Algebra and Discrete Mathematics
Number 4. **(2004).** pp. 60 – 65

RESEARCH ARTICLE

Basic semigroups: theory and applications

J. S. Ponizovskii

Abstract. A concept of basic matrix semigroups over fields (with some variations) is introduced and throughly investigated. Sections 1 and 2 contain main definitions, Section 3 treats some properties of basic semigroups, Section 4 is devoted to some application of basic semigroups: matrix representations (including faithful representations), finiteness theorems, the problem of Korjakov (when a matrix semigroup over field K is conjugate to a matrix semigroup over a proper subfield of K). The paper is a survey and contains no proofs (which may be found in papers from References).

1. Notions

In what follows $n > 1$, K is a fixed field; $M_n(K)$ denotes the multiplicative semigroup of all $n \times n$-matrices over K. If S is a semigroup then $T \leq S$ ($T \trianglelefteq S$) means that T is a subsemigroup (an ideal) of S. If $S \leq M_n(K)$ then $J(S) = \{x \in M_n(K) \mid xS \cup Sx \leq S\}$ is the idealizer of S in $M_n(K)$.

Let $S \leq M_n(K)$. Define $H(S)$ as follows. If $S = \{0\}$ (0 is the zero matrix) then $H(S) = \{0\}$. If $S \neq \{0\}$ and r is the least natural such that $r = \text{rank}(x)$ for some $x \in S$, then $H(S) = \{x \in S \mid \text{rank}(x) \leq r\}$.

Clearly $H(S) \trianglelefteq S$. $H(S)$ is called the homogeneous ideal of S. A semigroup T with zero 0 is called 0-prime if and only if the following holds:

$$x, y \in T, \quad xTy = 0 \Longrightarrow [x = 0 \vee y = 0].$$

Let W denote an n-dimensional linear space over K consisting of all row-vectors of dimension n. Elements from $M_n(K)$ act on W as right operators. If V is a subspace of W then $(V : K)$ stands for the dimension of V. If $A \subseteq M_n(K)$ then $L(A)$ is the K-linear envelope of A.

2. Basic, strongly basic, weakly basic semigroups

Let $S \le M_n(K)$. Denote by $R(S)$ ($C(S)$) the row-space (the column-space) of S: $R(S)$ is a subspace of W spanned by all the rows of all matrices from S ($C(S)$ is defined similarly).

Theorem 1. *The followings conditions are equivalent for any $S \le M_n(K)$:*
(i) $(R(S) : K) = (C(S) : K) = n$;
(ii) $W \cdot L(S) = W$ and if $w \in W$ is such that $w \cdot S = 0$ then $w = 0$;
(iii) if $x \in M_n(K)$ is such that either $xS = 0$ or $Sx = 0$ then $x = 0$.

Definition. Let $S \le M_n(K)$. Then:

$$S \text{ is basic} \Longleftrightarrow \text{any of (i), (ii), (iii) from Theorem 1 holds;}$$

$$S \text{ is strongly basic} \Longleftrightarrow H(S) \text{ is basic } 0\text{−prime;}$$

$$S \text{ is weakly basic} \Longleftrightarrow \text{for any } x \in M_n(K),\ xS = Sx = 0 \text{ implies } x = 0.$$

The class of all basic (strongly basic, weakly basic) subsemigroups of $M_n(K)$ is denoted by $B(K)$ ($SB(K), WB(K)$). The following holds:

$$SB(K) \subset B(K) \subset WB(K) \quad (SB(K) \ne B(K) \ne WB(K)).$$

Examples.
(i) Any irreducible $S \le M_n(K)$ is strongly basic.
(ii) Any indecomposable inverse $S \le M_n(K)$ is strongly basic.
(iii) Any nonzero indecomposable commutative $S = S^2 \le M_n(K)$ is basic but not necessarily strongly basic.
(iv) Let S be a set of all matrices from $M_n(K)$ with nonzero entries in the last row only. Then S is a weakly basic subsemigroup of $M_n(K)$, but S is not basic.
(v) Any $S \le M_n(K)$ containing the identity matrix is basic but not necessarily strongly basic.

