Algebra and Discrete Mathematics
Number 4. **(2004).** pp. 1 – 11

RESEARCH ARTICLE

Clones of full terms

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Communicated by V. I. Sushchansky

Abstract. In this paper the well-known connection between hyperidentities of an algebra and identities satisfied by the clone of this algebra is studied in a restricted setting, that of n-ary full hyperidentities and identities of the n-ary clone of term operations which are induced by full terms. We prove that the n-ary full terms form an algebraic structure which is called a Menger algebra of rank n. For a variety V, the set $Id_n^F V$ of all its identities built up by full n-ary terms forms a congruence relation on that Menger algebra. If $Id_n^F V$ is closed under all full hypersubstitutions, then the variety V is called $n-F-$solid. We will give a characterization of such varieties and apply the results to $2-F-$solid varieties of commutative groupoids.

1. Full terms

Here we consider algebras of n-ary type, that is, all operation symbols have the same fixed arity n. Let τ_n be such a fixed n-ary type with operation symbols $(f_i)_{i\in I}$ indexed by some set I. Let $X_n = \{x_1, \ldots, x_n\}$ and let $X = \{x_1, \ldots, x_n, \ldots\}$ be a countably infinite set of variables. Then $W_{\tau_n}(X_n)$ is the set of all n-ary terms of type τ_n. Together with n-ary operations $\overline{f}_i$ defined by

$$\overline{f}_i : W_{\tau_n}(X_n)^n \longrightarrow W_{\tau_n}(X_n) \text{ with}$$
$$(t_1, \ldots, t_n) \mapsto \overline{f}_i(t_1, \ldots, t_n) := f(t_1, \ldots, t_n)$$

Research of the second author supported by the Ministry of University Affairs, Thailand

2000 Mathematics Subject Classification: *08A40, 08A60, 08A02, 20M35.*

Key words and phrases: *Clone, unitary Menger algebra of type τ_n, full hyperidentity, n-F-solid variety .*

$W_{\tau_n}(X_n)$ forms the absolutely free algebra

$$\mathcal{F}_{\tau_n}(X_n) := (W_{\tau_n}(X_n); (\overline{f}_i)_{i\in I})$$

of type τ_n. There is another possibility to define an operation on the set $W_{\tau_n}(X_n)$, namely by
$S^n(x_j, t_1, \ldots, t_n) := t_j$ for $1 \leq j \leq n$ and
$S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n) \quad := \quad f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, , t_1, \ldots, t_n))$.

We consider a subset of $W_{\tau_n}(X_n)$, the set of all full terms. Let H_n be the set of all mappings $s : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$. Full terms of type τ_n are inductively defined by:

Definition 1. *(i) Let $s \in H_n$ be an arbitrary function and let f_i be an operation symbol of type τ_n. Then $f_i(x_{s(1)}, \ldots, x_{s(n)})$ is a full term of type τ_n.*

(ii) If $t_1, \ldots, t_n$ are full terms of type τ_n, then $f_i(t_1, \ldots, t_n)$ is a full term of type τ_n.

Let $W^F_{\tau_n}(X_n)$ be the set of all n-ary full terms of type τ_n. By definition, the set $W^F_{\tau_n}(X_n)$ is closed under the operations $\overline{f}_i$. Therefore $(W^F_{\tau_n}(X_n); (\overline{f}_i)_{i\in I})$ is a subalgebra of $\mathcal{F}_{\tau_n}(X_n)$. Clearly, the restriction of S^n to $W^F_{\tau_n}(X_n)$ is an operation on this set. Therefore we define a superposition operation S^n on $W^F_{\tau_n}(X_n)$ by:

Definition 2. *(i)* $S^n(f_i(x_{s(1)}, \ldots, x_{s(n)}), t_1, \ldots, t_n)$
$:= f_i(t_{s(1)}, \ldots, t_{s(n)})$,

(ii) $S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n)$
$:= f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, , t_1, \ldots, t_n))$.

Now we consider the algebra $\text{clone}_F\tau_n := (W^F_{\tau_n}(X_n); S^n)$ of type $n+1$. Then we have:

Proposition 1. *The algebra $\text{clone}_F\tau_n$ satisfies the so-called superassociative law*

(C) $\widetilde{S}^n(X_0, \widetilde{S}^n(Y_1, X_1, \ldots, X_n), \ldots, \widetilde{S}^n(Y_n, X_1, \ldots, X_n))$
$\approx \widetilde{S}^n(\widetilde{S}^n(X_0, Y_1, \ldots, Y_n), X_1, \ldots, X_n)$, *where $\widetilde{S}^n$ is an $(n+1)$-ary operation symbol and X_i, Y_j are variables.*

Proof. We give a proof by induction on the complexity of the full term which is substituted for X_0.

