Algebra and Discrete Mathematics
Number 4. **(2004).** pp. 48 – 59

RESEARCH ARTICLE

Subsets of defect 3 in elementary Abelian 2-groups

B. V. Novikov, L. Yu. Polyakova

1. Introduction

It is well-known [1] that linear codes over a two-element field are precisely subgroups of an elementary Abelian 2-group G. It is naturally to consider subsets in G which are close to subgroups, as codes which are close to linear ones. In this connection in [3] the notion of a defect of a subset of a group G has been introduced as a measure of a deviation from a subgroup (so that a subset has the defect 0 only if it is a subgroup).

The subsets of defect 1 and 2 are described in [3]. In this description so called *standard* subsets play a leading role (see definition in section 2): all subsets of defect 1 are standard, and among subsets of defect 2 there is only one non-standard. In this article we show, that all subsets of defect 3 containing not less than 12 elements, are standard, and we describe all non-standard ones.

One can suppose that this situation is kept in the general case: large subsets of the fixed defect are standard. However now we do not know, whether this assumption is true.

2. Properties of the defect

Everywhere further G denotes a finite elementary Abelian 2-group, T its subset containing the identity , $|T|$ number of elements in T, $\langle T\rangle$ the subgroup of G, generated by T. Besides for any element $a \in T \setminus 1$ we put $T_a = T \setminus aT$.

A defect of a subset T is a number $\operatorname{def} T = \max_{a\in T} |T_a|$.

If H is a subgroup of G and $T \subset H$ then $\operatorname{def} T \le |H\backslash T|$. In particular, putting $H = \langle T\rangle$, we get inequality:

$$|T| + \operatorname{def} T \le |\langle T\rangle|.$$

We call T *standard*, if $|T| + \operatorname{def} T = |\langle T\rangle|$.

For example, if F is a subgroup of G, H is a subgroup of F and $T = (F \setminus H) \cup 1$ then T is standard and $\operatorname{def} T = |H| - 1$. Subsets of the form $T = (F \setminus H) \cup 1$ will be called *strictly standard*.

Obviously, subsets of defect 0 are exactly subgroups. The following results for defect 1 and 2 have been obtained in [3]:

Theorem 1. Each subset of defect 1 is of the form $T = H \setminus a$, where H is a subgroup of G, $a \in H$.

Theorem 2. Let $\operatorname{def} T = 2$. Then either T is standard or $|T| = 4$ and $|\langle T\rangle| = 8$ (so $T \setminus 1$ is a basis of $\langle T\rangle$).

Thus, subsets of defect 1 are strictly standard, and subsets of defect 2, except the single one in essence, are standard (but are not strictly standard).

In [3] the following result also has been received: if a, b, c are different non-identity elements of G then $G \setminus \{a, b, c\}$ has defect 3. We shall use this statement below.

It is useful to interpret the notion of defect in terms of graphs [2]. To a subset T we compare a graph $\Gamma(T)$ in the following way: vertices of $\Gamma(T)$ are elements of $T \setminus 1$ and edges are such pairs of vertices (a, b) that $ab \notin T$. Then the degree of the vertex a equals $\deg a = |T_a|$ and $\operatorname{def} T = \max\limits_{a\in T} \deg a$.

In this section we obtain some general properties of subsets of any defect.

Theorem 3. Let $C_1, C_2, \dots, C_r$ be connected components of the graph $\Gamma = \Gamma(T)$, $1 \le i \ne j \le r$. Then

1) There is such $k \le r$ that $C_iC_j \subset C_k$.

2) If in 1) $k \ne i$ then $aC_j = C_k$ for every $a \in C_i$.

Proof. 1) It follows from definition of Γ that $C_iC_j \subset T$. Let $a \in C_i$. Since aC_j is connected, it is contained in some component C_k. Similarly, if $x \in C_j$ then $C_ix \subseteq C_l$ for some $l \le r$. But since $ax \in aC_j \cap C_ix$ then $k = l$ and k does not depend on a choice of a and x. Hence, $C_iC_j \subset C_k$.

2) Let $a \in C_i$, $aC_j \subset C_k$. Then $C_j \subset aC_k$. As $i \ne k$, by the first part of Theorem $aC_k \subset C_j$. Hence, $aC_k = C_j$. □

We shall call a subset T *homogeneous*, if $\operatorname{def} T = \deg a$ for all $a \in T \setminus 1$ (i. e. if $\Gamma(T)$ is homogeneous). Theorem 4 gives more detailed information about structure of homogeneous subsets. We shall preliminary prove several assertions.

