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Finite groups with a system of generalized central elements

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Abstract. Let H be a normal subgroup of a finite group G. A number of authors have investigated the structure of G under the assumption that all minimal or maximal subgroups in Sylow subgroups of H are well-situated in G. A general approach to the results of that kind is proposed in this article. The author has found the conditions for p-elements of H under which G-chief p-factors of H are $\mathfrak{F}$-central in G.

1. Introduction

All groups considered in this article will be finite. A number of authors have investigated the structure of a non-nilpotent group G under the assumption that all minimal or maximal subgroups in Sylow subgroups of G are well-situated in G. The first result in this direction was obtained by Ito [1]; he proved that a group G of odd order is nilpotent provided that all minimal subgroups of G lie in the center of G. This result was developed by Gaschütz in the following way: if every minimal subgroup of G is normal in G, then a Sylow 2-subgroup P of G' is normal and G'/P is nilpotent (see [2], Theorem IV.5.7). Buckley [3] also considered the situation when minimal subgroups are normal; this means that these subgroups are $\mathfrak{U}$-central normal subgroups where $\mathfrak{U}$ is the formation of supersoluble groups. Later, some authors [4], [5], [6], [7] extended the mentioned results using formation theory; they investigated groups in which minimal subgroups lie in $\mathfrak{F}$-hypercenter of the group. Other generalizations were

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obtained in [8], [9] using the concept of a c-normal subgroup introduced in [10]. A subgroup H of G is called c-normal if there exists a normal subgroup N of G such that $G = HN$ and $H \cap N \subseteq H_G = \mathrm{Core}_G(H)$. It is clear that if $H = \langle a \rangle$ is a c-normal primary cyclic subgroup of G, then H/H_G is either normal or normally complemented; in this case aB lies in a cyclic chief factor A/B of G. A more general approach was proposed in a paper [11] in which a concept of a $Q\mathfrak{F}$-central element was introduced. An element a of a group G is called $Q\mathfrak{F}$-central if there exists a $\mathfrak{F}$-central chief factor A/B of G such that $a \in A\backslash B$. Thus, the general line is to investigate a group with a system of $Q\mathfrak{F}$-central elements. It is interesting that groups with given systems of complemented, supplemented or c-supplemented [12] minimal subgroups actually appear groups with a system of $Q\mathfrak{F}$-central elements. We recall that Gorchakov [13] proved that a group is supersoluble if all its minimal subgroups are complemented. We also mention articles [14], [15], [16], [17] in which the groups with a given system of complemented and S-quasinormal minimal subgroups are investigated.

Analyzing the mentioned papers we can draw a conclusion that they are connected with the solution of the following question.

Question A. *Let $\mathfrak{F} = LF(F)$ be a saturated formation, H a normal subgroup of a group G, p a prime. Assume that all elements of order p in H are $Q\mathfrak{F}$-central in G. Assume also that if $p = 2$, then all elements of order 4 in H are $Q\mathfrak{F}$-central in G. Is it true that $G/C_G(A/B) \in F(p)$ for every G-chief factor A/B of H whose order $|A/B|$ is divided by p?*

Another line of investigations is concerned with maximal subgroups of Sylow subgroups. So, Srinivasan [18] proved that a group G is supersoluble if maximal subgroups of its Sylow subgroups are normal in G. Clearly, under assumptions of Srinivasan's theorem every Sylow subgroup P of G satisfies the following condition: every element in $P\backslash\Phi(P)$ is $Q\mathfrak{U}$-central in G. Srinivasan's theorem was generalized in [8], [10], [19] by replacing the normality with the weaker condition of c-normality; besides, in [8] the condition of c-normality of maximal subgroups of Sylow subgroups in a normal nilpotent subgroup is considered. We also recall S.N.Chernilov's result [20] on supersolubility of a group G with abelian Sylow subgroups having the following property: every primary cyclic subgroup complemented in a Sylow subgroup of G is complemented in G. Vedernikov and Kuleshov [21] established that a group G is supersoluble if every its primary cyclic subgroup having a non-trivial supplement in a Sylow p-subgroup of G possesses a non-trivial supplement in G. Analyzing this line of investigations we arrive at the conclusion that they are concerned with the following question.

