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RESEARCH ARTICLE

Partial resolutions in monoid cohomology

P. A. Grillet

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Abstract. Partial resolutions are constructed for the Eilenberg-MacLane cohomology of monoids, with applications to examples.

Introduction

The Eilenberg-MacLane cohomology groups of a monoid S are usually computed from projective resolutions of the trivial S-module $\mathbb{Z}$, as computation from cocycles tends to be very unwieldy [6], [7], [8]. When S has a complete (Church-Rosser) and reduced presentation, Squier's partial resolution [9] leads to simpler computations in dimensions $n \leq 2$ (in some cases, $n \leq 3$ [5]).

When S is commutative, its commutative cohomology groups are likewise difficult to compute, but the overpath method, introduced in [4], yields markedly simpler computations of H^2 by cocycles, directly from presentations of S [2], [3].

It turns out that the two results are closely related. When the overpath method is adapted to Eilenberg-MacLane cohomology, forsaking commutativity and using cycles rather than cocycles, the result is a partial resolution, constructed in Section 1, which is very similar to Squier's but applies to any presentation. Squier's resolution is retrieved in Section 2, with minor modifications, if the presentation is complete and reduced, or almost reduced.

Section 3 sets up the computation of H^0, H^1, H^2 by cocycles. Section 4 computes H^2 for five examples: finite cyclic groups and monoids; the bicyclic semigroup; the free commutative monoid on two generators; the

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monoid freely generated by two idempotents; and the free band with identity on two generators.

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1. First resolution

1. In what follows, S is an arbitrary monoid, determined by a presentation as the the quotient of the free monoid F on a set X (often denoted by X^*) by the congruence generated by a binary relation R on F (often called a *rewrite system*). The elements of R are ordered pairs $r = (r', r'')$ of elements of F. We denote the identity elements of S and F by 1, and the projection $F \longrightarrow S$ by $a \longmapsto \bar{a}$.

In F, a *connecting sequence* from $a \in F$ to $b \in F$ consists of sequences $a_0, a_1, \ldots, a_n, u_1, \ldots, u_n, v_1, \ldots, v_n$ of elements of F and a sequence $r_1, \ldots, r_n$ of elements of R, such that $n \geq 0$, $a = a_0$, $a_n = b$, and, for every $1 \leq i \leq n$, either $a_{i-1} = u_i\, r_i'\, v_i$ and $a_i = u_i\, r_i''\, v_i$, or $a_{i-1} = u_i\, r_i''\, v_i$ and $a_i = u_i\, r_i'\, v_i$. This definition is justified by the description of the congruence $\sim$ generated by R: namely, $a \sim b$ ($\bar{a} = \bar{b}$) if and only if there exists a connecting sequence from a to b.

In this section we construct a partial projective resolution

$$M_3 \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\epsilon} \mathbb{Z}$$

of the trivial §-module $\mathbb{Z}$, which resembles Squier's resolution [9] but has a different module M_3 and requires no hypothesis on R.

As in [9], $M_0 = \mathbb{Z}[S]$; M_1 is the free §-module with one basis element $[x]$ for each generator $x \in X$ of F; $\epsilon : \mathbb{Z}[S] \longrightarrow \mathbb{Z}$ is the augmentation homomorphism

$$\epsilon\big(\sum_{s \in S} n_s s\, s\big) \;=\; \sum_{s \in S} n_s s;$$

and $\partial_1 : M_1 \longrightarrow \mathbb{Z}[S]$ is the module homomorphism such that

$$\partial_1\, [x] \;=\; \bar{x} - 1$$

for all $x \in X$; equivalently, the additive homomorphism such that

$$\partial_1\, s[x] \;=\; s\bar{x} - s. \tag{1}$$

Lemma 1.1. *[9]* $\operatorname{Im} \partial_1 = \operatorname{Ker} \epsilon$.

Proof. First, $\epsilon\, \partial_1\, [x] = \epsilon\, (\bar{x} - 1) = 0$ for all $x \in X$, so that $\operatorname{Im} \partial_1 \subseteq \operatorname{Ker} \epsilon$. For the converse we construct some maps which will be used later.

The expansion mapping $\xi : F \longrightarrow M_1$ is defined by:

$$\xi\,(x_1\,x_2\,\dots\,x_n) \;=\; \sum_{1\le i\le n} \overline{x_1\,x_2\,\dots\,x_{i-1}}\,[x_i] \tag{2}$$

for all $x_1,\,x_2,\,\dots,\,x_n \in X$ (in particular, $\xi\,x = [x]$ and $\xi 1 = 0$). We see that $\partial_1\,\xi\,(x_1\,x_2\,\dots\,x_n) = \bar{x}_1 - 1 + \sum_{2\le i\le n}\big(\overline{x_1\,x_2\dots\,x_i} - \overline{x_1\,x_2\,\dots\,x_{i-1}}\big) = \overline{x_1\,x_2\dots\,x_n} - 1$, so that

$$\partial_1\,\xi\,a \;=\; \bar{a} - 1 \tag{3}$$

for all $a \in F$.

For each $s \in S$ choose any representative word $w_s s \in F$ such that $\bar{w_s} s = s$. Let $\sigma_0 : \mathbb{Z}[S] \longrightarrow M_1$ be the additive homomorphism such that

$$\sigma_0\,s \;=\; \xi w_s s \tag{4}$$

for all $s \in S$. Let $\zeta : \mathbb{Z} \longrightarrow \mathbb{Z}[S]$ be the additive homomorphism $\zeta(n) = n1$. By the above,

$$\partial_1\,\sigma_0\,s + \zeta\,\epsilon\,s \;=\; \bar{w_s} s - 1 + 1 = s$$

for all $s \in S$, so that $\partial_1\,\sigma_0 + \zeta\,\epsilon$ is the identity on $\mathbb{Z}[S]$. Hence $\operatorname{Ker}\epsilon \subseteq \operatorname{Im}\partial_1$. $\square$

2. As in [9], M_2 is the free §-module with one basis element $[r]$ for each $r \in R$; $\partial_2 : M_2 \longrightarrow M_1$ is the module homomorphism such that

$$\partial_2\,[r] \;=\; \xi r' - \xi r'' \tag{5}$$

for all $r \in R$, where ξ is the expansion mapping above.

To study M_2 we construct another trace map τ. Let P be a connecting sequence, consisting of sequences $a_0,\,a_1,\,\dots,\,a_n,\,u_1,\,\dots,\,u_n,\,v_1,\,\dots,\,v_n$ of elements of F and a sequence $r_1,\,\dots,\,r_n$ of elements of R, such that $n \ge 0$, $a = a_0$, $a_n = b$, and, for every $1 \le i \le n$, either $a_{i-1} = u_i\,r_i'\,v_i$ and $a_i = u_i\,r_i''\,v_i$, or $a_{i-1} = u_i\,r_i''\,v_i$ and $a_i = u_i\,r_i'\,v_i$. Then

$$\tau\,P \;=\; \sum_{1\le i\le n} \epsilon_i\,\bar{u}_i\,[r_i], \tag{6}$$

where $\epsilon_i = +1$ if $a_{i-1} = u_i r_i' v_i$, $a_i = u_i r_i'' v_i$, $\epsilon_i = -1$ if $a_{i-1} = u_i r_i'' v_i$, $a_i = u_i r_i' v_i$.