Proposition 1. Let T be a homogeneous subset, C_i, C_j, C_k such connected components of $\Gamma = \Gamma(T)$, that $C_iC_j \subset C_k$ and $i \neq j$. Then $aC_j = C_k$ for all $a \in C_i$.

Proof. Note that the graph aC_j is isomorphic to the graph C_j, hence the homogeneous graph C_k contains a homogeneous subgraph of the same degree. From here $aC_j = C_k$. □

Corollary 1. All connected components of the graph of a homogeneous subset T are isomorphic.

Proof. Let C_i, C_j be connected components of $\Gamma(T)$. According to Theorem 3 and Proposition 1 there is such a component C_k that $C_iC_j = C_k$. Moreover components C_j and $C_k = aC_j$ $(a \in C_i)$ are isomorphic. Similarly C_i and C_k are isomorphic. Therefore C_i and C_j are isomorphic too. □

Proposition 2. If the graph $\Gamma(T)$ of a homogeneous subset T is not connected then its components are complete graphs.

Proof. Let us assume that $\Gamma = \Gamma(T)$ is not connected and that among its connected components there is a non-complete one. Accordingly to Corollary 1 all components of Γ are isomorphic, so all of them are non-complete.

Let us consider components C_i, C_j, C_k, for which $i \neq j$ and $C_iC_j = C_k$. Since C_i is a non-complete connected component then $|C_i| \geq 3$ and there are such $a, b \in C_i$ that $ab \in T$. Then $ab \in C_m$ for some m. We shall prove that $m = i$. If it not so, $C_iC_m \subset C_i$, since, for example, $b = a \cdot ab \in C_iC_m$. Then accordingly to Corollary 1 $xC_m = C_i$ for all $x \in C_i$. In particular, for $x = a$ we have: $aC_m \not\ni a$ and $C_i \ni a$; the contradiction.

Thus $ab \in C_i$. Then $aC_j = bC_j = abC_j = C_k$, whence $C_j = bC_j = C_iC_j = C_k$. So $j = k$. Similar reasoning for the non-complete component C_j shows, that $i = k$. We get a contradiction again. □

Theorem 4. If T is homogeneous then either $\Gamma(T)$ is connected or T is strictly standard.

Proof. Suppose that $\Gamma(T)$ is not connected. Then by Proposition 2 all its components are complete.

Let $C_1 \neq C_2$ are components of $\Gamma(T)$, $x \in C_1$. We denote $H = xC_1$ and prove that H is a subgroup.

Let $C_1C_2 \subset C_3$. According to Proposition 1

$xC_2 = C_3 = C_1 y$ for any $y \in C_2$. Then $C_3 = Hxy$, whence $C_2 = C_3 x = Hy$. Therefore for $a, b \in H$ we have: $ax \cdot by \in C_3$, i. e. $aby \in C_2 = Hy$. From here $ab \in H$.

Besides, it follows from this reasoning that every component has a form $C_i = x_i H$. Since the product of two various components contains in some component (Theorem 3), then $F = T \cup H$ is a subgroup, and $T = (F \setminus H) \cup 1$. By the definition T is strict standard. □

We shall prove two more lemmas which will be used below for subsets of defect 3.

Lemma 1. $\deg a \equiv \operatorname{def} T (\operatorname{mod} 2)$ for every $a \in T \setminus 1$.

Proof. Since $a(T \cap aT) = T \cap aT$ then $T \cap aT$ contains, together with every x, an element ax and, hence, $|T \cap aT|$ is even. From here $|T| = |T \cap aT| + |T_a| \equiv |T_a| (\operatorname{mod} 2)$ for all $a \in T$. In particular, $|T| \equiv \operatorname{def} T (\operatorname{mod} 2)$. Thus, $\deg a \equiv \operatorname{def} T (\operatorname{mod} 2)$. □

Lemma 2. If $a, b, ab \in T$ then $\deg ab \leq \deg a + \deg b$.