3. Properties of basic (strongly basic, weakly basic) semigroups

3.1. Embedding theorem

For any abstract semigroup T, $\Omega(T)$ denotes the translational hull of T. Recall that a semigroup T is left weakly reductive if and only if following holds:

let $a, b \in T$; if $xa = xb$ for all $x \in T$ then $a = b$.

Right weakly reductive semigroups are defined similarly. A semigroup T is weakly reductive if the following holds:

let $a, b \in T$; if $xa = xb$ and $ax = ab$ for all $x \in T$ then $a = b$.

Clearly left (right) weakly reductive semigroup is weakly reductive but not vice versa.

It is well known that any weakly reductive semigroup has a standartd embedding into $\Omega(T)$ as an ideal. So if T is a weakly reductive semigroup we put $T \triangleleft \Omega(T)$.

Theorem 2. *Any weakly basic $S \leq M_n(K)$ is weakly reductive (hence we may take $S \trianglelefteq \Omega(S)$).*

Let $S \leq M_n(K)$ be weakly basic, $S \trianglelefteq \Omega(S)$ as abstract semigroups. Define a mapping $\omega : J(S) \to \Omega(S)$ as follows:

if $a \in J(S)$ then $\omega(a)$ is defined by the rule: $\omega(a) \cdot x = ax$, $x \cdot \omega(a) = xa$ for all $x \in S$.

It is easy to show that ω is a homomorphism of semigroups. The following fact is very important:

Theorem 3 (Embedding Theorem)**.** *If $S \leq M_n(K)$ is weakly basic then ω is a monomorphism. If $S \leq M_n(K)$ is basic then ω is an isomorphism.*

Remark. Theorem 3 shows that, for S basic, the pair $S \subset \Omega(S)$ may be included into $M_n(K)$. More exactly: there exists a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & \Omega(S) \\ \varepsilon \uparrow & & \uparrow \omega \\ S & \xrightarrow{g} & J(S) \end{array}$$

where ε is an identity mapping, f and g are inclusions.

3.2. Closure theorem

Let $S \leq M_n(K)$ be homogeneous (i.e. $S = H(S)$). A semigroup $\overline{S} \leq M_n(K)$ is called the closure of S if the following holds:

(i) $\overline{S}$ is completely 0-simple,

(ii) $S \subseteq \overline{S}$,

(iii) if $U \leq M_n(K)$ is completely 0-simple such that $S \subseteq U$ then $\overline{S} \leq U$.

Theorem 4 (Closure Theorem). *For any strongly basic $S \le M_n(K)$ there exists a closure $\overline{S}$; moreover $\overline{S}$ is unigue and $\overline{S}$ meets all $\mathcal{H}$-classes of S.*

Examples show that the condition "S is basic" cannot be omitted. The meaning of closure is rather evident: it is a sort of completely 0-simple approximation of a homogeneous semigroup.

3.3. Heritability properties

Theorem 5 (Heritability Theorem). *Let $S \le M_n(K)$ be strongly basic, and let T, U be such that $T \trianglelefteq S \trianglelefteq U \le M_n(K)$. Then T, U are strongly basic.*

The following theorem shows that an extension of a field K does not change the idealizer of a basic semigroup.

Theorem 6. *Let $K \subseteq F$ be fields, and let $S \le M_n(K)$ be basic. Then the idealizer of S in $M_n(K)$ is equal to the idealizer of S in $M_n(F)$.*

4. Applications

4.1. Matrix representations of semigroups

See [1].

4.2. Finiteness theorems

Let $S \le M_n(K)$ be strongly basic. Since $H(S) \trianglelefteq S$, then $H(S)$ is strongly basic by Theorem 5. Now we formulate

Theorem 7. *Let $S \le M_n(K)$ be strongly basic. If a maximal nonzero subgroup of $\overline{H(S)}$ is finite then S is finite (note that $\overline{H(S)}$ exists by Theorem 3).*

Theorem 8. *Let $S \le M_n(K)$ be periodic of bounded period. Assume that there exists a set of strongly basic representations of S (over some field F) which separates points of S. Then S is finite (a representation $f : S \to M(F)$ is strongly basic if $f(S)$ is strongly basic).*

This is a generalization of theorem of Y. Zalcstein [7].