Since $\xi(ab) = \xi a + \bar{a}\,\xi b$ for all $a, b \in F$, we have, in the above,

$$\begin{aligned} \xi a_{i-1} \;&=\; \xi u_i r_i' v_i \;=\; \xi u_i + \bar{u}_i\,\xi r_i' + \overline{u_i r_i'}\,\xi v_i, \\ \xi a_i \;&=\; \xi u_i r_i'' v_i \;=\; \xi u_i + \bar{u}_i\,\xi r_i'' + \overline{u_i r_i''}\,\xi v_i, \end{aligned}$$

or vice versa; since $u_i\bar{r}_i' = u_i\bar{r}_i''$ this yields

$$\xi a_{i-1} - \xi a_i \;=\; \epsilon_i\,\bar{u}_i\,(\xi r_i' - \xi r_i'') \;=\; \epsilon_i\,\bar{u}_i\,\partial_2[r_i]$$

and

$$\xi a - \xi b \;=\; \sum_{1 \le i \le n} (\xi a_{i-1} - \xi a_i) \;=\; \partial_2 \tau P, \tag{7}$$

for every connecting sequence P from a to b.

Lemma 1.2. *[9]* $\operatorname{Ker}\partial_1 = \operatorname{Im}\partial_2$.

Proof. By (3), (5), $\partial_1\,\partial_2[r] = \partial_1\,\xi r' - \partial_1\,\xi r'' = \bar{r}' - \bar{r}'' = 0$ for all $r \in R$; hence $\operatorname{Im}\partial_2 \subseteq \operatorname{Ker}\partial_1$.

To prove the converse we expand the partial contracting homotopy σ. For every $s \in S$ and $x \in X$ choose one arbitrary connecting sequence $P_{s,x}$ from $w_s s\,x$ to $w_{s\,\bar{x}}$. Let $\sigma_1 : M_1 \longrightarrow M_2$ be the additive homomorphism such that

$$\sigma_1\, s[x] \;=\; \tau P_{s,x} \tag{8}$$

for all $s \in S$ and $x \in X$. Then

$$\begin{aligned}
\partial_2\,\sigma_1\, s[x] + \sigma_0\,\partial_1\, s[x] \;&=\; \partial_2\,\tau P_{s,x} + \sigma_0(s\bar{x} - s) \quad \text{by (8), (1)} \\
&=\; \xi(w_s s\,x) - \xi w_{s\,\bar{x}} + \xi w_{s\,\bar{x}} - \xi w_s s \quad \text{by (7), (4)} \\
&=\; \xi(w_s s\,x) - \xi w_s s \;=\; \bar{w}_s s[x] \;=\; s[x], \quad \text{by (2),}
\end{aligned}$$

so that $\partial_2\,\sigma_1 + \sigma_0\,\partial_1$ is the identity on M_1. Hence $\operatorname{Ker}\partial_1 \subseteq \operatorname{Im}\partial_2$. $\square$

3. *Coterminal* connecting sequences are connecting sequences P, Q from the same $a \in F$ to the same $b \in F$. M_3 is the free S-module generated by all ordered pairs $[P, Q]$ of coterminal connecting sequences; ∂_3 is the module homomorphism such that

$$\partial_3[P, Q] \;=\; \tau Q - \tau P, \tag{9}$$

where τ is the trace map above. (One may further assume that the generators of M_3 satisfy $[P, P] = 0$ and $[Q, P] = -[P, Q]$.)

Lemma 1.3. $\operatorname{Ker}\partial_2 = \operatorname{Im}\partial_3$.

Proof. By (7), $\partial_2\,\tau P = \xi a - \xi b$ when P is a connecting sequence from a to b. If now P and Q are connecting sequences from a to b, then $\partial_2\,\partial_3[P, Q] = \partial_2(\tau Q - \tau P) \;=\; 0$; hence $\operatorname{Im}\partial_3 \subseteq \operatorname{Ker}\partial_2$.

For the converse inclusion we use some properties of the trace map τ. Combining a connecting sequence P from a to b with a connecting sequence Q from b to c yields a connecting sequence $P + Q$ from a to c;

if P consists of sequences $a_0, a_1, \dots, a_n, u_1, \dots, u_n, v_1, \dots, v_n \in F$ and a sequence $r_1, \dots, r_n \in R$, and Q consists of sequences $b_0, b_1, \dots, b_m$, $t_1, \dots, t_m, w_1, \dots, w_m \in F$ and a sequence $s_1, \dots, s_n \in R$, then $P+Q$ consists of the sequences $a_0, a_1, \dots, a_n = b_0, b_1, \dots, b_m$; $u_1, \dots, u_n, t_1, \dots, t_m$; $v_1, \dots, v_n, w_1, \dots, w_m$; and $r_1, \dots, r_n, s_1, \dots, s_n$. We see that

$$\tau\,(P+Q) \;=\; \tau P + \tau Q. \tag{10}$$

Reversing a connecting sequence P from a to b yields a connecting sequence $-P$ from b to a; if P consists of sequences $a_0, a_1, \dots, a_n, u_1, \dots, u_n, v_1, \dots, v_n \in F$ and $r_1, \dots, r_n \in R$, then $-P$ consists of $a_n, a_{n-1}, \dots, a_1, a_0, u_n, \dots, u_1, v_n, \dots, v_1$ and $r_n, \dots, r_1$. We see that

$$\tau\,(-P) \;=\; -\,\tau P. \tag{11}$$

When $c, d \in F$ and P is a connecting sequence from a to b, consisting of sequences $a_0, a_1, \dots, a_n, u_1, \dots, u_n, v_1, \dots, v_n \in F$ and a sequence $r_1, \dots, r_n \in R$, then cPd is a connecting sequence from cad to cbd, consisting of sequences $ca_0d, ca_1d, \dots, ca_nd, cu_1, \dots, cu_n, v_1d, \dots, v_nd$, and the same $r_1, \dots, r_n$. We see that

$$\tau\,(cPd) \;=\; \bar{c}\,\tau P. \tag{12}$$

For every $a = x_1\,x_2 \dots x_n \in F$, where $x_1, x_2, \dots, x_n \in X$, we have a connecting sequence

$$P_s a \;=\; P_{1,x_1}\,x_2 \dots x_n + P_{\bar{x}_1,x_2}\,x_3 \dots x_n + \dots + P_{x_1 \dots \bar{x}_{n-1},x_n} \tag{13}$$

from a to $w_{\bar{a}}$ obtained by combining the connecting sequences P_{1,x_1} from x_1 to $w_{\bar{x}_1}$, $P_{\bar{x}_1,x_2}$ from $w_{\bar{x}_1}\,x_2$ to $w_{x_1\bar{x}_2}$, $P_{x_1\bar{x}_2,x_3}$ from $w_{x_1\bar{x}_2}\,x_3$ to $w_{x_1\bar{x}_2x_3}$, $\dots$, $P_{x_1\dots\bar{x}_{n-1},x_n}$ from $w_{x_1\dots\bar{x}_{n-1}}\,x_n$ to $w_{x_1\bar{\dots}x_n}$. By the above,

$$\tau P_s a \;=\; \tau P_{1,x_1} + \tau P_{\bar{x}_1,x_2} + \dots + \tau P_{x_1\dots\bar{x}_{n-1},x_n} \;=\; \sigma_1\,\xi a. \tag{14}$$