Proof. Suppose the opposite: let $\deg a = k$, $\deg b = m$, $\deg ab = p > k + m$. Then there are $x_1, \ldots, x_p \in T \setminus 1$ for which $abx_1, \ldots, abx_p \notin T$. Not less than $p - m$ elements among elements bx_j $(1 \leq j \leq p)$ are contained in T; let, for example, $bx_1, \ldots, bx_{p-m} \in T$. Since $p - m > k$ by hypothesis, there is such x_i $(1 \leq i \leq p - m)$ that $abx_i \in T$, and we obtain a contradiction. □

From Lemmas 1 and 2 it follows

Corollary 2. If $\operatorname{def} T$ is odd and $a, b, ab \in T$ then $\deg ab \leq \deg a + \deg b - 1$. □

In particular,

Corollary 3. If $a, b, ab \in T$, and $\deg a = \deg b = 1$ then $\deg ab = 1$. □

3. Non-homogeneous subsets of defect 3

From Lemma 1 it follows that a subset of defect 3 can contain only elements of the degree 1 and 3. A number of following statements of this section is right for any subsets of odd defect, containing elements of the degree 1; therefore we shall assume, that T is just such a subset. If T will be a subset of defect 3 we shall stipulate it.

We introduce the following designations: $T_1 = \{a \in T | \deg a = 1\}$, $H = \langle T_1 \rangle$, $S = T_1 \cup 1$.

Lemma 3. $|H \setminus T_1| \le 2$.

Proof. If $aS = S$ for every $a \in T_1$ then S, obviously, coincides with H and the lemma is proved. Suppose it is not so, i. e. $aS \setminus S \ne \emptyset$ for some $a \in T_1$. Let us fix some $x \in aS \setminus S$. Then $x = ab$, where $b \in T_1$. By Corollary 3 $x \notin T$ (otherwise $\deg x = 1$ and $x \in S$), hence $x \in aT \setminus T$. In view of the fact that $\deg a = 1$, we obtain $|aS \setminus S| = 1$ and $\operatorname{def} S = 1$. However, by Theorem 1 S is standard, $|H \setminus S| = 1$, so $|H \setminus T_1| = 2$. □

Thus, two cases are possible. We shall consider them separately:

1) $S = H \setminus f$, where f is an element from H;

2) $S = H$.

Proposition 3. If $S = H \setminus f$ then $T \setminus S$ is the join of cosets of H.

Proof. We shall prove that the equality $h(T \setminus S) = T \setminus S$ is right for every $h \in H$. Notice that for any $a \in T_1$ the degree of af also is equal 1. Therefore $f \notin T$ (otherwise $f \in T_1$ by Corollary 3), so $T_a = \{af\}$. Hence, $a(T \setminus S) \subset T$. Besides $a(T \setminus S) \cap S = \emptyset$. Really, if it is not so, there is such $t \in T \setminus S$, that $at \in T_1$, but this contradicts Corollary 3.

Thus $a(T \setminus S) = T \setminus S$ for all $a \in T_1$. Since $af \in T_1$ then $f(T \setminus S) = fa \cdot a(T \setminus S) = fa(T \setminus S) = T \setminus S$. □

Corollary 4. If $S = H \setminus f$ and $\operatorname{def} T \ge 3$ then $\operatorname{def} T \ge |T_1| + 3$.

Proof. $T \cup H = T \cup \{f\}$ is not a subgroup, otherwise $\operatorname{def} T = 1$ by Theorem 1. It follows out of Corollary 3 that there are $x, y \in T \setminus H$ such that $xy \notin T \cup H$. But then $xyH \cap (T \cup H) = \emptyset$, so $T_x \supset yH$. Besides the element $fx \notin yH$ also is contained in T_x. Hence, $\deg x \ge |H| + 1 \ge |T_1| + 3$. □

$>$From here we obtain immediately that if $\operatorname{def} T = 3$ then the case 1) is impossible, so, $H = S = T_1 \cup 1$. In this situation (the case 2)) for every $a \in T_1$ there is an unique $x \in T \setminus T_1$ such that $w = xa \notin T$. Fix the elements a and x.

Proposition 4. Let $\operatorname{def} T \ge 3$, $T_1 \cup 1 = H$. Then one of the following statements takes place:

1) $T_1 \subset T_x$.

2) If $b \in T_1$, $y \in T$ and $by \notin T$ then $by = w$. Besides $T \cup w$ is the join of cosets of H.

