb{Z}_2$-module. The group ring $\mathbb{Z}_2G$ is *Gorenstein*, i.e. of self-injective dimension 1. The results of [1] imply that every $\mathbb{Z}_2G$-lattice is a direct summand of a free $\mathbb{Z}_2G$-module and an $\boldsymbol{A}$-lattice, where $\boldsymbol{A}$ is a unique minimal over-ring of $\mathbb{Z}_2G$. In our case $\boldsymbol{A} = \mathbb{Z}_2G + \mathbb{Z}_2z$, where $z = \frac{1+a+b+ab}{2}$. Let $x = a - 1$, $y = b - 1$. Then $\operatorname{rad}\mathbb{Z}_2G = \langle\, x, y\,\rangle$ and $\operatorname{rad}\boldsymbol{A} = \langle\, x, y, z\,\rangle$. The maximal over-ring $\boldsymbol{R}$ of $\boldsymbol{A}$ is generated (as $\mathbb{Z}_2$-module) by the primitive orthogonal idempotents e_{ij} $(i, j \in \{+, -\}$, where

$$\begin{aligned} e_{++} &= \frac{1+a+b+ab}{4},\\ e_{+-} &= \frac{1+a-b-ab}{4},\\ e_{-+} &= \frac{1-a+b-ab}{4},\\ e_{--} &= \frac{1-a-b+ab}{4}. \end{aligned}$$

Moreover, $\operatorname{rad}\boldsymbol{R} = \operatorname{rad}\boldsymbol{A}$, so $\boldsymbol{A}$ is a so called *Backström order* in the sense of [7]. So we can apply the results of [2, 7] that relate the description of $\boldsymbol{A}$-lattices with representaions of a quiver.

Namely, let $\boldsymbol{R}_{ij} = e_{ij}\boldsymbol{R}$ $(i, j \in \{+, -\}$. The $\mathbb{F}_2$-algebra $\boldsymbol{R}/J$ is isomorphic to $\mathbb{F}_2^4$ with the basis consisting of the classes of these idempotents. Now the results of [2, 7] imply that there is a one-to-one correspondence between the $\boldsymbol{A}$-lattices and diagrams of vector spaces (representations of the quiver of type $\tilde{D}_4$)

$$\begin{tikzcd} & V_{++} \\ & V_{+-} \\ V \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \\ & V_{-+} \\ & V_{--} \end{tikzcd} \tag{1}$$

such that all arrows are surjective and the induced map $\iota : V_0 \to \oplus_{ij} V_{ij}$ is injective. Namely, M corresponds to the diagram with $V_0 = M/JM$, $V_{ij} =$

$e_{ij}(\boldsymbol{R}M/JM)$ $(i,j \in \{+,-\})$ and arrows denote the natural projections of $V_0 \subseteq \boldsymbol{R}M/JM$ onto the components. On the contrary, given such diagram with $\dim V_{ij} = d_{ij}$, we identify the direct sum $\oplus_{ij} V_{ij}$ with N/JN, where $N = \oplus_{ij}\boldsymbol{R}_{ij}^{d_{ij}}$, and take the preimage M of V_0 considered as the subspace of $\oplus_{ij} V_{ij}$ via the map ι. Note that the only indecomposable diagrams of the shape (1) which do not correspond to $\boldsymbol{A}$-lattices are the "trivial" diagrams, where one of the spaces is 1-dimensional and the others are zero. We call the *dimension* of such a diagram the vector $(d_0, d_{++}, d_{+-}, d_{-+}, d_{--})$, where $d_0 = \dim V_0$, $d_{ij} = \dim V_{ij}$, and usually arrange this quintuple as

$$\boxed{\begin{matrix} & d_{++} \\ & d_{+-} \\ d_0 & d_{-+} \\ & d_{--} \end{matrix}}\,.$$