For every $r \in R$ there is also a connecting sequence, which may be denoted by r, from r' to r'', with trace $\tau r = [r]$. This yields two connecting sequences $P_{r'}$ and $r + P_{r''}$ from r' to $w_{\bar{r}'} = w_{\bar{r}''}$. We can now extend our partial contracting homotopy with the module homomorphism $\sigma_2 : M_2 \longrightarrow M_3$ such that

$$\sigma_2[r] \;=\; [P_{r'}\,,\, r + P_{r''}]. \tag{15}$$

For every $r \in R$,

$$\partial_3\,\sigma_2[r] + \sigma_1\,\partial_2[r] \;=\; \tau r + \tau P_{r''} - \tau P_{r'} + \sigma_1\,\xi r' - \sigma_1\,\xi r'' \;=\; [r],$$

by (9), (5), and (14). Hence $\operatorname{Ker}\partial_2 \subseteq \operatorname{Im}\partial_3$. $\square$

Since M_0, M_1, M_2, M_3 are free $\mathbb{Z}[S]$-modules, Lemmas 1.1, 1.2, 1.3 yield

Theorem 1.4. *$M_3 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow \mathbb{Z}$ is a partial projective resolution of the trivial §-module $\mathbb{Z}$.*

2. Squier's resolution

1. In F, a *path* P from $a \in F$ to $b \in F$ consists of sequences $a_0, a_1, \dots, a_n, u_1, \dots, u_n, v_1, \dots, v_n$ of elements of F and a sequence $r_1, \dots, r_n$ of elements of R, such that $n \geq 0$, $a = a_0$, $a_n = b$, and, for every $1 \leq i \leq n$, $a_{i-1} = u_i\, r_i'\, v_i$ and $a_i = u_i\, r_i''\, v_i$ (always going from r' to r''); P is *trivial* if $n = 0$, *simple* if $n = 1$. We write $a \xrightarrow{P} b$ when P is a path from a to b, $a \longrightarrow b$ when there is a path from a to b (the latter is often written $a \xrightarrow{*} b$, with $a \longrightarrow b$ denoting a simple path). A connecting sequence from a to b (as in Section 1) can be decomposed into paths running in alternating directions:

$$a \longleftarrow \ . \longrightarrow \ . \longleftarrow \ . \longrightarrow \ . \longleftarrow \ . \longrightarrow \ b,$$

with the first and last paths possibly trivial.

A presentation, or its set R of defining relations, is *terminating* or *Noetherian* if there is no infinite sequence $. \longrightarrow . \longrightarrow \dots \longrightarrow . \longrightarrow \dots$ of simple paths; equivalently, no infinite sequence $. \longrightarrow . \longrightarrow \dots \longrightarrow . \longrightarrow \dots$ of nontrivial paths. In a terminating presentation, there is for every $a \in F$ a path $a \longrightarrow w$ in which w is *irreducible*, that is, there is no simple path $w \longrightarrow c$; equivalently, there is no equality $w = ur'v$ with $r \in R$.

A *wedge* is a pair of paths $a \longrightarrow b$ and $a \longrightarrow c$. A *diamond* is a quadruple of four paths $a \longrightarrow b \longrightarrow d$, $a \longrightarrow c \longrightarrow d$.

$$\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & \\
b \arrow[dr] & & c \arrow[dl] \\
 & d &
\end{tikzcd}$$

R is *confluent* if every wedge can be completed to a diamond: if for every paths $a \longrightarrow b$ and $a \longrightarrow c$ there exist paths $b \longrightarrow d$ and $c \longrightarrow d$. R is *complete* if it is terminating and confluent.

R is *Church-Rosser* if for every $a, b \in F$ such that $\bar{a} = \bar{b}$ there exist paths $a \longrightarrow c$, $b \longrightarrow c$. A terminating presentation is Church-Rosser if and only if it is complete, if and only if for every $a \in F$ there is a unique irreducible $w \in F$ with a path $a \longrightarrow w$.

We call R *semi-reduced* if $r' = us'v$ implies $r' = s'$ when $r, s \in R$; R is *reduced* if R is semi-reduced and r'' is irreducible for every $r \in R$ [9]. Every terminating Church-Rosser presentation is equivalent to a reduced terminating Church-Rosser presentation [9]. Squier's resolution assumes a reduced terminating Church-Rosser presentation, and can therefore be applied, after reduction, to any terminating Church-Rosser presentation.

2. Presentations with some of these properties are readily constructed from existing presentations by means of order relations on F. When $<$ is a strict order relation on F, a presentation is *ordered* (by $<$) if $r' > r''$ for every $r \in R$. If $<$ is *compatible* (if $a < b$ implies $uav < ubv$ for all $u, v \in F$), then paths are descending ($a \longrightarrow b$ implies $a \geq b$).

Proposition 2.1. *Every free monoid F has a compatible well order. Relative to any such:*

(a) every monoid presentation $R \subseteq F \times F$ is equivalent to an ordered presentation; every ordered presentation is terminating;

(b) every ordered monoid presentation is equivalent to a complete, ordered presentation;

(c) every complete, ordered monoid presentation is equivalent to a semi-reduced complete, ordered presentation;

(d) every semi-reduced, complete, ordered monoid presentation is equivalent to a reduced complete, ordered presentation.

Parts (c) and (d) are due to [9]. Part (a) has a converse of sorts: any terminating presentation is ordered for the corresponding order relation

$$a > b \text{ if and only if there exists a nontrivial path } a \longrightarrow b,$$

which is a compatible strict partial order relation on F; moreover, the descending chain condition holds in F.

Proof. The generating set X can be well ordered; words of fixed length can be well ordered lexicographically; then

$$a > b \text{ if and only if either } |a| > |b|, \text{ or } |a| = |b| \text{ and } a > b$$

(where $|a|$ denotes the length of a) is a compatible well order on F.

(a) Given $R \subseteq F \times F$, we can delete from R all trivial pairs (w, w), and replace $(r', r'') \in R$ by (r'', r') whenever $r' < r''$. This yields a presentation which is ordered and equivalent to R. Ordered presentations are terminating since the descending chain condition holds in F.

(b) Given an ordered presentation R, let $\bar{R}$ be the union of R and the set of all $r = (r', r'') \in F \times F$ such that $r' > r''$ and there exists a wedge $x \longrightarrow r'$, $x \longrightarrow r''$ which cannot be completed to a diamond. Since $x \longrightarrow r'$, $x \longrightarrow r''$ implies $\bar{x} = \bar{r}' = \bar{r}''$, R and $\bar{R}$ generate the same congruence and are equivalent. Moreover, $\bar{R}$ is ordered by definition, and every wedge $a \longrightarrow b$, $a \longrightarrow c$ which cannot be completed to a diamond in R yields some $r \in \bar{R}$ and can be completed to a diamond in $\bar{R}$: to $a \longrightarrow b \xrightarrow{1,(b,c),1} c$, $a \longrightarrow c \longrightarrow c$ if $b > c$, to $a \longrightarrow b \longrightarrow b$, $a \longrightarrow c \xrightarrow{1,(z,y),1} b$ if $b < c$.