Proof. Assume that 1) is not executed and $b \in T_1 \setminus T_x$, such that $b \neq a$ and $b \in T_y$ for some $y \neq x$. Since $\deg a = \deg b = 1$ then $xb, ya \in T$. As $T_1 \cup 1$ is a subgroup, $ab \in T_1$ and by Corollary 3 $\deg ab = 1$. But $xb \cdot ab, ya \cdot ab \notin T$, so $xb = ya$, whence $yb = w$. From here it follows also, that $T \cup w$ is the join of cosets of H. □

Corollary 5. If the condition 2) of Proposition 4 is executed then $\operatorname{def} T \geq |T_1| + 2$.

Proof. Since $\operatorname{def} T \neq 1$, Theorem 1 implies that $T \cup w$ is not a subgroup. Therefore such $u, v \in T$ exist that $uv \notin T \cup w$, and at the same time at least one of these elements, for example u, is not contained in H. Then $uH \cdot v \cap T = \emptyset$, whence $\deg v \geq |H|$. As $|H|$ is even, we get from here $\operatorname{def} T \geq |H| + 1 = |T_1| + 2$. □

Corollary 6. If $\operatorname{def} T = 3$ and $T_1 \subset T_x$ then either $|T_1| = 3$ or $|T_1| \leq 1$.

Proof. According to the condition $T_x \supset T_1$, therefore $|T_1| \leq 3$. Since $T_1 \cup 1$ is a subgroup for a subset T of defect 3, $|T_1| \neq 2$. □

Proposition 5. If $\operatorname{def} T = 3$, $T_1 \subset T_x$ and $|T_1| = 3$ then T is standard.

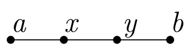
Proof. It is enough to show, that $\langle T \rangle = T \cup xT_1$. Indeed, $T_1 T \subset T \cup xT_1$. Besides, since $T_x \supset T_1$ and $|T_1| = 3$ then $T_x = T_1$. Hence, $xy \in T$ for every $y \in T \setminus T_1$. Consider an arbitrary element $a \in T_1$. Notice that $axy \in T$, otherwise $xy \in T_a = \{x\}$. So $xyT_1 \subset T$ and $T_y = xyT_1$. But then $yT \subset T \cup xT_1$. □

Corollary 7. If $\operatorname{def} T = 3$ and T is non-standard then $|T_1| \leq 1$. □

The next theorem is applicable both to homogeneous and to non-homogeneous subsets of defect 3 and essentially confines a class of graphs which can correspond to these subsets.

Theorem 5. If T is non-standard and $\operatorname{def} T = 3$ then diameters of connected components of $\Gamma(T)$ do not exceed 2.

Proof. We shall prove by contradiction, using an induction on $|T|$. Let $a, b \in T$ and



(1)

is the shortest way from a to b in the graph Γ. Then $ax, xy, yb \notin T$, $ay, xb, ab \in T$.

Let H be a subgroup generated by elements a, x, y, b. We shall prove some auxiliary statements (Lemmas 4 – 7).

Lemma 4. Elements a, x, y, b form a basis in H.

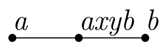
Proof. If in the subgroup H it holds $w = 1$ for some word w in the alphabet $\{a, x, y, b\}$, then the length of w should be not less than 3 because all elements a, x, y, b are different. Therefore w coincides with one of the words $axyb, axy, axb, ayb, xyb$. If $axyb = 1$, then $ab = xy$, but $ab \in T$, and $xy \notin T$; the contradiction. If $axy = 1$ then $y = ax \notin T$. The other variants are similarly impossible. □

Lemma 5. $T \not\subset H$.

Proof. Assume that $T \subset H$. Since T is non-standard, it is contained in $H \setminus T$ (in addition to ax, xy, yb) even one of elements $axyb$, axy, axb, ayb, xyb. Consider the possible cases.

1) $\underline{axb \notin T}$. Then $xb \in T_a$ and $ab \in T_x$. Hence $T_x = \{a, y, ab\}$ and therefore $axy \in T$. Similarly from $T_{ab} = \{x, xb, ay\}$ it follows $ayb \in T$. But then $x, xb, ayb, axy \in T_a$. By Lemma 4 all these elements are different, so $|T_a| \geq 4$, that is impossible. Hence $axb \in T$ and similarly $ayb \in T$.

2) $\underline{axy \notin T, axb, ayb \in T}$. Then $T_x\{a, y, ay\}$. Therefore $ayb \notin T_x$, i.e. $axyb \in T$. If $xyb \notin T$ there would be a way of length 2:



contrary to the assumption. Hence $xyb \in T$. But then $T_x \supseteq \{a, y, ay, xyb\}$. The contradiction. Therefore $axy \in T$ and similarly $xyb \in T$.