Question B. *Let $\mathfrak{F}$ be a saturated formation, H a normal subgroup of a group G. Assume that every Sylow subgroup P of H satisfies the following condition: every element in $P\backslash(\Phi(P) \cup \Phi(G))$ is $Q\mathfrak{F}$-central in G. Is it true that every non-Frattini G-chief factor of H is $\mathfrak{F}$-central in G?*

The main aim of the present article is to give a positive answer to Questions A and B. Moreover, we give the answer even in a more general form fixing our attention to the behaviour of p-elements with a fixed prime p.

2. Preliminaries

We use the standard notations [22], [23]. For a prime p, G_p denotes a Sylow p-subgroup of G; $\pi(G)$ is the set of primes dividing $|G|$; $\pi(\mathfrak{F}) = \bigcup_{G\in\mathfrak{F}} \pi(G)$; $F^*(G)$ is the generalized Fitting subgroup of G, i.e. the quasinilpotent radical of G [24]. A subgroup M is called a minimal supplement to a normal subgroup H of G if $MH = G$ and $M_1H \neq G$ for every proper subgroup M_1 of M.

We need some information from the theory of formations. A formation is a class of groups closed under taking homomorphic images and subdirect products. We denote by $G^{\mathfrak{F}}$ a $\mathfrak{F}$-residual of a group G, i.e. the smallest normal subgroup with quotient in $\mathfrak{F}$. A formation $\mathfrak{F}$ is called saturated if $G/\Phi(G) \in \mathfrak{F}$ always implies $G \in \mathfrak{F}$. A function $f : \{primes\} \to \{formations\}$ is called a local satellite. A chief factor H/K of G is called f-central in G if $G/C_G(H/K) \in f(p)$ for every prime p dividing $|H/K|$.

If $\mathfrak{F}$ is the class of all groups whose chief factors are f-central, then we say that f is a local satellite of $\mathfrak{F}$ and write $\mathfrak{F} = LF(f)$. A local satellite f of a formation $\mathfrak{F}$ is called: 1) full if $f(p) = \mathfrak{N}_p f(p)$ for every prime p (here $\mathfrak{N}_p$ is the class of p-groups); 2) integrated if $f(p) \subseteq \mathfrak{F}$ for every prime p; 3) semi-integrated if for every prime p, $f(p)$ is either a subformation in $\mathfrak{F}$ or the class $\mathfrak{E}$ of all groups; 4) canonical if it is full and integrated; 5) semicanonical if it is full and semi-integrated. The notation $LF(F)$ means that F is a canonical local satellite of $LF(F)$. A chief factor H/K of G is called $\mathfrak{F}$-central in G if it is f-central, where f is the canonical local satellite of $\mathfrak{F}$.

In the proofs of our results we will use the following theorems.

2.1. *Every non-empty saturated formation possesses a semicanonical local satellite and the unique canonical local satellite* (see [22], [23], [25]).

2.2. *Let f be a local satellite such that $f(p) = (1)$ and $f(q) = \mathfrak{E}$ for every prime $q \neq p$. Then $LF(f)$ is the class of p-nilpotent groups* [2].

2.3.(a) *Let f be a satellite such that $f(p)$ is the class of all abelian groups with exponents dividing $p-1$, and $f(p) = \mathfrak{E}$ for every prime $q \neq p$. Then $LF(f)$ is the class p-$\mathfrak{U}$ of p-supersoluble groups.*

(b) *Let a prime p divide the order of a chief factor H/K of G. Then H/K is p-$\mathfrak{U}$-central if and only if $|H/K| = p$* ([2], Kapitel VI).

2.4. *Let $\mathfrak{F}$ be a saturated formation and H a normal subgroup of a group G such that $H/H \cap \Phi(G) \in \mathfrak{F}$. Then $H = A \times B$ where $A \in \mathfrak{F}$, $B \subseteq \Phi(G)$, $\pi(B) \cap \pi(\mathfrak{F}) = \emptyset$* ([23], Theorem 4.2).

2.5. *Let H be a normal subgroup of G such that $H/H \cap \Phi(G)$ is p-nilpotent. Then H is p-nilpotent* [2], [23].

2.6. *Let $\mathfrak{F} = LF(f)$ where f is semi-integrated. Let H/L be a G-chief factor of $G^{\mathfrak{F}}$ such that $f(p) \subseteq \mathfrak{F}$ for some $p \in \pi(H/L)$. Then H/L is f-eccentric in G if one of the following conditions holds: 1) H/L is non-Frattini in G; 2) a Sylow p-subgroup in $G^{\mathfrak{F}}$ is abelian* ([23], Theorems 8.1 and 11.6).