Theorem 9. *Let $S \le M_n(K)$ be regular irreducible with finite subgroups. Then S is finite.*

Applying the well known theorem by Shur we get

Theorem 10. *Let $S \leq M_n(K)$ be irreducible, periodic and regular. Then S is finite.*

Theorem 8 is in [3]. Theorem 9 is published in [2].

4.3. Reduction to smaller fields

Results concern the problem:

Let $F \subset K$ be a field extension, and let $S \leq M_n(K)$; when S is conjugate to a subsemigroup of $M_n(F)$?

Some sufficient conditions are given in the following

Theorem 11. *Let $F \subset K$ be fields. Let $S \leq M_n(K)$ be strongly basic and G be a maximal nonzero subgroup of $\overline{H(S)}$. Then S is conjugate to a subsemigroup of $M_n(F)$ provided G has this property.*

This theorem is a generalization of a result from [4].

Theorem 11 gives a positive answer to the question 3.39 of Korjakov [6].

5. Faithful matrix representations of semigroups

Let S be a semigroup having a faithful matrix representation $f : S \to M_n(K)$. Assume that T is a semigroup such that $S \trianglelefteq T \leq \Omega(S)$. When f may be extended to a faithful representation $F : S \to M_n(K)$?

It is always possible if $f(S)$ is basic since then one can take $F = \omega$ (see Theorem 3). But it is not so in general if $f(S)$ is only weakly basic because in this case ω maps $J(S)$ into $M_n(K)$ (more exactly into $J(f(S))$), a part of $\Omega(S)$ only. The following theorem shows that sometimes such F may be constructed in parts.

Theorem 12 ([5])**.** *Let S be a weakly reductive semigroup (so that we put $S \triangleleft \Omega(S)$). Let $\{S_i \mid i \in I\}$ be a family of subsemigroups of $\Omega(S)$ such that the following holds:*

(i) S is an ideal of S_i for all $i \in I$;

(ii) there exists a faithful representation $f : S \to M_n(K)$ such that $f(S)$ is weakly basic;

(iii) for any $i \in I$, there exists a faithful representation $f_i : S_i \to M_n(K)$ such that

$$x \in S \Longrightarrow f_i(x) = f(x),$$

$$x \in S,\ y \in S_i \Longrightarrow f(xy) = f(x)f_i(y),\ f(yx) = f_i(y)f(x).$$

Let T be a subsemigroup of $\Omega(S)$ generated by all S_i ($i \in I$). Then there exists a faithful representation $F : T \to M_n(K)$ such that F extends f and all f_i ($i \in I$), i.e.

$F(x) = f(x)$ for all $x \in S$,
$F(y) = f_i(y)$ for arbitrary $i \in I$ and for all $y \in S_i$.

References

[1] J. Okninski and J. S. Ponisovskii, *A New Matrix representation Theorem for Semigroups.* Semigroup Forum, **52**(1996), no.3, 293-305.

[2] J. S. Ponisovskii, *On matrix semigroups*, Sovremennaja Algebra, Semigroups with additional structures. Leningrad, 1986, 64-71 (in Russian).

[3] J. S. Ponisovskii, *On a theorem by Y. Zalcstein*, Semigroups, Automata and Languages, Proc. Conf. University of Porto 1994, World Sci. Singapore – New Jersey – London – Hong Kong, 225-231.

[4] J. S. Ponisovskii. *On Matrix Semigroups over a Field K conjugate to Matrix semigroups over a proper subfield of K.* Proc. Conf. "Semigroups with Applications", Oberwolfach, 1991, 1-5.

[5] J. S. Ponisovskii, *On faithful representations of semigroups*, Sovremennaja Algebra **3**(23). Leningrad, 1998, 103-106 (in Russian).

[6] *Sverdlovskaja tetrad III.*Sverdlovsk, 1989, Problem 3.39 (in Russian).

[7] Y. Zalcstein. *A finiteness theorem for linear semigroups*, Proc. Amer. Math. Soc. **103**(1988), no.2, 399-400.

Contact information

J. S. Ponizovskii

Russian State Hydrometeororlogical University, Department of Mathematics, Malookhtinsky pr. 98, 195196 St-Peterburg, Russia
E-Mail: `JP@JP4518.spb.edu`

Received by the editors: 21.05.2004
and final form in 06.12.2004.