Starting with $R_0 = R$, construct R_n by induction as $R_{n+1} = \bar{R}_n$, and let $R_\omega = \bigcup R_n$. The congruence generated by R contains R_1, R_2, ..., and R_ω; hence R and R_ω are equivalent. Moreover, R_ω is ordered, hence terminating, and a wedge of R_ω contains only finitely many edges, is a wedge of some R_n, and can be completed to a diamond of R_{n+1}; hence R_ω is complete.

(c),(d) Given a complete presentation R, the proof of Theorem 2.4 of [9] first constructs a semi-reduced complete presentation R' which is equivalent to R, by deleting some pairs from R (namely, all s such that $s' = ur'v$ for some $r \in R$ and $u, v \in F$ not both equal to 1). If R is ordered, then so is R'. From R' the second part of the proof then constructs a reduced complete presentation R'' which is equivalent to R', and hence to R, which consists of pairs (u, v) with a nontrivial path $u \longrightarrow v$ in R' (namely, all pairs (r', w) such that $r \in R'$, there is a nontrivial path $r' \longrightarrow w$, and w is irreducible); if R' is ordered then so is R''. □

Parts (a), (c), and (d) hold verbatim for finite presentations [9], but not part (b): it is not true that every finite ordered presentation is equivalent to a finite complete ordered presentation; [9] provides a counterexample.

3. We now let R be a semi-reduced complete presentation and obtain Squier's resolution

$$M_3' \xrightarrow{\partial_3} M_2 \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} M_0 \xrightarrow{\epsilon} \mathbb{Z}$$

from a suitable submodule M_3' of M_3.

An *essential* wedge is a wedge $a \xrightarrow{1,r,v} b$, $a \xrightarrow{u,s,1} c$, with $r, s \in \mathcal{R}$, $u, v \in F$, $r' = ut$, $s' = tv$ for some $t \in F$, $t \neq 1$, and $u, v \neq 1$ in case $r = s$; then $a = r'v = us' = utv$ is an *essential* word. For each essential wedge $a \longrightarrow b$, $a \longrightarrow c$ we choose any one diamond $a \longrightarrow b \longrightarrow d$, $a \longrightarrow c \longrightarrow d$ (for instance, with d irreducible, as in [9]). An *essential* diamond is one of these chosen diamonds. Thus every essential wedge can be completed to an essential diamond. The two paths $a \longrightarrow b \longrightarrow d$, $a \longrightarrow c \longrightarrow d$ in an essential diamond are an *essential* pair of paths. M_3' is the submodule of M_3 (freely) generated by all essential pairs $[P, Q]$ of paths. (As before one may further assume that $[P, P] = 0$ and $[Q, P] = -[P, Q]$.)

Lemma 2.2. *When R is a semi-reduced complete presentation, $\partial_3(M_3') = \operatorname{Ker} \partial_2$.*

Proof. By Lemma 1.3 it suffices to show that

(*) $\tau Q - \tau P \in \partial_3(M_3')$

for every coterminal paths P, Q. We note the following:

(a) (*) holds for every essential diamond, by definition of M_3'.

(b) if (*) holds for P and Q, then (*) holds for $T + P$ and $T + Q$, for $P + U$ and $Q + U$, and for uPv and uQv;

(c) c if in the diagram of paths

$$\begin{tikzcd}
& a \arrow[dl, "P"'] \arrow[dr, "Q"] & & \\
b \arrow[dr, "T"'] & & c \arrow[dl, "U"] \arrow[dr, "V"] & \\
& d \arrow[dr, "W"'] & & e \arrow[dl, "Y"] \\
& & f &
\end{tikzcd}$$

(*) holds for the two inner diamonds, then (*) holds for the outer diamond.

Claim 1. Every wedge $a \xrightarrow{p,r,q} b$, $a \xrightarrow{u,s,v} c$ of simple paths can be completed to a diamond for which (*) holds. We have $a = pr'q = us'v$, so that r' and s' are subwords of a. If $r = s$, then we may assume that $p \neq u$ and $q \neq v$, otherwise $b = c$ and adding trivial paths $a \xrightarrow{p,r,q} b \longrightarrow b$, $a \xrightarrow{u,s,v} b \longrightarrow b$ yields a diamond for which (*) holds.

If the subwords r' and s' do not overlap in a, say, $a = pr'ts'v$ for some $t \in F$, there is a diamond $a \xrightarrow{p,r,ts'v} b \xrightarrow{pr''t,s,v} d$, $a \xrightarrow{pr't,s,v} c \xrightarrow{p,r,ts''v} d$, where $d = pr''ts''v$, for which (*) holds: with $P = p\,r\,ts'v + pr''t\,s\,v$, $Q = pr't\,s\,v + p\,r\,ts''v$ we have

$$\tau Q - \tau P \;=\; \overline{pr'}t[s] + \bar{p}[r] - \bar{p}[r] - \overline{pr''}t[s] \;=\; 0$$

since $p\bar{r}'t = p\bar{r}''t$ in S.

Now assume that the subwords r' and s' overlap in a. Since R is semi-reduced, neither of r', s' is a proper subword of the other. Hence the end of one overlaps the beginning of the other. Let $t \neq 1$ be the common part. If the end of r' overlaps the beginning of s', then $r' = ht$, $s' = tk$, $u = ph$, $q = kv$, $a = phtkv$; if $r = s$, then $h, k \neq 1$, since in that case we may assume that $p \neq u$ and $q \neq v$. Then $htk \xrightarrow{1,r,k} r''k$, $htk \xrightarrow{h,s,1} hs''$ is an essential wedge, and can be completed to an essential diamond $htk \xrightarrow{1,r,k} r''k \xrightarrow{P''} d$, $htk \xrightarrow{h,s,1} hs'' \xrightarrow{Q''} d$. Hence the given wedge $a \xrightarrow{p,r,q} b$, $a \xrightarrow{u,s,v} c$, in which $q = kv$, $b = pr''kv$, $u = ph$, $c = phs''v$, and $a = phtkv$, can be completed to a diamond $phtkv \xrightarrow{p,r,kv} pr''kv \xrightarrow{pP''v} pdv$, $phtkv \xrightarrow{ph,s,v} phs''v \xrightarrow{pQ''v} pdv$ for which (*) holds by (b), since it holds for the essential diamond. The case where the end of s' overlaps the beginning of r' is similar.