Substituting for X_0 a term of the form $f_i(x_{s(1)}, \ldots, x_{s(n)})$ for a function $s \in H_n$, then

$$\begin{aligned} & S^n(f_i(x_{s(1)}, \ldots, x_{s(n)}), S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\ & \quad = f_i(S^n(t_{s(1)}, s_1, \ldots, s_n), \ldots, S^n(t_{s(n)}, s_1, \ldots, s_n)) \\ & \quad = S^n(f_i(t_{s(1)}, \ldots, t_{s(n)}), s_1, \ldots, s_n) \\ & \quad = S^n(S^n(f_i(x_{s(1)}, \ldots, x_{s(n)}), t_1, \ldots, t_n), s_1, \ldots, s_n) \text{ by Definition 2.} \end{aligned}$$

If we substitute for X_0 a term $t = f_i(r_1, \ldots, r_n)$ and assume that (C) is satisfied for $r_1, \ldots, r_n$, then

$$\begin{aligned} & S^n(f_i(r_1, \ldots, r_n), S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\ & \quad = f_i(S^n(r_1, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)), \ldots, \\ & \qquad S^n(r_n, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n))) \\ & \quad = S^n(f_i(S^n(r_1, t_1, \ldots, t_n), \ldots, S^n(r_n, t_1, \ldots, t_n)), s_1, \ldots, s_n) \\ & \quad = S^n(S^n(f_i(r_1, \ldots, r_n), t_1, \ldots, t_n), s_1, \ldots, s_n). \end{aligned}$$

□

The algebra $clone_F \tau_n$ is generated by the set

$$F_{s\tau_n} := \{f_i(x_{s(1)}, \ldots, x_{s(n)}) \mid i \in I,\ s \in H_n\}$$

of so-called "fundamental terms".
The algebra $clone_F \tau_n$ is an example of a Menger algebra of rank n.

Definition 3. *([3]) An algebra $\mathcal{M} = (M;\ S^n)$ of type $\tau = (n+1)$ is called a Menger algebra of rank n if it satisfies the axiom (C).*

Let $V_{M\_}$ be the variety of all algebras satisfying (C) and let $\mathcal{F}_{V_{M\_}}(Y)$ be the free algebra with respect to $V_{M\_}$, freely generated by $Y = \{y_j \mid j \in J\}$, where Y is a new alphabet of individual variables indexed by the index set $J = \{(i, s) \mid i \in I, s \in H_n\}$. The operation of $\mathcal{F}_{V_{M\_}}(Y)$ is denoted by $\widetilde{S}^n$. Then we can prove:

Theorem 1. *The algebra $clone_F \tau_n$ is free with respect to the variety $V_{M\_}$ of Menger algebras of rank n, freely generated by the set Y.*

Proof. We prove that $clone_F \tau_n$ is isomorphic to $\mathcal{F}_{V_{M\_}}(Y)$ under the mapping $\varphi : W^F_{\tau_n}(X_n) \longrightarrow \mathcal{F}_{V_{M\_}}(Y)$, inductively defined by

(i) $\varphi(f_i(x_{s(1)}, \ldots, x_{s(n)})) = y_{(i,s)}$,

(ii) $\varphi(f_i(t_{s(1)}, \ldots, t_{s(n)})) = \widetilde{S}^n(y_{(i,s)}, \varphi(t_1), \ldots, \varphi(t_n))$
where $i \in I$ and $s \in H_n$.

The homomorphism property can be proved by induction on the complexity of the term t_0. If $t_0 = f_i(x_{s(1)}, \ldots, x_{s(n)})$ for some i and some mapping $s \in H_n$, then

$$\begin{aligned}\varphi(S^n(f_i(x_{s(1)},\ldots,x_{s(n)}),t_1,\ldots,t_n)) &= \varphi(f_i(t_{s(1)},\ldots,t_{s(n)})\\ &= \widetilde{S}^n(y_{(i,s)},\varphi(t_1),\ldots,\varphi(t_n))\\ &= \widetilde{S}^n(\varphi(f_i(x_{s(1)},\ldots,x_{s(n)})),\\ &\quad \varphi(t_1),\ldots,\varphi(t_n)).\end{aligned}$$