3. Description of representations

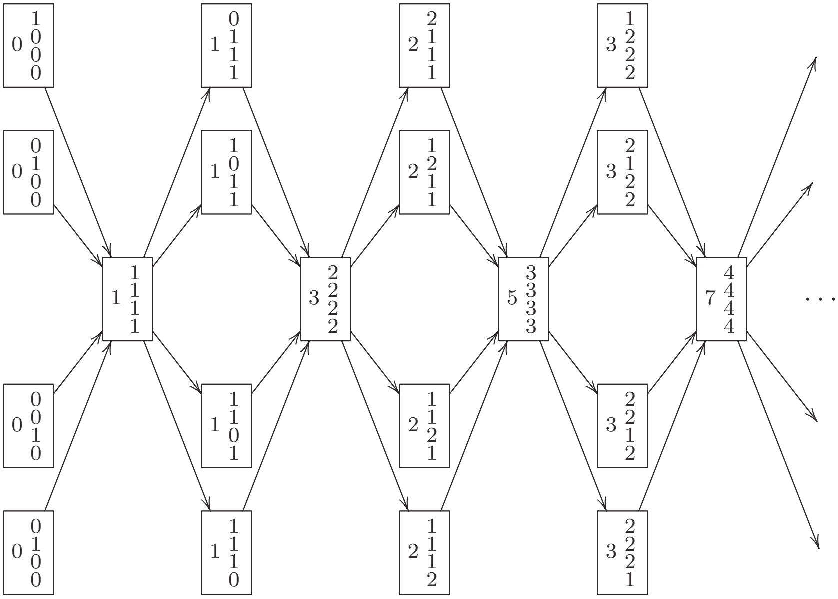
The indecomposable diagrams are arranged into the *Auslander–Reiten quiver* [6]. Its vertices are just the indecomposable diagrams and arrows are the *irreducible maps*, i.e. non-invertible morphisms of the diagrams which cannot be presented as sums of compositions of two non-invertible morphisms. The structure of this diagram is described in [2] and [6, Sec. 3.6]. It consists of three parts:

$$\boxed{\text{preprojective}} \longrightarrow \boxed{\text{regular}} \longrightarrow \boxed{\text{preinjective}}$$

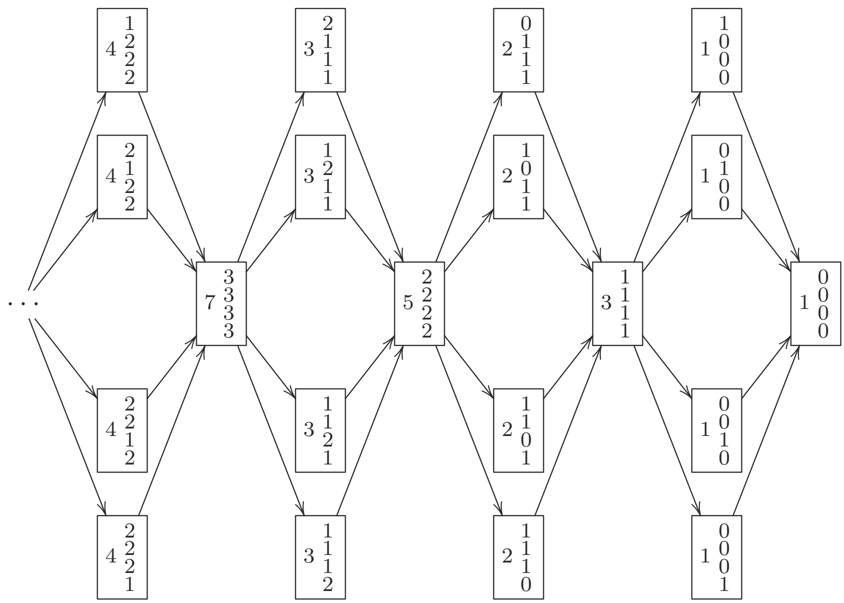
with morphisms going "from left to right", i.e. there are no morphisms from the preinjective part to the regular and preprojective parts and no morphisms from the regular part to the preprojective part. Moreover, there are no arrows (irreducible morphisms) between different parts. In the preprojective and preinjective parts the representations are uniquely defined by their dimensions, which are the *positive real roots* of the Tits form, i.e. non-negative integral solutions of the equation

$$d_{00}^2 + \sum_{ij} d_{ij}^2 - d_{00} \sum_{ij} d_{ij} = 1.$$

Namely, the preprojective component is



And the preinjective component is



The regular part consists of *tubes*. Most of them are *regular*, i.e. behave as indecomposable finite dimensional modules over the ring $\Bbbk[[t]]$, where $\Bbbk$ is an extension of the field $\mathbb{F}_2$: $\Bbbk = \mathbb{F}_2[t]/(f(t))$, where $f(t)$ is an irreducible polynomial over the field representations, weak equivalence, Auslander–Reiten quiver$\mathbb{F}_2$ and $f(t) \notin \{t, t-1\}$. Each such polynomial $f(t)$ of degree d gives rise to one tube T_f of the form

$$\begin{tikzcd}
R(1,f) \arrow[r, bend left] & R(2,f) \arrow[l, bend left] \arrow[r, bend left] & R(3,f) \arrow[l, bend left] \arrow[r, bend left] & \dots \arrow[l, bend left]
\end{tikzcd}$$

where $R(n,f)$ is the representation of dimension $\boxed{2nd\,\begin{smallmatrix} nd\\ nd\\ nd\\ nd\end{smallmatrix}}$, which is given by the diagram

$$\begin{tikzcd}
 & V \\
 & V \\
V^2 \arrow[uur, "\begin{pmatrix} I & 0 \end{pmatrix}"] \arrow[ur, "\begin{pmatrix} 0 & I \end{pmatrix}"'] \arrow[dr, "\begin{pmatrix} I & I \end{pmatrix}"] \arrow[ddr, "\begin{pmatrix} I & F_{f^n} \end{pmatrix}"'] & \\
 & V \\
 & V
\end{tikzcd}$$