Claim 2. (*) holds for all paths $a \xrightarrow{P} w$ and $a \xrightarrow{Q} w$ where w is irreducible. This is proved by artinian induction on a, the induction hypothesis being that (*) holds for all paths $b \longrightarrow w$ and $b \longrightarrow w$ where w is irreducible and b is lower than a (= there is a nontrivial path $a \longrightarrow b$). If P is trivial, then $a = w$, Q is trivial (otherwise there is an infinite sequence $a \xrightarrow{Q} a \xrightarrow{Q} a \dots$ of nontrivial paths), and $\tau Q - \tau P = 0 - 0 = 0$. Similarly (*) holds if Q is trivial. We may now assume that P and Q are nontrivial, and are the paths in a diamond $a \xrightarrow{p,r,q} b \xrightarrow{P'} w$, $a \xrightarrow{u,s,v} c \xrightarrow{Q'} w$ in which $a \longrightarrow b$, $a \longrightarrow c$ are simple paths and P', Q' are shorter that P and Q. By the above there is a diamond $a \xrightarrow{p,r,q} b \xrightarrow{P''} d$, $a \xrightarrow{u,s,v} c \xrightarrow{Q''} d$ for which (*) holds. Since R is complete there is a path $d \xrightarrow{W} w$: indeed there is a path $d \longrightarrow w'$ where w' is irreducible; this yields two paths $b \xrightarrow{P'} w$ and $b \xrightarrow{P''} d \xrightarrow{W} w'$, and implies $w' = w$. In the diagram

$$\begin{tikzcd}
 & a \arrow[dl, "r"'] \arrow[dr, "s"] & \\
b \arrow[d, "P'"'] \arrow[dr, "P''"] & & c \arrow[dl, "Q''"'] \arrow[d, "Q'"] \\
\omega \arrow[dr] & d \arrow[d, "W"] & \omega \arrow[dl] \\
 & \omega &
\end{tikzcd}$$

in which the two $w \longrightarrow w$ paths are trivial, (*) holds for every inner diamond, by the induction hypothesis; by (c), (*) holds for the given outer diamond.

With Claim 2 we now show that (*) holds for all coterminal connecting sequences $a \xrightarrow{P} b$ and $a \xrightarrow{Q} b$. Any connecting sequence $a \xrightarrow{P} b$ can be analyzed into a sequence

$$\begin{tikzcd}[column sep=small]
& c_1 \arrow[dl, "P1"'] \arrow[dr, "Q_1"] & & c_2 \arrow[dl, "P_2"'] \arrow[dr, "Q_2"] & & & & c_n \arrow[dl, "P_n"'] \arrow[dr, "Q_n"] & \\
a = a_0 & & a_1 & & a_2 & \cdots & a_{n-1} & & a_n = b
\end{tikzcd}$$

of possibly trivial paths whose directions alternate; that is,

$$P \;=\; -P_1 + Q_1 - P_2 + Q_2 \cdots - P_n + Q_n.$$

As above there are paths $a_i \xrightarrow{W_i} w$ to the same irreducible element w. By Claim 2, (*) holds for $c_i \xrightarrow{P_i} a_{i-1} \xrightarrow{W_{i-1}} w$ and $c_i \xrightarrow{Q_i} a_i \xrightarrow{W_i} w$. Hence

$$\tau Q_i + \tau W_i - \tau P_i - \tau W_{i-1} \in \partial_3(M_3').$$

Adding from $i = 1$ to $i = n$ yields

$$\tau P - \tau W_0 + \tau W_n \;=\; -\tau P_1 + \tau Q_1 \cdots - \tau P_n + \tau Q_n - \tau W_0 + \tau W_n \;\in\; \partial_3(M_3').$$

If now Q is another path from a to b, then $\tau Q - \tau W_0 + \tau W_n \in \partial_3(M_3')$; hence $\tau Q - \tau P \in \partial_3(M_3')$ and (*) holds for P and Q. $\square$

Since M_0, M_1, M_2, and M_3' are free $\mathbb{Z}[S]$-modules, Lemmas 1.1, 1.2, 2.2 yield

Theorem 2.3. *When R is a semi-reduced complete presentation, $M_3' \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow \mathbb{Z}$ is a partial projective resolution of the trivial §-module $\mathbb{Z}$.*

3. Cochains

1. Let $R \subseteq F \times F$ be a presentation of a monoid S and A be an §-module. For $n \leq 2$ a *minimal n-cochain* with values in A is an §-homomorphism $U : M_n \longrightarrow A$. (Minimal 2-cochains are functions of one variable and are called minimal 1-cochains in [2], [3], [4].) The coboundary of $U : M_n \longrightarrow A$ is $\delta U = U \circ \partial_{n+1}$. Under pointwise addition, minimal n-cochains, cocycles, and coboundaries constitute abelian groups $MC^n(R, A) \supseteq MZ^n(R, A) \supseteq MB^n(R, A)$. By Theorem 1.4,

$$H^n(S, A) \;\cong\; Ext^n_{\mathbb{Z}[S]}(\mathbb{Z}, A) \;\cong\; MZ^n(R, A)/\, MB^n(R, A),$$

by isomorphisms which are natural in A.

Since M_n is free, there is an isomorphism $U \longmapsto \widehat{U}$ which is natural in A and sends a mapping U of the set of generators of M_n to the minimal n-cochain $\widehat{U}$ with the same value on every generator.

This isomorphism $MC^0(R, A) \cong A$ takes $MZ^0(R, A)$ to $B = \{\, a \in A \bigm| sa = a \text{ for all } s \in S \,\}$, so that $H^0(S, A) \cong A/B$, as with the bar resolution.

For 1-cochains, recall that a crossed homomorphism is a mapping $U : S \longrightarrow A$ such that $U(st) = U(s) + sU(t)$ for all $s, t \in S$. For every $a \in A$, $a' : s \longmapsto sa - a$ is a crossed homomorphism.

Proposition 3.1. *Up to isomorphisms which are natural in A, a minimal 1-cochain is a mapping of X into A; a minimal 1-cocycle is a crossed homomorphism of S into A; and a minimal 1-coboundary is a crossed homomorphism of the form $s \longmapsto sa - a$ for some $a \in A$.*

Proof. A minimal 1-cochain is a homomorphism $\widehat{U} : M_1 \longrightarrow A$ induced by a mapping U of X into A, so that $\widehat{U}[x] = U(x)$. Then

$$\delta\widehat{U}[r] \;=\; \widehat{U}(\partial_2[r]) \;=\; \widehat{U}(\xi r' - \xi r'').$$

If $\hat{U}$ is a minimal 1-cocycle, then $\widehat{U}(\xi r') = \widehat{U}(\xi r'')$ for every $r \in R$. Since $\xi(pq) = \xi(p) + \bar{p}\xi(q)$ for all $p, q \in F$, it follows that $\widehat{U}(p) = \widehat{U}(q)$ whenever $p \sim q$. Hence $\widehat{U}$ induces a mapping $U' : S \longrightarrow A$, which is well defined by: $U'(\bar{p}) = \widehat{U}(\xi p)$ for all $a \in F$. Then U' is a crossed homomorphism, since $\xi(pq) = \xi(p) + \bar{p}\xi(q)$ for all $p, q \in F$, and uniquely determines $\widehat{U}$, by $\widehat{U}[x] = U'(\bar{x})$.