Inductively, assume that $t_0 = f_i(r_1,\ldots,r_n)$ and that

$$\varphi(S^n(r_j,t_1,\ldots,t_n)) = \widetilde{S}^n(\varphi(r_j),\varphi(t_1),\ldots,\varphi(t_n))$$

for all $1 \leq j \leq n$. Then

$$\begin{aligned}&\varphi(S^n(f_i(r_1,\ldots,r_n),t_1,\ldots,t_n))\\ &\quad= \varphi(f_i(S^n(r_1,t_1,\ldots,t_n),S^n(r_2,t_1,\ldots,t_n),\ldots,\\ &\qquad S^n(r_n,t_1,\ldots,t_n)))\\ &\quad= \widetilde{S}^n(y_{(i,id)},\varphi(S^n(r_1,t_1,\ldots,t_n)),\varphi(S^n(r_2,t_1,\ldots,t_n)),\ldots,\\ &\qquad \varphi(S^n(r_n,t_1,\ldots,t_n)))\\ &\quad= \widetilde{S}^n(y_{(i,id)},\widetilde{S}^n(\varphi(r_1),\varphi(t_1),\ldots,\varphi(t_n)),\ldots,\\ &\qquad \widetilde{S}^n(\varphi(r_n),\varphi(t_1),\ldots,\varphi(t_n)))\\ &\quad= \widetilde{S}^n(\widetilde{S}^n(y_{(i,id)},\varphi(r_1),\ldots,\varphi(r_n)),\varphi(t_1),\ldots,\varphi(t_n)))\\ &\quad= \widetilde{S}^n(\varphi(f_i(r_1,\ldots,r_n),\varphi(t_1),\ldots,\varphi(t_n))).\end{aligned}$$

This shows that φ is a homomorphism. The mapping φ is bijective since $\{y_{(i,s)} \mid i \in I, s \in H_n\}$ is a free independent set. Therefore we have $y_{(i,s_1)} = y_{(j,s_2)} \Rightarrow (i,s_1) = (j,s_2) \Rightarrow i = j, s_1 = s_2$
$\Rightarrow f_i(x_{s_1(1)},\ldots,x_{s_1(n)}) = f_j(x_{s_2(1)},\ldots,x_{s_2(n)})$.

Thus φ is bijective on the generating sets of both algebras and therefore φ is an isomorphism. □

In [2] strongly full terms of type τ_n were defined in the following way:

(i) $f_i(x_1,\ldots,x_n)$ is strongly full for every $i \in I$,

(ii) if $t_1,\ldots,t_n$ are strongly full, then $f_i(t_1,\ldots,t_n)$ is strongly full.

Let $W^{SF}_{\tau_n}(X_n)$ be the set of all strongly full terms of type τ_n. That means, we obtain strongly full terms by Definition 1 if we allow for s only the identity function. Since S^n is closed on $W^{SF}_{\tau_n}(X_n)$ we obtain an algebra $\text{clone}_{SF}\tau_n := (W^{SF}_{\tau_n}(X_n); S^n)$.

It is clear that full terms can be expressed as strongly full terms if we change the type from τ_n to τ_n^*. The operation symbols of the new type τ_n^* are all n-ary and indexed by a set J which has the same cardinality as $\{(i,s) \mid i \in I, s \in H_n\}$. As a result we obtain that $\text{clone}_F\tau_n$ is isomorphic to $\text{clone}_{SF}\tau_n^*$.

2. Full hypersubstitutions and substitutions of $\mathbf{clone}_F\tau_n$

For a full term t we need the full term t_s arising from t if we map all variables corresponding to a mapping $s \in H_n$. This can be defined inductively by the following steps:

(i) If $t = f_i(x_{r(1)}, \ldots, x_{r(n)})$ for $i \in I, r \in H_n$,
then $t_s = f_i(x_{s(r(1))}, \ldots, x_{s(r(n))})$.

(ii) If $t = f_i(t_1, \ldots, t_n)$, then $t_s = f_i((t_1)_s, \ldots, (t_n)_s)$.

It is clear that t_s is a full term for any full term t and $s \in H_n$.

Hypersubstitutions are important to describe hyperidentities and solid varieties. We restrict this concept to full hypersubstitutions.

Definition 4. *A full hypersubstitution σ of type τ_n is a mapping*

$$\sigma : \{f_i \mid i \in I\} \longrightarrow W^F_{\tau_n}(X_n).$$

Note that hypersubstitutions can be defined for arbitrary terms. Every full hypersubstitution σ can be extended to a mapping $\hat{\sigma}$ defined on $W^F_{\tau_n}(X_n)$ by the following steps:

(i) $\hat{\sigma}[f_i(x_{s(1)}, \ldots, x_{s(n)})] \;:=\; (\sigma(f_i))_s$ for every $s \in H_n$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_n)] := S^n(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$.