3) $\underline{axyb \notin T, axy, axb, ayb, xyb \in T}$. Then $T_x \supseteq \{a, y, ayb, xyb\}$, that is impossible. □

Remark. Proving in Lemma 5 the inequality $|T_t| \geq 4$ for some $t \in T$, we base each time on Lemma 4. Further we shall use this lemma without the reference to it.

Denote $\overline{T} = H \cap T$, $\overline{\Gamma} = \Gamma(\overline{T})$.

Lemma 6. $\operatorname{def} \overline{T} = 3$ and $\overline{T}$ is standard.

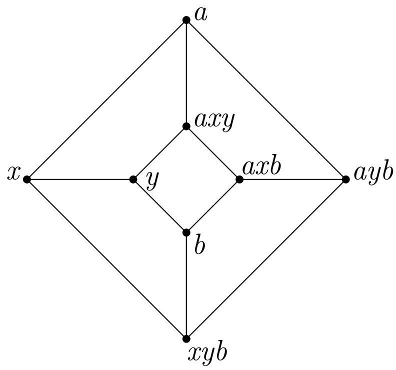
Proof. For any $t \in T$ we have:

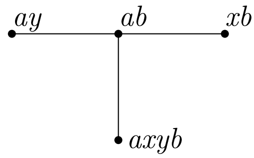
$$\overline{T} \setminus t\overline{T} = (T \cap H) \setminus (tT \cap H) = (T \cap H) \setminus tT,$$

whence $|\overline{T} \setminus t\overline{T}| \leq |T \setminus tT| \leq 3$. Suppose that $\operatorname{def} \overline{T} = 2$. From $\overline{T}_x = \{a, y\}$ and $ay \in \overline{T}$ it follows $axy \in \overline{T}$. But then $\overline{T}_y \supseteq \{x, b, axy\}$ and $\operatorname{def} \overline{T} \geq 3$.

Thus $\operatorname{def} \overline{T} = 3$. Since $|\overline{T}| < |T|$ (Lemma 5) and the way (1) is contained in $\overline{T}$, then by the assumption of induction $\overline{T}$ is standard. □

Evidently, $\overline{T} = H \setminus \{ax, xy, yb\}$ and $\overline{\Gamma}$ has the form





Denote these components by C_1 and C_2.

Lemma 7. *$zH \subset T$ for every $z \in T \setminus \overline{T}$.*

Proof. Since all vertices of C_1 have the degree 3, z is not connected with any of them by an edge, i.e. $zC_1 \subset T$, and for the same reason $abz \in T$. If $z\overline{T} \not\subset T$, let, for example, $ayz \notin T$. Then $az \in T_y = \{x, b, axy\} \subset H$ contrary to $z \notin H$. Therefore $z\overline{T} \subset T$.

It remains to show that $z(H \setminus \overline{T}) \subset T$. If $zax \notin T$ then $za \in T_x = \{a, x, xyb\}$. The contradiction. Hence, $zax \in T$ and similarly $zxy, zyb \in T$. □

Returning to the proof of the theorem, we note, that in each coset $zH \subset T$ there is an element u, such that $|T_u| = 3$ (e.g., $T_u = \{xz, axyz, aybz\}$ for $u = az$).

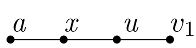
Denote by K the join of all cosets of H which have nonempty intersection with T (in fact, by Lemma 7 all of them, except H, are contained in T). We shall prove, that K is a subgroup. Indeed, let uH and vH be two different cosets, such that $uH \neq H \neq vH$, $uH \cup vH \subset T$. Besides, let their representatives u and v be chosen in such a way that $|T_u| = |T_v| = 3$. If $uv \notin T$ then $v \in T_u = \{axu, xuy, ybu\}$. This is impossible, since $uH \neq vH$. Hence $uv \in T$ and $uvH \subset T$.