If $\widehat{U}$ is a coboundary, $\widehat{U} = \delta\hat{a}$ for some $a \in A$, then $\widehat{U}s[x] = \hat{a}\partial_1 s[x] = \hat{a}(s\bar{x} - s) = s\bar{x}a - sa$; hence

$$\begin{aligned}
\widehat{U}\xi\,(x_1 \dots x_n) \;&=\; \widehat{U}\Big(\sum_{1 \le i \le n} \overline{x_1 \dots x_{i-1}}[x_i]\Big) \\
&=\; \sum_{1 \le i \le n} \big(\overline{x_1 \dots x_{i-1}}\,\bar{x}_i\, a - \overline{x_1 \dots x_{i-1}}\, a\big) \;=\; \overline{x_1 \dots x_n}\, a - a
\end{aligned}$$

and $U'(s) = sa - a$ for all $s = \overline{x_1 \dots x_n} \in S$. □

A minimal 2-cochain is a homomorphism $\widehat{V} : M_2 \longrightarrow A$ induced by a mapping V of R into A, so that $\widehat{V}[r] = V(r)$. If P is a connecting sequence in F, consisting of sequences $a_0, a_1, \dots, a_n, u_1, \dots, u_n, v_1, \dots, v_n \in F$ and $r_1, \dots, r_n \in R$, let

$$V(P) \;=\; \widehat{V}(\tau P) \;=\; \sum_{1 \le i \le n} \epsilon_i\, \bar{u}_i\, V(r_i), \tag{16}$$

where $\epsilon_i = +1$ if $a_{i-1} = u_i r'_i v_i$, $a_i = u_i r''_i v_i$, $\epsilon_i = -1$ if $a_{i-1} = u_i r''_i v_i$, $a_i = u_i r'_i v_i$ ($\epsilon_i = +1$ for a path). V is *independent of path* if $V(P) = V(Q)$ whenever P and Q are coterminal.

Proposition 3.2. *Up to isomorphisms which are natural in A, a minimal 2-cochain is a mapping of R into A; a minimal 2-cocycle is a mapping of R into A which is independent of path; a minimal 2-coboundary is a mapping of R into A of the form $V(r) = \widehat{U}(\xi r') - \widehat{U}(\xi r'')$ for some mapping $U : X \longrightarrow A$.*

By (2),

$$\widehat{U}\xi(x_1 \dots x_n) \;=\; \sum_{1 \le i \le n} x_1\, x_2 \,\dots\, x_{i-1}\, U(x_i). \tag{17}$$

Very similar characterizations of minimal cocycles and coboundaries are given in [2] and [3].

Proof. When V is a minimal 2-cochain and P, Q are coterminal chains,

$$\delta\widehat{V}[P,Q] \;=\; \widehat{V}\partial_3[P,Q] \;=\; \widehat{V}(\tau Q - \tau P) \;=\; V(Q) - V(P),$$

by (16); therefore $\widehat{V}$ is a minimal 2-cocycle if and only if V is independent of path. If furthermore $\widehat{U}$ is a minimal 2-cochain, then

$$\delta\widehat{U}[r] \;=\; \widehat{U}\partial_2[r] \;=\; \widehat{U}(\xi r' - \xi r'') \;=\; \widehat{U}(\xi r') - \widehat{U}(\xi r'').$$

□

Proposition 3.3. *If R is semi-reduced and complete, then V is independent of path if and only if $V(P) = V(Q)$ whenever P, Q is an essential pair of paths.*

Proof. By Lemma 2.2, $\widehat{V}\partial_3[P,Q] = 0$ for all coterminal pairs $[P,Q]$ (generators of M_3) if and only if $\widehat{V}\partial_3[P,Q] = 0$ for all essential pairs $[P,Q]$ (generators of M'_3). □

If R is finite in Proposition 3.3, then there are only finitely many essential words, essential diamonds, and essential pairs, so that minimal 2-cocycles are characterized by finitely many conditions. It is easy to devise an algorithm which yields all these conditions.

4. Examples

1. **Example 1** is the finite cyclic monoid $C = \langle c \mid c^m = c^{m+n} \rangle$ of index $m \geq 0$ and period $d > 0$; for instance, a cyclic group of order d. (Infinite cyclic monoids and groups have $H^2(C, A) = 0$ for all A.)

In this example F is free on $X = \{c\}$, with $\bar{c} = c$, and $\mathcal{R} = \{r\}$, where $r = (c^{m+d}, c^m)$. A minimal 2-cochain $\widehat{V} \in MC^1(F, A)$ may be identified with $V(r)$; this identifies $MC^2(R, A)$ with A.

If $p \longrightarrow q$ and $p \longrightarrow t$ are simple paths, then $p = c^n$, where $n \geq m + d$, and $q = t = c^{n-d}$. Therefore we have a complete and reduced presentation.

Let $V = V(r) \in A$. An essential word $p = utv$ has $u = c^i$, $t = c^j \neq 1$, $v = c^k$, $r' = ut$, $s' = tv$, and $u, v \neq 1$ if $r = s$; hence $i, j, k > 0$, $i + j = j + k = m + d$, and $i = k = m + d - j$, where $0 < i, j, k < m + d$. An essential diamond consists of an essential wedge $c^{i+m+d} \xrightarrow{1,\, r,\, c^i} c^{i+m}$ and $c^{i+m+d} \xrightarrow{c^i,\, r,\, 1} c^{i+m}$, followed by paths $c^{i+m} \xrightarrow{R} d$ and $c^{i+m} \xrightarrow{S} d$. We may choose $d = c^{i+m}$ and trivial paths $c^{i+m} \xrightarrow{R} d$ and $c^{i+m} \xrightarrow{S} d$; then the paths in the diamond are rc^i and $c^i r$. Hence V is a cocycle if and only if $V(r) = \bar{c}^i V(r) = c^i\, V(r)$ for all $0 < i < m + d$, if and only if $V = cV$. Thus

$$MZ^2(R, A) \;=\; A_c \;=\; \{a \in A \mid ca = a\}.$$

If C acts trivially on A (if $ga = a$ for all $g \in G$ and $a \in A$), then $MZ^2(R, A) \;=\; A_c \;=\; A$.

Minimal 1-cochains may also be identified with elements of A. If $a = U(c) \in A$, then $\widehat{U}(\xi c^k) = \sum_{1 \leq i \leq k} \bar{c}^{i-1}\, a = \sum_{1 \leq i \leq k} c^{i-1}\, a$ by (17) and

$$\begin{aligned}(\delta a)(r) \;&=\; \bar{a}(r') - \bar{a}(r'') \;=\; \sum_{1 \leq i \leq m+d} c^{i-1}\, a - \sum_{1 \leq i \leq m} c^{i-1}\, a \\ &=\; \sum_{m+1 \leq i \leq m+d} c^{i-1}\, a \;=\; c^m \sum_{1 \leq i \leq d} c^{i-1}\, a.\end{aligned}$$

Hence

$$MB^2(R, A) \;=\; B \;=\; \{c^m \sum_{1 \leq i \leq d} c^{i-1}\, a \mid a \in A\}.$$

If C acts trivially on A, then $B = dA = \{da \mid a \in A\}$. This yields the well known result:

Proposition 4.1. *When $C = \langle c \mid c^m = c^{m+d} \rangle$ is a finite cyclic monoid, or C is a cyclic group of order d, then*

$$H^2(G, A) \cong A_c / B,$$

where $A_c = \{a \in A \mid ca = a\}$ *and* $B = \{c^m \sum_{1 \le i \le d} c^{i-1}\, a \mid a \in A\}$. *If* C *acts trivially on* A, *then* $H^2(C, A) \cong A/dA$.

When C is a cyclic group, comparable results hold in higher dimensions [6], [1].

2. **Example 2** is the bicyclic semigroup, which has the monoid presentation $G = \langle p, q \mid pq = 1\rangle$.

F is free on $X = \{p, q\}$, with $\bar{p} = p$, $\bar{q} = q$, and $\mathcal{R} = \{r\}$, where $r = (pq,\, 1)$; $MC^2(R, A)$ may be identified with A.