Let $Hyp^F(\tau_n)$ be the set of all full hypersubstitutions of type τ_n. On $Hyp^F(\tau_n)$ we define a binary operation $\circ_h$ by $\sigma_1 \circ_h \sigma_2 \;:=\; \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of functions. Together with the identity hypersubstitution σ_{id} defined by $\sigma_{id}(f_i) := f_i(x_1, \ldots, x_n)$ one has a monoid $(Hyp^F(\tau_n); \circ_h, \sigma_{id})$. For more background on hypersubstitutions see [1]. Then we have:

Proposition 2. *Let $\sigma \in Hyp^F(\tau_n)$. Then $\hat{\sigma}$ is an endomorphism on the algebra $(W^F_{\tau_n}(X_n); S^n)$.*

Proof. Indeed $\hat{\sigma} : (W^F_{\tau_n}(X_n); S^n) \to (W^F_{\tau_n}(X_n); S^n)$ is a function from $(W^F_{\tau_n}(X_n); S^n)$ into itself. Now we prove by induction on the complexity of a term t_0 that for any $t_1, \ldots, t_n \in W^F_{\tau_n}(X_n)$,

$$\hat{\sigma}(S^n(t_0, t_1, \ldots, t_n)) = S^n(\hat{\sigma}(t_0), \hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)). \tag{1}$$

First we consider $t_0 = f_i(x_{s(1)}, \ldots, x_{s(n)})$ where $i \in I$ and $s \in H_n$. Then

$\hat{\sigma}(S^n(f_i(x_{s(1)}, \ldots, x_{s(n)}), t_1, \ldots, t_n))$

$$
\begin{aligned}
&= \hat{\sigma}(f_i(t_{s(1)}, \ldots, t_{s(n)})) \\
&= S^n(\sigma(f_i), \hat{\sigma}(t_{s(1)}), \ldots, \hat{\sigma}(t_{s(n)})) \\
&= S^n((\sigma(f_i))_s, \hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)) \\
&= S^n(\hat{\sigma}(t_0), \hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)).
\end{aligned}
$$

Now assume that $t_0 = f_i(r_1, \ldots, r_n)$ where r_i are full terms and that (1) holds for each r_i, $1 \le i \le n$. Then

$$
\begin{aligned}
&\hat{\sigma}(S^n(f_i(r_1, \ldots, r_n), t_1, \ldots, t_n)) \\
&\quad = \hat{\sigma}(f_i(S^n(r_1, t_1, \ldots, t_n), S^n(r_2, t_1, \ldots, t_n), \ldots, \\
&\qquad S^n(r_n, t_1, \ldots, t_n))) \\
&\quad = S^n(\sigma(f_i), \hat{\sigma}(S^n(r_1, t_1, \ldots, t_n)), \hat{\sigma}(S^n(r_2, t_1, \ldots, t_n)), \ldots, \\
&\qquad \hat{\sigma}(S^n(r_n, t_1, \ldots, t_n))) \\
&\quad = S^n(\sigma(f_i), S^n(\hat{\sigma}(r_1), \hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)), \ldots, \\
&\qquad S^n(\hat{\sigma}(r_n), \hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n))) \\
&\quad = S^n(S^n(\sigma(f_i), \hat{\sigma}(r_1), \ldots, \hat{\sigma}(r_n)), \hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)) \\
&\quad = S^n(\hat{\sigma}(f_i(r_1, \ldots, r_n)), \hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)).
\end{aligned}
$$

Therefore $\hat{\sigma}$ is an endomorphism. □

We have seen that the free algebra $\text{clone}_F \tau_n$ is generated by the set $F_{s\tau_n} = \{f_i(x_{s(1)}, \ldots, x_{s(n)}) \mid i \in I,\ s \in H_n\}$. Therefore any mapping η from $F_{s\tau_n}$ into $W^F_{\tau_n}(X_n)$ can be uniquely extended to an endomorphism $\overline{\eta}$ of $\text{clone}_F \tau_n$. Such mappings are called full clone substitutions. Let $Subst_{FC}$ be the set of all such full clone substitutions. Together with a binary composition operation $\odot$ defined by $\eta_1 \odot \eta_2 := \overline{\eta}_1 \circ \eta_2$ where $\circ$ is the usual composition of functions and with the identity mapping $id_{F_{s\tau_n}}$ on $F_{s\tau_n}$ we see that $(Subst_{FC}; \odot, id_{F_{s\tau_n}})$ is a monoid. Let $End(\text{clone}_F \tau_n)$ be the monoid of endomorphisms on the algebra $\text{clone}_F \tau_n$. Then we examine the connection between these monoids and the monoid of full hypersubstitutions of type τ_n.

Clearly the monoids $End(\text{clone}_F \tau_n)$ and $(Subst_{FC}; \odot, id_{F_{s\tau_n}})$ are isomorphic.