2.7. *If $G = AB$, then for every prime p there exist Sylow p-subgroups A_p, B_p and G_p in A, B and G such that $G_p = A_p B_p$* ([23], p. 134).

2.8. *Let H be a normal subgroup of a group G. Let α and β be G-chief series of H. Then there exists a one-to-one correspondence between the chief factors of α and those of β such that the corresponding factors are G-isomorphic and such that the Frattini chief factors of α correspond to the Frattini chief factors of β* ([22], Theorem A.9.13).

2.9. *Let H be a normal subgroup of a group G, and let M be a minimal supplement to H in G. If M contains at least one Sylow p-subgroup of H for some prime p, then H is p-nilpotent* ([26]; [23], Theorem 12.4).

2.10. *Let $\mathfrak{F}$ be a saturated formation, and H a normal subgroup of a group G such that every G-chief factor of H is $\mathfrak{F}$-central in G. Then $G/C_G(H) \in \mathfrak{F}$* (see [23], Theorem 9.5).

2.11. *If G is a group, then $C_G(F^*(G)) \subseteq F(G)$* (see [25], Theorem 15.22).

2.12. (a) *Let p be an odd prime. A group G is p-nilpotent if every its element of order p is Q-central in G.*

(b) *A group G is 2-nilpotent if every its 2-element of order ≤ 4 is Q-central in G* (see [11], Theorem 2).

2.13. *Let $\mathfrak{F}$ be a saturated formation and H a normal subgroup of a group G. Let ω be the set of primes p such that $H^{\mathfrak{F}}$ possesses an abelian Sylow p-subgroup. Then there exists a subgroup C of G such that $G = CH^{\mathfrak{F}}$ and $\pi(C \cap H^{\mathfrak{F}}) \cap \omega = \emptyset$* (see [23], Theorem 11.8).

2.14. *If a Sylow p-subgroup of a p-soluble group G is abelian, then* $l_p(G) \leq 1$ (see [23], Theorem 5.11).

2.15. *Let* $G = \langle a \rangle B$, *where* $\langle a \rangle \neq 1$ *is a* 2*-subgroup and* $B \neq G$. *Then there exists a normal maximal subgroup* M *of* G *such that* $G = \langle a \rangle M$ (see [21], Lemma 1).

2.16. *Let* $G = \langle a \rangle B = HB$, *where* $B \neq G$, $H \trianglelefteq G$, $\langle a \rangle \subseteq H$, *and* $\langle a \rangle$ *is a p-subgroup. Assume that* H_p *is abelian and* G *is p-soluble. Then a is a* $Q\mathfrak{U}$*-central element of* G.

Proof. Let G be a counterexample of minimal order. Then we can assume that $B_G = O_{p'}(H) = 1$. By 2.14, H_p is normal in G. We have that

$$G = \langle a \rangle B = H_p B, H_p = \langle a \rangle (H_p \cap B).$$

Evidently, $H_p \cap B$ is normal in G. Since $B_G = 1$, it follows that $H_p \cap B = 1$, and $H_p = \langle a \rangle$ is normal in G.

3. Main results

For a prime p and a group H, we use the following conventions:
$W_p(H) = \{x : x \in H, |x| = p\}$ if p is odd,
$W_2(H) = \{x : x \in H, |x| \in \{2, 4\}\}$,
$W(H) = \{x : x \in H, |x|$ is a prime or $|x| = 4\}$.

Definition 3.1 (see [11], Definition 3). *Let f be a local satellite. An element a of a group G is called Qf-central in G if there exists a f-central chief factor* A/B *of* G *such that* $a \in A \backslash B$. *The identity element is always regarded as a Qf-central element.*

Definition 3.2. *Let* $\mathfrak{F} = LF(f)$ *be a saturated formation, where f is an integrated local satellite of* $\mathfrak{F}$. *An element a of G is called* $Q\mathfrak{F}$*-central if it is Qf-central.*

It is easy to show that Definition 3.2 does not depend on the choice of an integrated local satellite.