But then $T = K \setminus \{ax, xy, yb\}$ is standard. □

Now we can prove the main result of this section:

Theorem 6. *If $\operatorname{def} T = 3$ and T is non-homogeneous then T is standard.*

Proof. Assume the opposite. Let $T_1 \neq \emptyset$. Then according to Corollary 7 $T_1 = \{a\}$ for some $a \in T$. Let $x \in T \setminus T_1$ and $ax \notin T$. Since $\deg x = 3$, there is such an element $u \in T \setminus T_1$ that $xu \notin T$. Furthermore, there are such $v_1, v_2 \in T \setminus T_1$ that $v_i \neq x$, $v_i u \notin T$ $(i = 1, 2)$. We obtain a way



By Theorem 5 $az, zv_1 \notin T$ for some $z \in T$. Since $\deg a = 1$ then $z = x$ and $xv_1 \notin T$. Similarly $xv_2 \notin T$. But then $\deg x \geq 4$. This is impossible. □

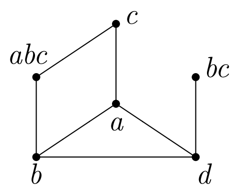
4. Homogeneous subsets of defect 3

In this section we shall assume, that T is a non-standard homogeneous subset of defect 3. In this case its graph $\Gamma(T)$ is connected by Theorem 4. We shall find out, how $\Gamma(T)$ looks and show that there are only 3 non-standard homogeneous subsets.

We need the following lemma:

Lemma 8. Let $a \in T$ and $T_a = \{b, c, d\}$. Then either $bc, bd, cd \notin T$ or $bc, bd, cd \in T$.

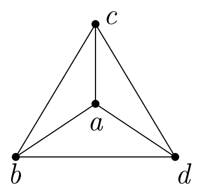
Proof. Obviously, if $bcd = 1$, the lemma is right. Let $bcd \neq 1$. Assume opposite, let, e.g., $bc \in T$, $bd \notin T$. Since $bcd \neq 1$ then $bc \notin T_a$. Therefore $abc \in T$, whence $abc \in T_b \cap T_c$. Consider the shortest way from b to bc (it exists because $\Gamma(T)$ is connected). By Theorem 5 it contains not more than two edges. As $bc \notin T_a \cup T_b \cup T_{abc}$, this way consists of edges (b, d) and (d, bc), so $bcd \notin T$ (see fig.).



Since $abc \notin T_d = \{a, b, bc\}$, $abcd \in T$, but then $abcd \in T_a = \{b, c, d\}$, what leads to the contradiction. □

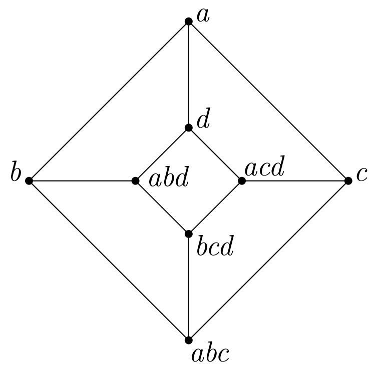
Consider two cases for the graph $\Gamma(T)$.

1) Let such a vertex a exist in Γ, that $T_a = \{b, c, d\}$ and $bcd \neq 1$. If $bc, bd, cd \notin T$ then by Lemma 8 we get that Γ is the complete graph K_4 with four vertices:



On the other hand, if $bc, bd, cd \in T$ we get $abc, abd, acd \in T$ (because $bc, bd, cd \notin T_a$). From here $T_b = \{a, abc, abd\}$, $T_c = \{a, abc, acd\}$, $T_d = \{a, abd, acd\}$. We note also that $bcd \in T$, otherwise $cd \in T_b =$

$\{a, abc, abd\}$, what is impossible. Therefore the graph Γ in this case should look so:



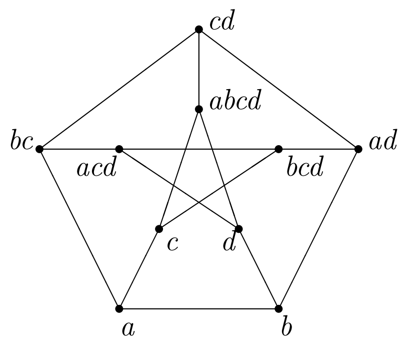
However, diameter of this graph equals 3, what contradicts Theorem 5. Thus, this case is impossible.