Proposition 3. *The monoid* $(Hyp^F(\tau_n); \circ_h, \sigma_{id})$ *can be embedded into the monoid* $(Subst_{FC}; \odot, id_{F_{s\tau_n}})$.

Proof. Let $\sigma \in Hyp^F(\tau_n)$. Then by Proposition 2, $\hat{\sigma}$ is an endomorphism on the algebra $\text{clone}_F \tau_n$. Since $\mathcal{F}_{s\tau_n} = \{f_i(x_{s(1)}, \ldots, x_{s(n)}) \mid i \in I,\ s \in H_n\}$ is a generating set of $\text{clone}_F \tau_n$, the mapping $\hat{\sigma}_{/F_{s\tau_n}}$ is a substitution with $\overline{\hat{\sigma}_{/F_{s\tau_n}}} = \hat{\sigma}$. We define the mapping $\psi : Hyp^F(\tau_n) \longrightarrow Subst_{FC}$ by $\psi(\sigma) = \hat{\sigma}_{/F_{s\tau_n}}$. Injectivity of ψ is clear. We will show that ψ is a homomorphism. Let $\sigma_1, \sigma_2 \in Hyp^F(\tau_n)$. Then $\psi(\sigma_1 \circ_h \sigma_2) = (\sigma_1 \circ_h \sigma_2)\hat{}_{/F_{s\tau_n}} = (\hat{\sigma}_1 \circ \hat{\sigma}_2)_{/F_{s\tau_n}} = \hat{\sigma}_1 \circ \hat{\sigma}_{2/F_{s\tau_n}} = \overline{\hat{\sigma}_{1/F_{s\tau_n}}} \circ \hat{\sigma}_{2/F_{s\tau_n}} = \overline{\psi(\sigma_1)} \circ \psi(\sigma_2) = \psi(\sigma_1) \odot \psi(\sigma_2)$. Clearly, the mapping ψ preserves the identity element. □

3. Full hyperidentities and identities in $clone_F\tau_n$

Let V be a variety of type τ_n and let $Id_n^F V := W_{\tau_n}^F(X_n)^2 \cap IdV$ be the set of all identities of V consisting of n-ary full terms. Then we have

Proposition 4. *$Id_n^F V$ is a congruence on $clone_F\tau_n$.*

Proof. We will prove that from $r \approx t, r_i \approx t_i \in Id_n^F V, i = 1, \ldots, n$, there follows $S^n(r, r_1, \ldots, r_n) \approx S^n(t, t_1, \ldots, t_n) \in Id_n^F V$. At first we prove by induction on the complexity of the term $t \in W_{\tau_n}^F(X_n)$ that for every $n \in \mathbb{N}^+$ from $t_i \approx r_i \in Id_n^F V, i = 1, \ldots, n$ there follows $S^n(t, t_1, \ldots, t_n) \approx S^n(t, r_1, \ldots, r_n) \in Id_n^F V$. Indeed, if $t = f_i(x_{s(1)}, \ldots, x_{s(n)}),\ i \in I,\ s \in H_n$, then

$$\begin{aligned} &S^n(f_i(x_{s(1)}, \ldots, x_{s(n)}), t_1, \ldots, t_n) \\ &\quad = f_i(t_{s(1)}, \ldots, t_{s(n)}) \\ &\quad \approx f_i(r_{s(1)}, \ldots, r_{s(n)}) \\ &\quad = S^n(f_i(x_{s(1)}, \ldots, x_{s(n)}), r_1, \ldots, r_n) \in Id_n^F V \end{aligned}$$

since IdV is compatible with the operation $\overline{f}_i$ of the absolutely free algebra $\mathcal{F}_{\tau_n}(X)$ and by the definition of full terms. Assume now that $t = f_i(l_1, \ldots, l_n) \in W_{\tau_n}^F(X_n)$ and that for $l_j, 1 \leq j \leq n$, we have already

$$S^n(l_j, t_1, \ldots, t_n) \approx S^n(l_j, r_1, \ldots, r_n) \in Id_n^F V.$$

Then

$$\begin{aligned} S^n(f_i(l_1, \ldots, l_n), t_1, \ldots, t_n) &= f_i(S^n(l_1, t_1, \ldots, t_n), \ldots, \\ &\qquad S^n(l_n, t_1, \ldots, t_n)) \\ &\approx f_i(S^n(l_1, r_1, \ldots, r_n), \ldots, \\ &\qquad S^n(l_n, r_1, \ldots, r_n)) \\ &= S^n(f_i(l_1, \ldots, l_n), r_1, \ldots, r_n) \\ &\qquad \in Id_n^F V. \end{aligned}$$