Definition 3.3. *An element a of G is called Q-central if it is* $Q\mathfrak{N}$*-central (this means that there exists a central chief factor* A/B *of* G *such that* $a \in A \backslash B$*).*

Theorem 3.1. *Let p be a prime, and* $\mathfrak{F} = LF(f)$ *a saturated formation, where f is a semicanonical local satellite such that* $f(p) \subseteq \mathfrak{F}$ *and* $f(q) = \mathfrak{E}$ *for every prime* $q \neq p$. *Let H be a normal subgroup of a group G. Assume that every element in* $W_p(H)$ *is Qf-central in G. Then every G-chief factor of H is f-central in G.*

Proof. We will use induction on $|G|+|H|$. Let $W = W_p(H) = \{x_i : i \in I\}$. We may assume that $W \neq \emptyset$. Assume that there is a normal p'-subgroup $K \neq 1$ in G. Consider the natural epimorphism $\alpha : G \to G/K$. Evidently, $W^\alpha = W_p(HK/K)$. If $x_i \in W$, then by assumption,there is a f-central chief factor A/B of G such that $x_i \in A\backslash B$. The factors AK/BK and $A/B(A \cap K)$ are G-isomorphic; besides, it follows from $x_i \in A\backslash B$ that $A \neq B(A \cap K)$ because every p-element in $B(A \cap K)$ is contained in B. Hence, $B = B(A \cap K)$. We have that $x_i \in AK\backslash BK$, and AK/BK is a f-central chief factor of G. But then, $(AK)^\alpha/(BK)^\alpha$ is a f-central chief factor of G^α. Clearly, $x_i^\alpha \in (AK)^\alpha\backslash(BK)^\alpha$. By the inductive hypothesis, the theorem is true for G/K. Then it is also true for G. So, we may assume that $O_{p'}(G) = 1$.

Consider an arbitrary element x_i in W. By assumption, there is a f-central chief factor A/B of G such that $x_i \in A\backslash B$. We have

$$AH/BH \simeq A/A \cap BH = A/B(A \cap H).$$

Since $x_i \in A\backslash B$, the equality $B = B(A\cap H)$ is impossible. Therefore, $A = B(A\cap H)$, and we have that A/B and $A\cap H/B\cap H$ are G-isomorphic G-chief factors. So, we showed that for each $x_i \in W$, there is a G-chief factor X_i/Y_i of H such that $x_i \in X_i\backslash Y_i$ and X_i/Y_i is f-central in G. Set

$$C = \bigcap_{i\in I} C_G(X_i/Y_i).$$

Clearly, $G/C \in f(p) \in \mathfrak{F}$. Therefore, every G-chief factor of HC/C is f-central. If $H\cap C \neq H$, then, by the inductive hypothesis, every G-chief factor of $H\cap C$ is f-central in G, and the theorem is proved. So, we may assume that $H \subseteq C$. This means that every element in W is Q-central in C. By 2.12, H is p-nilpotent. Since $O_{p'}(G) = 1$, we have that H is a p-group. Let Q be a Sylow q-subgroup in C, $q \neq p$. Then we have that every element in W is Q-central in QH. By 2.12, QH is p-nilpotent. Since $G/C \in f(p) = \mathfrak{N}_p f(p)$ and C_q centralizes every G-chief p-factor of H for every prime $q \neq p$, it follows that every G-chief p-factor of H is f-central in G. The theorem is proved.

Corollary 3.1.1. *Let $\mathfrak{F}$ be a saturated formation, and G a group such that every element in $W(G)$ is $Q\mathfrak{F}$-central in G. Then $G \in \mathfrak{F}$.*

Proof. Applying Theorem 1 for the case $H = G$ and for the arbitrary prime p, we obtain that every chief factor of G is $\mathfrak{F}$-central. So, $G \in \mathfrak{F}$, and the result is true.

Corollary 3.1.2. *Let $\mathfrak{F}$ be a saturated formation, and H a normal subgroup of a group G such that $G/H \in \mathfrak{F}$ and every element in $W(F^*(H))$*

is $Q\mathfrak{F}$*-central in* G*. Then* $G \in \mathfrak{F}$*.*

Proof. By Theorem 3.1, every G-chief factor of $F^*(H)$ is $\mathfrak{F}$-central in G. By 2.10, $G/C_G(F^*(H))$ belongs to $\mathfrak{F}$. From this and from $G/H \in \mathfrak{F}$ it follows that $G/C_H(F^*(H)) \in \mathfrak{F}$. By 2.11, $C_H(F^*(H))$ is contained in $F^*(H)$. Therefore $G/F^*(H)$ belongs to $\mathfrak{F}$, and we have that $G \in \mathfrak{F}$.