Here $V = \Bbbk^{dn}$ and F_{f^n} is the Frobenius matrix with the minimal polynomial $f^n(t)$.

There are also 3 special tubes T_k $(2 \leqslant k \leqslant 4)$ of period 2. The tube T_4 is of the form

$$\begin{tikzcd}
\boxed{1\,\begin{smallmatrix}1\\1\\0\\0\end{smallmatrix}} \arrow[r] & \boxed{2\,\begin{smallmatrix}1\\1\\1\\1\end{smallmatrix}} \arrow[r] \arrow[dl] & \boxed{3\,\begin{smallmatrix}2\\2\\1\\1\end{smallmatrix}} \arrow[r] \arrow[dl] & \boxed{4\,\begin{smallmatrix}2\\2\\2\\2\end{smallmatrix}} \arrow[r] \arrow[dl] & \dots \arrow[dl] \\
\boxed{1\,\begin{smallmatrix}0\\0\\1\\1\end{smallmatrix}} \arrow[r] & \boxed{2\,\begin{smallmatrix}1\\1\\1\\1\end{smallmatrix}} \arrow[r] \arrow[ul] & \boxed{3\,\begin{smallmatrix}1\\1\\2\\2\end{smallmatrix}} \arrow[r] \arrow[ul] & \boxed{4\,\begin{smallmatrix}2\\2\\2\\2\end{smallmatrix}} \arrow[r] \arrow[ul] & \dots \arrow[ul]
\end{tikzcd}$$

where the representations of the dimension $\boxed{2n\ \begin{smallmatrix} n\\ n\\ n\\ n\end{smallmatrix}}$ in the first (in the second) row are, respectively,

$$\begin{tikzcd}[row sep=small] & V \\ & V \\ V^2 \arrow[uur, "(I\ 0)"] \arrow[ur, "(0\ I)"'] \arrow[dr, "(I\ I)"] \arrow[ddr, "(I\ J_n)"'] & \\ & V \\ & V \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[row sep=small] & V \\ & V \\ V^2 \arrow[uur, "(I\ I)"] \arrow[ur, "(I\ J_n)"'] \arrow[dr, "(I\ 0)"] \arrow[ddr, "(0\ I)"'] & \\ & V \\ & V \end{tikzcd}$$

where $V = \Bbbk^n$, and J_n is the nilpotent Jordan $n \times n$ matrix. The other representations are uniquely defined by their dimensions.representations, weak equivalence, Auslander–Reiten quiver The tubes T_2 and T_3 are obtained from T_4 just by transposing V_{--} with, respectively, V_{+-} or V_{-+} and also transposing the corresponding maps.

4. Weak equivalence

Now we have to find how automoprhisms of G act on the representations. The group of automorphisms of G is identified with S_3. If we consider its action on the representations, it permutes the components V_{ij}, where $i, j \in \{+,-\}$, $(ij) \neq (++)$. Namely, the transposition of $(--)$ with $(+-)$ corresponds to the automorphism $a \mapsto ab$, $b \mapsto b$, while the transposition of $(--)$ with $(-+)$ corresponds to the automorphism $a \mapsto a$, $b \mapsto ab$ and the transposition of $(-+)$ with $(+-)$ corresponds to the automorphism $a \mapsto b$, $b \mapsto a$. Thus these automorphisms permute the components V_{ij} of the corresponding diagrams. Therefore, for the preprojective and preinjective components it leaves untouched the central and the first rows and permutes the other three. It also permutes the special tubes T_k mapping the first row to the first and the second row to the second. One can also check that the action of S_3 on the regular tubes coincides with its classical action on polynomials. Namely, the transposition of $(-+)$ and $(--)$ maps $f(t)$ to $c^{-1}t^d f(1/t)$, where $d = \deg f$, $c = f(0)$; the transposition of

$(+-)$ and $(-+)$ maps $f(t)$ to $c^{-1}(t-1)^d f(1/(t-1))$.representations, weak equivalence, Auslander–Reiten quiver

It accomplishes the description of indecomposable representations of G up to weak equivalence.

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