Now we prove the implication

$$t \approx r \in Id_n^F V \Rightarrow S^n(t, r_1, \ldots, r_n) \approx S^n((r, r_1, \ldots, r_n) \in Id_n^F V.$$

This is a consequence of the fully invariance of $Id_n V$ as a congruence on the absolutely free algebra $\mathcal{F}_{\tau_n}(X_n)$ and the definition of full terms. Assume now that $t \approx r, t_i \approx r_i \in Id_n^F V$. Then $S^n(t, t_1, \ldots, t_n) \approx S^n(r, t_1, \ldots, t_n) \approx S^n(r, r_1, \ldots, r_n) \in Id_n^F V$. □

Full hypersubstitutions can be used to define F-hyperidentities in a variety V of type τ_n.

Definition 5. *Let V be a variety of type τ_n and let $Id_n^F V$ be the set of all identities of V consisting of n-ary full terms. Then $s \approx t \in Id_n^F V$ is called* *an n-F-hyperidentity in V if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id_n^F V$ for every $\sigma \in Hyp^F(\tau_n)$. If every identity in $Id_n^F V$ is satisfied as an n-F-hyperidentity, the variety V is called n-F-solid.*

We will give a sufficient condition for the n-F-solidity of a variety V.

Proposition 5. *If $Id_n^F V$ is a fully invariant congruence relation on $clone_F \tau_n$, then the variety V is n-F-solid.*

Proof. Let $s \approx t \in Id_n^F V$ and let $\sigma \in Hyp^F(\tau_n)$ be a full hypersubstitution. Since by Proposition 2 the extension $\hat{\sigma}$ of σ is an endomorphism of $\mathrm{clone}_F \tau_n$, we have $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id_n^F V$. □

As we will show later, the opposite is not true.

By Proposition 4 we can form the quotient algebra

$$clone_F V := clone_F \tau_n / Id_n^F V$$

which belongs to the variety of Menger algebras of rank n. There is the following connection between clone identities and n-F-hyperidentities in V.

Proposition 6. *Let V be a variety of type τ_n and let $s \approx t \in Id_n^F V$. If $s \approx t$ is an identity in $clone_F V$, then it is an n-F-hyperidentity in V.*

Proof. Let $s \approx t \in Id_n^F V$ be an identity in $\mathrm{clone}_F V$ and let $\sigma \in Hyp^F(\tau_n)$. Then $\hat{\sigma} \in End(\mathrm{clone}_F \tau_n)$ and $\hat{\sigma}_{/F_{s\tau_n}} \in Subst_{FC}$ with $\overline{\hat{\sigma}_{/F_{s\tau_n}}} = \hat{\sigma}$. By the natural mapping $natId_n^F V$ we have

$$natId_n^F V \circ \hat{\sigma}_{/F_{s\tau_n}} : \{f_i(x_{s(1)}, \ldots, x_{s(n)}) \mid i \in I, s \in H_n\} \to \mathrm{clone}_F V$$

and this is a valuation mapping with

$$\overline{natId_n^F V \circ \hat{\sigma}_{/F_{s\tau_n}}} = natId_n^F V \circ \hat{\sigma}.$$

Then

$$\left.\begin{array}{rcl} s \approx t \in Id(\mathrm{clone}_F V) & \Rightarrow & (\overline{natId_n^F V \circ \hat{\sigma}_{/F_{s\tau_n}}})(s) \\ & = & (\overline{natId_n^F V \circ \hat{\sigma}_{/F_{s\tau_n}}})(t) \\ & \Rightarrow & (natId_n^F V \circ \hat{\sigma})(s) \\ & = & (natId_n^F V \circ \hat{\sigma})(t) \\ & \Rightarrow & [\hat{\sigma}[s]]_{Id_n^F V} = [\hat{\sigma}[t]]_{Id_n^F V} \\ & \Rightarrow & \hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id_n^F V \end{array}\right. \quad \text{for every } \sigma \in Hyp^F(\tau_n).$$

This means, $s \approx t$ is satisfied as an n-F-hyperidentity in V. □

Conversely, not every n-F-hyperidentity in V is an identity in $\mathrm{clone}_F V$ as the following examples show.

4. 2-F-solid varieties of type (2)

We ask for the greatest and the least 2-F-solid varieties of groupoids.