Corollary 3.1.3. *Let* p *be a prime, and* H *a normal subgroup of a group* G*. If every element in* $W_p(H)$ *is* Q*-central in* G*, then* H *is* p*-nilpotent, and* $H/O_{p'}(H)$ *lies in the hypercenter of* $G/O_{p'}(H)$*.*

Corollary 3.1.4. *Let* p *be a prime, and* H *a normal subgroup of a group* G*. Assume that every element in* $W_p(H)$ *is* $Q\mathfrak{U}$*-central in* G*. Then* H *is* p*-supersoluble, and every* G*-chief* p*-factor of* H *is cyclic.*

We introduce the subgroup $O_{p',\Phi}(G)$ as follows:

$$O_{p',\Phi}(G)/O_{p'}(G) = \Phi(G/O_{p'}(G)).$$

Theorem 3.2. *Let* p *be a prime, and* $\mathfrak{F} = LF(f)$ *a saturated formation, where* f *is a semicanonical local satellite such that* $f(p) \subseteq \mathfrak{F}$ *and* $f(q) = \mathfrak{E}$ *for every prime* $q \neq p$*. Let* H *be a normal subgroup of a group* G*. Assume that every element in* $H_p \backslash (\Phi(H_p) \cup \Phi(G))$ *is* Qf*-central in* G*. Then every* G*-chief factor of* $H/H \cap O_{p',\Phi}(G)$ *is* f*-central in* $G/H \cap O_{p',\Phi}(G)$*.*

Proof. We will prove this theorem using induction on $|H| + |G|$. Assume that there is a normal p'-subgroup $K \neq 1$ in G such that $K \subseteq H$. Consider the natural epimorphism $\alpha : G \to G/K$. If $a \in H_p$, then a^α belongs to a Sylow p-subgroup H_p^α of HK/K. Assume that a^α is not contained in $\Phi(H_p^\alpha) \cup \Phi(G^\alpha)$. Since $H_pK/K \simeq H_p$, it follows that a is not contained in $\Phi(H_p)$. Furthermore, it follows from $(\Phi(G))^\alpha \subseteq \Phi(G^\alpha)$ that a is not contained in $\Phi(G)$. By assumption, there is a f-central chief factor A/B of G such that $a \in A \backslash B$. The factors AK/BK and $A/B(A \cap K)$ are G-isomorphic; besides, it follows from $a \in A \backslash B$ that $A \neq B(A \cap K)$ because every p-element in $B(A \cap K)$ is contained in B. Hence, $B = B(A \cap K)$. We have that $a \in AK \backslash BK$, and AK/BK is a f-central chief factor of G. But then, $(AK)^\alpha/(BK)^\alpha$ is a f-central chief factor of G^α. Clearly, $a^\alpha \in (AK)^\alpha \backslash (BK)^\alpha$. By the inductive hypothesis, the theorem is true for G/K. Then it is also true for G.

So, we may assume that $O_{p'}(H) = 1$. Consider $H \cap G^{\mathfrak{F}}$. We may assume that $H \cap G^{\mathfrak{F}}$ has non-Frattini G-chief factors. We call a normal subgroup L of G f-limit if $L/L \cap \Phi(G)$ is a f-eccentric G-chief factor. The set Σ of f-limit subgroups contained in $H \cap G^{\mathfrak{F}}$ is not empty. Really, if $L/(\Phi(G) \cap H \cap G^{\mathfrak{F}})$ is a minimal normal subgroup in $G/(\Phi(G) \cap H \cap G^{\mathfrak{F}})$, where $L \subseteq H \cap G^{\mathfrak{F}}$, then L is f-limit by 2.6. So, let L be a subgroup of minimal order in Σ. Set $\Phi = L \cap \Phi(G)$. It follows from $O_{p'}(H) = 1$ and 2.5 that p divides $|L/\Phi|$. Let M/Φ be a minimal supplement for L/Φ in G/Φ. Then by 2.7 we have $G_p = M_pL_p$. By 2.9, M_p/Φ does not contain L_p/Φ; hence, $M \neq G$. From $G_p = M_pL_p$ it follows that $H_p = G_p \cap H = (H_p \cap M_p)L_p$, where $H_p \cap M_p$ does not contain L_p. Hence, there is an element a in $L_p \backslash (H_p \cap M_p)$ such that $a \notin \Phi(H_p)$. Since $H_p \cap M_p \supseteq \Phi = L \cap \Phi(G)$, we have that $a \notin \Phi(G)$. So, we get