2) Consider now the case when for all $t \in T$, from $T_t\{x, y, z\}$ it follows $xyz = 1$. Let $a \in T$. Then T_a has a form $T_a = \{b, c, bc\}$ for some $b, c \in T$. Besides $T_b = \{a, d, ad\}$ for some $d \in T$. From here it follows

$$ab, ac, abc, bd, abd \notin T. \tag{2}$$

We shall consider several subcases:

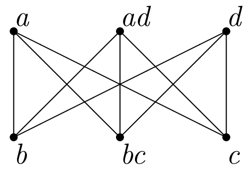
a) Suppose that $cd \in T$. Then $cd \in T_{bc} \cap T_{ad}$. Since $T_{cd} \supset \{ad, bc\}$ then $T_{cd} = \{ad, bc, abcd\}$, and similarly $T_{bc} = \{a, cd, acd\}$, $T_{ad} = \{b, cd, bcd\}$. It follows from (2) that $T_{acd} = \{d, bc, bcd\}$, $T_{bcd} = \{c, acd, ad\}$, $T_{abcd} = \{c, d, cd\}$, $T_d = \{acd, b, abcd\}$, $T_c = \{a, bcd, abcd\}$. Therefore the graph looks so:



It is so-called Petersen graph [2].

b) Analogously, if $abcd \in T$, we obtain the same graph.

c) If $cd, abcd \notin T$, $\Gamma(T) = K_{3,3}$, a complete bipartite graph:



Thus, we proved

Theorem 7. If T is a non-standard homogeneous subset of defect 3, then $\Gamma(T)$ is either the complete graph K_4, or the Petersen graph, or the complete bipartite graph $K_{3,3}$. □

To formulate the main result of this section, we need the next definition.

Let T, U are subsets of the group G. We say that T is *isomorphic* to U if there exists such a bijection $f : T \to U$ that $f(ab) = f(a)f(b)$, as soon as $a, b, ab \in T$ (this definition means that $\Gamma(T)$ and $\Gamma(U)$ are isomorphic).

Theorem 8. Each homogeneous subset T of defect 3 is either standard, or isomorphic to one of the following subsets

1) $\{1, x, y, z, w\}$,

2) $\{1, x, y, z, w, xw, yz\}$,

3) $\{1, x, y, z, w, xy, xz, xw, yz, yw, zw\}$,

where x, y, z, w are linearly independent elements of the group G.

Proof. Let T be non-standard. By Theorem 7 its graph $\Gamma(T)$ is:
either the complete graph K_4, and then $T = \{1, a, b, c, d\}$;
or the complete bipartite graph $K_{3,3}$, and then $T = \{1, a, b, c, d, ad, bc\}$;
or the Petersen graph, and then $T = \{1, a, b, c, d, ad, bc, cd, acd, bcd, abcd\}$.
The last subset is isomorphic to the subset 3) from the condition of the theorem. Indeed, isomorphism between them is realized by function f, for which

$$f(a) = x,\ f(b) = yz,\ f(c) = yw,\ f(d) = w.$$

□

From the description of subsets of defect 3, and also from Theorems 1 and 2, we obtain the following

Corollary 8. Let a, b, c, d be different elements from $G \setminus 1$. Then for the set $T = G \setminus \{a, b, c, d\}$ the following statements are fulfilled:

If $|G| = 8$ then T is either a subgroup of order 4 or a non-standard subset of defect 2.

If $|G| > 8$ then T is a (standard) subset of defect 4.

Proof. Evidently, $\operatorname{def} T \leq 4$. Let $|G| = 8$. Then T contains, besides 1, three more elements. If they are linearly dependent, T is a subgroup if not then T is a subset of defect 2 by Theorem 2.

Let $|G| > 8$. Note that T cannot be a standard subset of defect, smaller than 4. Then by Theorem 1 $\operatorname{def} T \neq 1$. Non-standard subsets of defect 2 contain 4 elements, and non-standard ones of defect 3 can contain only 5, 7 or 11 elements. Since $|T| = 2^k - 4$ for some natural $k \geq 4$ then $\operatorname{def} T \neq 2$ and $\operatorname{def} T \neq 3$. □

References

[1] E. R. Berlekamp. *Algebraic Coding Theory.* Moscow, Mir, 1971 (in Russian).

[2] F. Harary. *Graph theory.* Moscow, Mir, 1973 (in Russian).

[3] B. V. Novikov. *On a characteristic of subsets of Abelian groups.* Discrete Mathematics and Applications (Bulgaria), **6**(2002), 97-101.

Contact information

B. V. Novikov Saltovskoye shosse 258 apt. 20, 61178 Kharkov, Ukraine
E-Mail: boris.v.novikov@univer.kharkov.ua

L. Yu. Polyakova B. Chichibabina str. 2 apt. 84, 61022 Kharkov, Ukraine
E-Mail: nataliya.yu.tyutryumova@univer.kharkov.ua

Received by the editors: 24.06.2004
and final form in 06.12.2004.