Theorem 2. *The variety* $V_{big} = Mod\{x_1x_2 \approx x_2x_1, x_1^2 \approx x_2^2\}$ *is the greatest* 2-F*-solid variety of commutative groupoids and the variety* $Z = Mod\{x_1x_2 \approx x_3x_4\}$ *of all zero semigroups is the least non-trivial one.*

Proof. The class of all groupoids which satisfy the commutative law as a full hyperidentity is the greatest 2-F-solid variety of commutative groupoids. We denote this variety by $H_FMod\{f(x_1,x_2) \approx f(x_2,x_1)\}$ where f is a binary operation symbol and call it the full hypermodel-class of the commutative law. Since this variety is 2-F-solid, it is closed under the application of full hypersubstitutions. The application of the full hypersubstitution $\sigma_{x_1{}^2}$ to the commutative identity gives $x_1^2 \approx x_2^2$ and the identity hypersubstitution gives the commutative law. (We notice that instead of $f(x_1,x_2)$ we write x_1x_2.) This shows $H_FMod\{f(x_1,x_2) \approx f(x_2,x_1)\} \subseteq V_{big}$. We can prove the opposite inclusion by showing that V_{big} is 2-F-solid since $H_FMod\{f(x_1,x_2) \approx f(x_2,x_1)\}$ is the greatest 2-F-solid variety of commutative groupoids. To do so we need all full hypersubstitutions. But we can restrict ourselves to all full hypersubstitutions of type (2) which are essential for V_{big}. An easy observation shows that for any variety V, if $s \approx t$ is an identity in V, if for a hypersubstitution σ_1 the equation $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t]$ is an identity in V and if σ_2 is a hypersubstitution such that $\sigma_1(f) \approx \sigma_2(f)$ is an identity in V, then also $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t]$ is an identity in V. For an arbitrary term $t \in W^F_{(2)}(X_2)$ we define the term t^d inductively by $f(x_1,x_2)^d = f(x_2,x_1), f(x_2,x_1)^d = f(x_1,x_2), f(x_1,x_1)^d = f(x_1,x_1), f(x_2,x_2)^d = f(x_2,x_2)$ and if t has the form $t = f(t_1,t_2), t_1,t_2 \in W^F_{(2)}(X_n)$ we set $t^d = S^2(f(x_2,x_1),t_1^d,t_2^d)$. We show by induction on the complexity of $t \in W^F_{(2)}(X_n)$, that $t \approx t^d \in Id_2^FV_{big}$. For terms of complexity 1, that is, with one binary operation symbol this is clear. Assume that $t = f(t_1,t_2)$ and that $t_i^d \approx t \in Id_2^FV_{big}, i = 1,2$. Then $t^d = S^2(f(x_2,x_1)t_1^d,t_2^d) \approx S^2(f(x_2,x_1),t_1,t_2) = f(t_2,t_1) \approx f(t_1,t_2) = t \in Id_2^FV_{big}$. Now we prove that for every full hypersubstitution σ_t we have $\hat{\sigma}_t[f(x_2,x_1)] \approx t^d \in Id_2^FV_{big}$. By definition, we have $\hat{\sigma}_t[f(x_2,x_1)] = \sigma_t(f)_s$ where s is the permutation (01). If t has complexity 1, that is, if $t \in \{f(x_1,x_1), f(x_1,x_2), f(x_2,x_1), f(x_2,x_2)\}$, then

$$
\begin{aligned}
&f(x_1,x_1)^d = f(x_1,x_1) \approx f(x_2,x_2) = \sigma_{f(x_1,x_1)}(f)_s \in Id_2^FV_{big},\\
&f(x_2,x_2)^d = f(x_2,x_2) \approx f(x_1,x_1) = \sigma_{f(x_2,x_2)}(f)_s \in Id_2^FV_{big},\\
&f(x_1,x_2)^d = f(x_2,x_1) = \sigma_{f(x_1,x_2)}(f)_s, f(x_2,x_1)^d = f(x_1,x_2)\\
&\qquad = \sigma_{f(x_2,x_1)}(f)_s.
\end{aligned}
$$

Assume that $t = f(t_1, t_2)$, then $\hat{\sigma}_t[f(x_2, x_1)] = f((t_1)_s, (t_2)_s)$ and assume that $(t_i)_s = t_i^d \in Id_2^F V_{big}$. Then $f((t_1)_s, (t_2)_s) \approx f(t_1^d, t_2^d) \approx f(t_2^d, t_1^d) = \hat{\sigma}_t[f(x_2, x_1)]$. For an arbitrary full hypersubstitution σ_t we have $\hat{\sigma}_t[f(x_2, x_1)] = t^d \approx t = \hat{\sigma}_t[f(x_1, x_2)] \in Id_2^F V_{big}$ and the commutative law is satisfied as a 2-F-hyperidentity in V_{big}.