$$a \in H_p \backslash (\Phi(H_p) \cup \Phi(G)).$$

By assumption, G possesses a f-central chief factor A/B such that $a \in A\backslash B$. Consider $AL/BL \simeq A/B(A \cap L)$. Since A/B is a chief factor, $B(A \cap L)$ is either equal B or else A. Since a belongs to $A \cap L$ and does not belong to B, we have that $B \neq B(A \cap L)$. Hence, $A = B(A \cap L)$. So, we have G-isomorphic G-chief factors A/B and $A \cap L/B \cap L$; besides, $a \in (A \cap L)\backslash(B \cap L)$.

Suppose that $A\cap L/B\cap L$ is a non-Frattini chief factor of G. Then, by 2.8, $A\cap L/B\cap L$ is G-isomorphic with L/Φ. In this case, L/Φ is f-central in G. This contradicts 2.6. So, we obtained that $A\cap L/B\cap L$ is a Frattini chief factor of G. If $B \cap L$ is not contained in $\Phi(G)$, we have that $B \cap L$ possesses a f-limit normal subgroup of G; this contradicts the minimality of $|L|$. Therefore, $B \cap L \subseteq \Phi(G)$. We get $A \cap L \subseteq \Phi(G)$. Hence, $a \in A \cap L \subseteq \Phi(G)$. We arrive at a contradiction, because $a \notin \Phi(G)$. The theorem is proved.

Corollary 3.2.1. *Let $\mathfrak{F}$ be a saturated formation, H a normal subgroup of a group G such that $G/H \in \mathfrak{F}$ and for every prime p the following condition holds: each element in $H_p\backslash(\Phi(H_p) \cup \Phi(G))$ is $Q\mathfrak{F}$-central in G. Then $G \in \mathfrak{F}$.*

Corollary 3.2.2. *Let $\mathfrak{F}$ be a saturated formation, H a normal soluble subgroup of a group G such that $G/H \in \mathfrak{F}$ and the following condition holds: if P is a Sylow subgroup of $F(H)$, then each element in $P\backslash(\Phi(P) \cup \Phi(G))$ is $Q\mathfrak{F}$-central in G. Then $G \in \mathfrak{F}$.*

Proof. Let $\Phi = \Phi(G) \cap F(H)$. By Theorem 3.2, every G-chief factor of $F(H)/\Phi$ is $\mathfrak{F}$-central in G. By 2.5, $F(H)/\Phi = F(H/\Phi)$. By 2.10, $G/\Phi/C_{G/\Phi}(F(H/\Phi)) \in \mathfrak{F}$. Therefore, $G/C_G(F(H)/\Phi) \in \mathfrak{F}$. From this and from $G/H \in \mathfrak{F}$ it follows that $G/C_H(F(H)/\Phi) \in \mathfrak{F}$. But $C_H(F(H)/\Phi) \subseteq F(H)$. Hence, $G/\Phi \in \mathfrak{F}$. Since $\mathfrak{F}$ is saturated, we have that $G \in \mathfrak{F}$.

Corollary 3.2.3. *Let p be a prime, and H be a normal subgroup of a group G. Assume that every element in $H_p\backslash(\Phi(H_p) \cup \Phi(G))$ is Q-central* *in G. Then H is p-nilpotent, and every its non-Frattini G-chief p-factor is central in G.*

Corollary 3.2.4. *Let p be a prime, and H be a normal subgroup of a group G. Assume that every element in $H_p\backslash(\Phi(H_p)\cup\Phi(G))$ is $Q\mathfrak{U}$-central in G. Then H is p-supersoluble, and every its non-Frattini G-chief p-factor is cyclic.*

Corollary 3.2.5. *A group G is supersoluble if for every non-cyclic Sylow subgroup P of G the following condition holds: every element in $P\backslash(\Phi(P)\cup\Phi(G))$ is $Q\mathfrak{U}$-central in G.*

Proof. If G_2 is non-cyclic, then by Theorem 3.2, G is 2-supersoluble. If G_2 is cyclic, then G is 2-nilpotent. Thus, G is soluble, and by Theorem 3.2, G is p-supersoluble for all prime p such that G_p is non-cyclic.