Since pq cannot overlap itself, there are no essential words and no essential diamonds. R is a complete, reduced presentation, and every minimal 2-cochain is a minimal 2-cocycle.

When $U : \{p, q\} \longrightarrow A$,

$$\delta\widehat{U}[r] \;=\; \widehat{U}\xi(pq) - \widehat{U}\xi(1) \;=\; U(p) + p\,U(q)$$

by (18) and (17); if we identify $MC^2(R, A)$ with A, then $MB^2(R, A) = A + pA = A$. Hence:

Proposition 4.2. *When* S *is the bicyclic semigroup,* $H^2(S, A) = 0$ *for all* A.

3. **Example 3** is the free commutative monoid $S = \langle c, d \mid cd = dc\rangle$.

F is free on $X = \{c, d\}$, with $\bar{c} = c$, $\bar{d} = d$; we let $\mathcal{R} = \{r\}$, where $r = (dc,\, cd)$; $MC^2(R, A)$ may be identified with A.

R is complete, with every path ending when all c's precede all d's. Since dc cannot overlap itself, there are no essential words and no essential diamonds; R is reduced, and every minimal 2-cochain is a minimal 2-cocycle.

When $U : \{c, d\} \longrightarrow A$,

$$\delta\widehat{U}[r] \;=\; \widehat{U}\xi(dc) - \widehat{U}\xi(cd) \;=\; U(d) + d\,U(c) - U(c) - c\,U(d);$$

if we identify $MC^2(R, A)$ with A, then $MB^2(R, A) = A(c) + A(d)$, where $A(c) = \{a - ca \mid a \in A\}$ and $A(d) = \{a - da \mid a \in A\}$. If S acts trivially on A, then $A(c) = A(d) = 0$. We have proved:

Proposition 4.3. *When* S *is the free commutative monoid* $S = \langle c, d \mid cd = dc\rangle$, *then*

$$H^2(S, A) \cong A \,/\, \big(A(c) + A(d)\big),$$

where $A(c) = \{a - ca \mid a \in A\}$ *and* $A(d) = \{a - da \mid a \in A\}$. *If* S *acts trivially on* A, *then* $H^2(S, A) \cong A$.

On the other hand, in the commutative cohomology, $H^n(S,A) = 0$ for every abelian group valued functor A and $n \geq 2$ [2].

4. **Example 4** is the monoid freely generated by two idempotents, $S = \langle e, f \mid e^2 = e,\ f^2 = f \rangle$.

F is free on $S = \{e, f\}$, with $\bar{e} = e$, $\bar{f} = f$, and $\mathcal{R} = \{r, s\}$, where $r = (ee,\ e)$ and $s = (ff,\ f)$. Minimal 2-cochains have two values $M(r)$ and $M(s)$, and $MC^2(F,A) \cong A \oplus A$.

Since ee and ff can only overlap themselves, there are two essential words eee and fff. This yields two essential wedges, four essential diamonds, and four essential pairs of paths:

$$\begin{gathered}
eee \xrightarrow{1,r,e} ee \xrightarrow{1,r,1} e, \quad eee \xrightarrow{e,r,1} ee \xrightarrow{1,r,1} e; \\
eee \xrightarrow{1,r,e} ee \longrightarrow ee, \quad eee \xrightarrow{e,r,1} ee \longrightarrow ee; \\
fff \xrightarrow{1,s,f} ff \xrightarrow{1,s,1} f, \quad fff \xrightarrow{f,s,1} ff \xrightarrow{1,r,1} f; \\
fff \xrightarrow{1,s,f} ff \longrightarrow ff, \quad fff \xrightarrow{f,s,1} ff \longrightarrow ff.
\end{gathered}$$

In particular, R is complete. By 3.3, a minimal 2-cochain V is a minimal 2-cocycle if and only if it $V(P) = V(Q)$ for every essential pair $[P,Q]$. This yields four conditions:

$$\begin{aligned}
V(r) + V(r) &= e\,V(r) + V(r), \\
V(r) &= e\,V(r), \\
V(s) + V(s) &= f\,V(s) + V(s), \\
V(s) &= f\,V(s);
\end{aligned}$$

equivalently, $V(r) = e\,V(r)$ and $V(s) = f\,V(s)$. If we identify $,C^1(F,A)$ with $A \oplus A$, then $MZ^1(F,A) = A_e \oplus A_f$, where $A_e = \{a \in A \mid ea = a\}$, $A_f = \{a \in A \mid fa = a\}$.

When $U : \{e, f\} \longrightarrow A$,

$$\delta\widehat{U}[r] \;=\; \widehat{U}\xi(ee) - \widehat{U}\xi(e) \;=\; U(e) + e\,U(e) - U(e) \;=\; e\,U(e)$$

and $\delta\widehat{U}[s] = f\,U(f)$. If we identify $MC^2(R,A)$ with $A \oplus A$, then

$$MZ^1(R,A) = eA \oplus fA.$$

Since $e^2 = e$ holds in S, $eea = ea$ for all $a \in A$, and $A_e = eA$; similarly $A_f = fA$. Hence $MZ^2(R,A) = MB^2(R,A)$. Thus:

Proposition 4.4. *When $S = \langle e, f \mid e^2 = e,\ f^2 = f \rangle$, then $H^2(S,A) = 0$ for all A.*

5. **Example 5** is the free Burnside monoid $S = M_{2,1,1}$, that is, the free band-with-identity-element on two generators e and f. The multiplication table of S is:

1	e	f	ef	fe	efe	fef
e	e	ef	ef	efe	efe	ef
f	fe	f	fef	fe	fe	fef
ef	efe	ef	ef	efe	efe	ef
fe	fe	fef	fef	fe	fe	fef
efe	efe	ef	ef	efe	efe	ef
fef	fe	fef	fef	fe	fe	fef

S has a finite presentation $S = \langle e, f \mid ee = e,\ ff = f,\ efef = ef,\ fefe = fe\rangle$, which we show is complete.

As in Example 4, F is free on $X = \{e, f\}$, with $\bar{e} = e$, $\bar{f} = f$. Then $\mathcal{R} = \{r, s, t, u\}$, where $r = (ee,\ e)$, $s = (efef,\ ef)$, $t = (fefe,\ fe)$, and $u = (ff,\ f)$ (arranged so the left hand sides are in alphabetic order). Minimal 2-cochains have four values and $MC^2(F, A) \cong A \oplus A \oplus A \oplus A$.