We denote by t_{x_i} the term arising from the full term t by replacing every occurrence of x_2 by x_1 if $i = 1$ and every occurrence of x_1 by x_2 if $i = 2$. We prove that for every full term t, the equations $t_{x_i} \approx t_{x_j}, i, j \in \{1, 2\}, i \neq j$, are identities in V_{big}. Indeed if t has complexity 1, then we have $f(x_1, x_1) \approx f(x_2, x_2) \in Id_2^F V_{big}$. If $t = f(t_1, t_2)$ and assume that $t_{i_{x_j}} \approx t_{i_{x_k}}, i, j, k \in \{1, 2\}, j \neq k$. Then $t_{x_1} = f(t_{1_{x_1}}, t_{2_{x_1}}) \approx f(t_{1_{x_2}}, t_{2_{x_2}}) = t_{x_2} \in Id_2^F V_{big}$.

Let σ_t be an arbitrary full hypersubstitution. Then $\hat{\sigma}_t[f(x_1, x_1)] = (\sigma_t(f))_{c_1} = t_{c_1}$ and $t_{c_1} = t_{x_1} \approx t_{x_2} = t_{c_2} = (\sigma_t(f))_{c_2} = \hat{\sigma}_t[f(x_2, x_1)] \in Id_2^F V_{big}$ where $c_i \in H_2$ with $c_i(1) = c_i(2) = i$, for $i = 1, 2$. This shows that $f(x_1, x_1) \approx f(x_2, x_2)$ is a 2-F-hyperidentity in V_{big}. Altogether, this shows the 2-F-solidity of V_{big} and the equality $H_F Mod\{f(x_1, x_2) \approx f(x_2, x_1)\} = V_{big}$.

The next step is to show that Z is the least non-trivial 2-F-solid variety V_l of commutative groupoids.

Clearly, $V_l \subseteq Mod(W_{(2)}^F(X_2)^2)$. But from

$$f(x_1, x_2) \approx f(x_1, x_1) \approx f(x_2, x_2) \in Id_2^F Mod(W_{(2)}^F(X_2)^2)$$

we obtain

$$f(x_1, x_2) \approx f(x_3, x_4) \in IdMod(W_{(2)}^F(X_2)^2)$$

and this means $V_l \subseteq Mod(W_{(2)}^F(X_2)^2) \subseteq Z$. From $(W_{(2)}^F(X_2))^2 \subseteq IdZ$ we obtain $Z = Mod(W_{(2)}^F(X_2)^2)$. There is only one full binary term over Z, namely $f(x_1, x_2)$. Therefore we have only to consider the identity hypersubstitution and this shows that Z is 2-F-solid. Since Z is an atom in the lattice of all varieties of groupoids, it is the least non-trivial 2-F-solid variety of commutative groupoids. □

The variety V_{big} is 2-F-solid but if we apply the clone endomorphism which maps the generator $f(x_1, x_2)$ to the full term $f(x_1, x_2)$ and the generator $f(x_2, x_1)$ to the full term $f(x_1, x_1)$ to the identity $f(x_1, x_2) \approx f(x_2, x_1) \in Id_2^F V_{big}$, then we get the equation $f(x_1, x_2) \approx f(x_1, x_1)$ which is not satisfied in V_{big} since $V_{big} \neq Z$. This means, $Id_2^F V_{big}$ is not fully invariant and the opposite of Proposition 5 is not satisfied. Since $f(x_1, x_2) \approx f(x_2, x_1), f(x_1, x_1) \approx f(x_2, x_2)$ are 2-F-hyperidentities in V_{big} and using the compatibility, we have that $f(f(x_1, x_2), f(x_1, x_1)) \approx$

$f(f(x_2, x_1), f(x_2, x_2))$ is a 2-F-hyperidentity in V_{big}. If we apply the valuation mapping which maps $f(x_1, x_2)$ and $f(x_1, x_1)$ to the full term $[f(x_1, x_2)]_{Id_2 V_{big}}$ and both $f(x_2, x_1)$ and $f(x_2, x_2)$ to $[f(x_1, x_1)]_{Id_2 V_{big}}$ to the equation $f(f(x_1, x_2), f(x_1, x_1)) \approx f(f(x_2, x_1), f(x_2, x_2))$, then we obtain the equation $[f(x_1, x_2)]_{Id_2 V_{big}} = [f(x_1, x_1)]_{Id_2 V_{big}}$ and so $f(x_1, x_2) \approx f(x_1, x_1)$ is an identity in V_{big}, which is a contradiction. This means, the equation $f(f(x_1, x_2), f(x_1, x_1)) \approx f(f(x_2, x_1), f(x_2, x_2))$ is a 2-F-hyperidentity in V_{big} but not an identity in $\text{clone}_F V_{big}$ and hence the opposite of Proposition 6 is not satisfied.

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Received by the editors: 23.02.2004
and final form in 17.12.2004.