Definition 3.4. *An element $a \neq 1$ of an abelian group P is called basic if there exists a subgroup B in P such that $P = \langle a\rangle \times B$. We denote by $\mathcal{B}(P)$ the set of all basic elements in P.*

The following theorem generalizes S.N.Chernikov's result [20] on a finite group with a system of complemented subgroups.

Theorem 3.3. *Let p be a prime, and $\mathfrak{F} = LF(f)$ a saturated formation of p-soluble groups, where f is a semicanonical local satellite such that $f(p) \subseteq \mathfrak{F}$ and $f(q) = \mathfrak{E}$ for every prime $q \neq p$. Let H be a normal subgroup of a group G. Assume that H_p is abelian and every element in $\mathcal{B}(H_p)$ is Qf-central in G. Then every G-chief factor of H is f-central in G.*

Proof. We will use induction on $|H| + |G|$. As well as in the proof of Theorem 3.2, it is easy to show that the assumption of the theorem is valid for $G/O_{p'}(H)$ and $H/O_{p'}(H)$. So, we may assume that $O_{p'}(H) = 1$. We will consider two cases: $H = G$ and $H \neq G$.

Case 1. Assume that $H = G$. Consider the $\mathfrak{F}$-residual R of G. By 2.13, there exists a subgroup C such that $G = CR$ and p does not divide $|C \cap R|$. By 2.7, $G_p = C_pR_p$ and $C_p \cap R_p \subseteq C \cap R$. So, $G_p = C_p \times R_p$. It follows from this that $\mathcal{B}(R_p) \subseteq \mathcal{B}(G_p)$.

It is clear that $\mathcal{B}(R_p)\backslash\Phi(G) \neq \emptyset$. Consider $a \in \mathcal{B}(R_p)\backslash\Phi(G)$. By assumption, there is a f-central chief factor A/B of G such that $a \in A\backslash B$. We have that AR/BR is G-isomorphic with $A/B(A \cap R)$. Since $a \in A\backslash B$, it follows that $A = B(A \cap R)$. Thus, A/B and $A \cap R/B \cap R$ are G-isomorphic f-central G-chief factors. This contradicts 2.6. So, the theorem is true for the case $H = G$.

Case 2. Now we assume that $H \neq G$. Let $\mathfrak{H}$ be the formation of p-soluble groups. By 2.13, there exists a subgroup C such that $G = CH^{\mathfrak{H}}$

and p does not divide $|C \cap H^{\mathfrak{H}}|$. By 2.7, $G_p = C_p H_p^{\mathfrak{H}}$, where G_p, C_p and $H_p^{\mathfrak{H}}$ are Sylow p-subgroups in G, C and $H^{\mathfrak{H}}$. Evidently, $G_p \cap H = H_p$ is a Sylow p-subgroup of H. Furthermore, $G_p \cap H^{\mathfrak{H}} = H_p^{\mathfrak{H}} = H_p \cap H^{\mathfrak{H}}$. We have $H_p = (C_p \cap H_p) \times H_p^{\mathfrak{H}}$. It follows from this that $\mathcal{B}(H_p^{\mathfrak{H}}) \subseteq \mathcal{B}(H_p)$.

Suppose that $\mathcal{B}(H_p^{\mathfrak{H}})$ is non-empty. If $a \in \mathcal{B}(H_p^{\mathfrak{H}})$ then by the assumption of the theorem there exists a f-central chief factor A/B of G such that $a \in A \backslash B$. Since all groups in $f(p)$ are p-soluble, A/B is a p-group. Consider

$$AH^{\mathfrak{H}}/BH^{\mathfrak{H}} \simeq A/A \cap BH^{\mathfrak{H}} = A/B(A \cap H^{\mathfrak{H}}).$$

Since $a \in (A \cap H^{\mathfrak{H}}) \backslash B$, we have that $B \neq B(A \cap H^{\mathfrak{H}})$. Therefore, $A = B(A \cap H^{\mathfrak{H}})$. We have that A/B and $A \cap H^{\mathfrak{H}}/B \cap H^{\mathfrak{H}}$ are G-isomorphic G-chief factors. Since H_p is contained in $C_H(A \cap H^{\mathfrak{H}}/B \cap H^{\mathfrak{H}})$ we have that

$$H/C_H(A \cap H^{\mathfrak{H}}/B \cap H^{\mathfrak{H}}) \in \mathfrak{H}.$$

Thus, there is a $\mathfrak{H}$-central chief p-factor D_1/D_2 of H such that

$$A \cap H^{\mathfrak{H}} \supseteq D_1 \supset D_2 \supseteq B \cap H^{\mathfrak{H}}.$$

This contradicts 2.6.