Overlaps in the left hand sides yield 12 essential words:

	ee	$efef$	$fefe$	ff
ee	eee	$eefef$	–	–
$efef$	–	$efefef$	$efefe, efefefe$	$efeff$
$fefe$	$fefee$	$fefef, fefefef$	$fefefe$	–
ff	–	–	$ffefe$	fff

This yields all essential diamonds:

1. $eee:$ $\quad eee \xrightarrow{1,r,e} ee \xrightarrow{1,r,1} e,\qquad eee \xrightarrow{e,r,1} ee \xrightarrow{1,r,1} e.$
2. $eefef:$ $\quad eefef \xrightarrow{1,r,fef} efef \xrightarrow{1,s,1} ef,\qquad eefef \xrightarrow{e,s,1} eef \xrightarrow{1,r,f} ef.$
3. $efefef:$ $\quad efefef \xrightarrow{1,s,ef} efef \xrightarrow{1,s,1} ef,\qquad efefef \xrightarrow{ef,s,1} efef \xrightarrow{1,s,1} ef.$
4. $efefe:$ $\quad efefe \xrightarrow{1,s,e} efe \longrightarrow efe,\qquad efefe \xrightarrow{e,t,1} efe \longrightarrow efe.$
5. $efefefe:$ $\quad efefefe \xrightarrow{1,s,efe} efefe \longrightarrow efe,\qquad efefefe \xrightarrow{efe,t,1} efefe \longrightarrow efe.$
6. $efeff:$ $\quad efeff \xrightarrow{1,s,f} eff \xrightarrow{e,u,1} ef,\qquad efeff \xrightarrow{efe,u,1} efef \xrightarrow{1,s,1} ef.$
7. $fefee:$ $\quad fefee \xrightarrow{1,t,e} fee \xrightarrow{f,r,1} fe,\qquad fefee \xrightarrow{fef,r,1} fefe \xrightarrow{1,t,1} fe.$
8. $fefef:$ $\quad fefef \xrightarrow{1,t,f} fef \longrightarrow fef,\qquad fefef \xrightarrow{f,s,1} fef \longrightarrow fef.$
9. $fefefef:$ $\quad fefefef \xrightarrow{1,t,fef} fefef \longrightarrow fef,\qquad fefefef \xrightarrow{fef,s,1} fefef \longrightarrow fef.$
10. $fefefe:$ $\quad fefefe \xrightarrow{1,t,fe} fefe \xrightarrow{1,t,1} fe,\qquad fefefe \xrightarrow{fe,t,1} fefe \xrightarrow{1,t,1} fe.$
11. $ffefe:$ $\quad ffefe \xrightarrow{1,u,efe} fefe \xrightarrow{1,t,1} fe,\qquad ffefe \xrightarrow{f,t,1} ffe \xrightarrow{1,u,e} fe.$
12. $fff:$ $\quad fff \xrightarrow{1,u,f} ff \xrightarrow{1,u,1} f,\qquad fff \xrightarrow{f,u,1} ff \xrightarrow{1,u,1} f$

In this list, $efefe \longrightarrow efe$ stands for either of the two paths in the $efefe$ entry, and similarly for $fefef \longrightarrow fef$. In particular, every essential wedge can be completed to a diamond. Since R is reduced, the proof of 2.3 shows that every wedge can be completed to a diamond, and R is confluent. R is also terminating, since length decreases along every path.

The list above yields 12 equalities which by 3.3 characterize minimal 2-cocycles:

(1) $V(r) + V(r) = e\,V(r) + V(r)$;

(2) $V(r) + V(s) = e\,V(s) + V(r)$;

(3) $V(s) + V(s) = ef\,V(s) + V(s)$;

(4) $V(s) = e\,V(t)$;

(5) $V(s) + V(s) = efe\,V(t) + V(s)$;

(6) $V(s) + e\,V(u) = efe\,V(u) + V(s)$;

(7) $V(t) + f\,V(r) = fef\,V(r) + V(t)$;

(8) $V(t) = f\,V(s)$;

(9) $V(t) + V(t) = fef\,V(s) + V(t)$;

(10) $V(t) + V(t) = fe\,V(t) + V(t)$;

(11) $V(u) + V(t) = f\,V(t) + V(u)$;

(12) $V(u) + V(u) = f\,V(u) + V(u)$;

where $efefe \xrightarrow{1,s,e} efe$, $fefef \xrightarrow{1,t,f} fef$.

Equation (8) reads

(T) $V(t) = f\,V(s)$

and shows that a minimal 1-cocycle is determined by its values on r, s, and u. The latter satisfy

(A) $V(r) = e\,V(r)$ and $f\,V(r) = fef\,V(r)$,

(B) $V(s) = ef\,V(s)$,

(C) $V(u) = f\,V(u)$ and $e\,V(u) = efe\,V(u)$

by (1) and (7), (2), (12) and (6) respectively. Conversely, (B) implies $e\,V(s) = V(s)$. since $\pi(eef) = \pi(ef)$, and it is immediate that (1) through (12) follow from (A),(B),(C), and (T). Hence

$$MZ^1(F,A) \;\cong\; A_r \oplus A_s \oplus A_u,$$

where $A_r = \{a \in A \mid a = ea,\ fa = fefa\}$, $A_s = \{a \in A \mid a = efa\}$, $A_u = \{a \in A \mid a = fa,\ ea = efea\}$; the isomorphism takes $V \in MZ^1(F,A)$ to $\big(V(r),\, V(s),\, V(u)\big)$. If G acts trivially on A, then

$$MZ^1(F,A) \;\cong\; Z \;=\; A \oplus A \oplus A.$$

When $U : \{e, f\} \longrightarrow A$,

$$\begin{aligned}
\delta\widehat{U}[r] &= \widehat{U}\xi(ee) - \widehat{U}\xi(e) \;=\; U(e) + e\,U(e) - U(e) \;=\; e\,U(e),\\
\delta\widehat{U}[s] &= \widehat{U}\xi(efef) - \widehat{U}\xi(ef)\\
&= U(e) + e\,U(f) + ef\,U(e) + efe\,U(f) - U(e) - e\,U(f)\\
&= ef\,U(e) + efe\,U(f),\\
\delta\widehat{U}[u] &= \widehat{U}\xi(ff) - \widehat{U}\xi(f) \;=\; U(f) + f\,U(f) - U(f) \;=\; f\,U(f);
\end{aligned}$$

then $\delta\widehat{U}$ is given by (T), since minimal 2-coboundaries are minimal 2-cocycles. Hence the isomorphism $MZ^2(R, A) \longrightarrow Z$ sends $MB^2(R, A)$ to

$$B \;=\; \{(ea,\ efa + efeb,\ fb) \mid a, b \in A\}.$$

If S acts trivially on A, then $B = \{(a,\ a + b,\ b) \mid a, b \in A\}$.

We note that $A_r = eA$: indeed $b = ea$ implies $b = eb$ and $fb = fefb$, since $\pi\,(ee) = \pi\,(e)$ and $\pi\,(fefe) = \pi\,(fe)$. Similarly $A_u = fA$ and $A_s = efA$; hence $Z = eA \oplus efA \oplus fA$. To find Z/B we note use the isomorphism

$$\theta : A \oplus A \oplus A \longrightarrow A \oplus A \oplus A,\ (a, b, c) \longmapsto (a,\ b - efea - efefc,\ c).$$

Then $\theta(Z) = Z$, since $b \in A_s$ $(b = efb)$ implies $b - efea - efefc \in A_s$. But $\theta\,(ea,\ efa + efeb,\ fb) = (ea,\ 0,\ fb)$, so that $\theta(B) = eA \oplus 0 \oplus fA$. Hence:

Proposition 4.5. *When S is the free band-with-identity-element with two generators e and f, $H^2(S, A) \cong efA$. If S acts trivially on A, then $H^2(S, A) \cong A$.*

Note that $efA \cong feA$: $x \longmapsto fx$ and $y \longmapsto ey$ provide mutually inverse isomorphisms.

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Contact information

P. A. Grillet Tulane University, New Orleans, La., U.S.A.

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