So, we assume that $\mathcal{B}(H_p^{\mathfrak{H}})$ is empty and H is p-soluble. Since $O_{p'}(H) = 1$ and H_p is abelian, it follows by 2.14 that H_p is normal. Clearly, we can assume that H is a p-group. Let $\mathcal{B}(H) = \{x_i : i \in I\}$. By assumption, for every $i \in I$ there exists a f-central G-chief factor A_i/B_i such that $x_i \in A_i \backslash B_i$. We set

$$X_i = A_i \cap H, Y_i = B_i \cap H.$$

Then factors $A_i H/B_i H$ and $A_i/B_i X_i$ are G-isomorphic. Since $x_i \in X_i \backslash B$, we have that $B_i \neq B_i X_i$. Thus, $A_i = B_i X_i$. So, A_i/B_i and X_i/Y_i are G-isomorphic f-central chief factors of G. It follows from this that $G/C_G(X_i/Y_i) \in f(p)$. We have that

$G/C \in f(p)$, where $C = \bigcap_{i \in I} C_G(X_i/Y_i) \supseteq H$.

It follows from this that every element x_i in $\mathcal{B}(H)$ is Q-central in HC_q for every prime $q \neq p$. For HC_q the theorem is true (we note that by Case 1, the theorem is true if a considered normal subgroup coincides with the whole group). Applying the proved part of the theorem to HC_q and the formation of p-nilpotent groups we have that HC_q is p-nilpotent. Therefore, $C_C(X/Y)$ is a p-group for every G-chief factor X/Y of H. But $G/C \in f(p)$. We see that $G/C_G(X/Y)$ belongs to $\mathfrak{N}_p f(p) = f(p)$, that is X/Y is f-central in G. The theorem is proved.

Corollary 3.3.1. *Let p be a prime. Assume that a normal subgroup H of a group G possesses an abelian Sylow p-subgroup P. Assume also that every element in $\mathcal{B}(P)$ is $Q\mathfrak{U}$-central in G. Then H is p-supersoluble, and every G-chief p-factor of H is cyclic.*

Corollary 3.3.2. *Let p be a prime. Assume that a normal subgroup H of a group G possesses an abelian Sylow p-subgroup P. Assume also that every element in $\mathcal{B}(P)$ is Q-central in G. Then H is p-nilpotent, and every G-chief p-factor of H is central in G.*

Corollary 3.3.3. *Let H be a normal subgroup of a group G. Assume that a Sylow 2-subgroup P of H is abelian and has the following property: $\langle a\rangle$ is complemented in G for every $a \in \mathcal{B}(P)$. Then H is 2-nilpotent, and every its G-chief 2-factor is central in G.*

Proof. By 2.15, every element in $\mathcal{B}(P)$ is Q-central in G. Now we apply Corollary 3.3.2.

Corollary 3.3.4. *Let H be a normal subgroup of a group G. Assume that for every Sylow subgroup P of G the following condition holds: P is abelian, and $\langle a\rangle$ is complemented in G for every $a \in \mathcal{B}(P)$. Then H is supersoluble, and every its G-chief factor is cyclic.*

Proof. By Corollary 3.3.3, H is 2-nilpotent. So, H is soluble. Let P be a Sylow p-subgroup of H, $p \in \pi(H)$. By assumption, for every $a \in \mathcal{B}(P)$ we have that

$$\langle a\rangle M = G, \langle a\rangle \cap M = 1.$$

By 2.16, a is $Q\mathfrak{U}$-central in G. Now we apply Corollary 3.3.1.

Corollary 3.3.5 (see [20]). *Assume that every Sylow p-subgroup P of G is abelian and satisfies the following condition: if $a \in \mathcal{B}(P)$, then $\langle a\rangle$ is complemented in G. Then G is supersoluble.*

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