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Description of the center of certain quotients of the Temperley-Lieb algebra of type $\widetilde{A}_N$

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Introduction

In this paper we consider a family of associative algebras, given via generators and relations. Namely, each algebra is generated by finite set of idempotents $\{p_i; 1 \leq i \leq N\}$, where for each pair of i, j $(i \neq j)$ either

$$p_i p_j = p_j p_i = 0 \tag{1}$$

or

$$p_i p_j p_i = \tau p_i \text{ and } p_j p_i p_j = \tau p_j \tag{2}$$

for some nonzero $\tau = \tau(i, j) \in \mathbb{C}$. One can see that, equipped with unity, this algebra becomes a quotient of Temperley-Lieb algebra, where we have commutation $p_i p_j = p_j p_i$ instead of orthogonality (1).

*-representations of some algebras defined as above were studied in [1, 2].

It will be useful to associate a non-oriented graph Γ with N vertices to the algebra above, where vertices i and j are connected by an edge if and only if (2) holds. So, there is no edge between the vertices correspondent to the orthogonal generating idempotents, and there is exactly one edge otherwise. We denote by Γ_0 and Γ_1 the sets of vertices and edges of the graph Γ correspondingly, and consider a nonzero map $\tau : \Gamma_1 \longrightarrow \mathbb{C}\backslash\{0\}$ on edges, where for the edge α between i and j one has $\tau(\alpha) = \tau(i, j)$ from (2). So, the pair (Γ, τ), consisting of a graph, Γ, and a map, τ, as above, defines an associative algebra. We denote this algebra $R = R(\Gamma, \tau)$. Note, that R may not contain the unit element $\mathbf{1}$.

As the generators corresponding to non-connected vertices are orthogonal, $R(\Gamma, \tau)$ is a direct sum of algebras, corresponding to the connected components of the graph Γ, i. e. $R(\Gamma, \tau) = \bigoplus_{\Gamma' \in \pi_0(\Gamma)} R(\Gamma', \tau|_{\Gamma'_1})$, where $\pi_0(\Gamma)$ is a set of connected components of graph Γ. Thus we can always assume Γ to be a connected graph.

In this paper we establish some results on the structure of $R(\Gamma, \tau)$. Section 1 contains a series of technical propositions, which are applied in next sections. We construct a homomorphism from R to the algebra of $N \times N$ matrices over a certain subalgebra $K \subset R$. It is known that $R(\Gamma, \tau)$ is finite dimensional if and only if Γ is a tree ([3]). In Section 2 we consider this case and prove

Theorem 1. *If Γ is a tree then the ring R is either isomorphic to $\mathcal{M}_N(\mathbb{C})$ or the center of R is $Z(R) = \{r \in R; Rr = rR = 0\}$.*

There exist a polynomial Q in $|\Gamma_1|$ variables such that $R \cong \mathcal{M}_N(\mathbb{C})$ if and only if $Q(\tau(\Gamma_1)) \neq 0$. The polynomial Q depends on the graph Γ only.

An isomorphism with $\mathcal{M}_N(\mathbb{C})$ was obtained by author in [4] for some special τ.

In Section 3 we consider the case when Γ contains exactly one cycle. In particular, all correspondent rings are PI-rings (Corollary 2).

Denote $\mathcal{A}nn_R R = \{r \in R; rR = 0\}$ — the left annihilator of R. We prove

Theorem 2. *If Γ is an arbitrary connected graph with exactly one cycle then there exist a polynomial P such that $Z(R/\mathcal{A}nn_R R)$ is isomorphic to the principal ideal in $\mathbb{C}[x, x^{-1}]$ generated by $P(x)$, i.e.*

$$Z(R/\mathcal{A}nn_R R) \cong \mathbb{C}[x, x^{-1}]P(x).$$

There are inclusions

$$\mathcal{M}_N(Z(R/\mathcal{A}nn_R R)) \subset R/\mathcal{A}nn_R R \subset \mathcal{M}_N(\mathbb{C}[x, x^{-1}]),$$

which are both proper if P is not a constant.

In particular $Z(R/\mathcal{A}nn_R R)$ is always a domain. In Section 3 we also present an algorithm for finding the polynomial P. Note that if $P = 0$ then $Z(R) \subset \mathcal{A}nn_R R$, thus $Z(R) = \{r \in R; Rr = rR = 0\}$.

In Section 4 we apply Theorem 2 to the case of a cyclic graph Γ. In this case $\mathcal{A}nn_R R = \{0\}$ (Proposition 8). Let nonzero numbers $\tau_0, \dots, \tau_{N-1}$ be arranged sequentially on edges along the cycle Γ. Consider the following matrix over $\mathbb{C}[x, x^{-1}]$

$$A = \begin{pmatrix} 1 & \tau_0 & 0 & \dots & x \\ 1 & 1 & \tau_1 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \tau_{N-2} \\ \frac{\tau_{N-1}}{x} & 0 & 0 & \dots & 1 \end{pmatrix}, \tag{3}$$

Put

$$\alpha(\tau) = \frac{\det A(1) + (-1)^N(1 + \tau_0 \dots \tau_{N-1})}{\sqrt{\tau_0 \dots \tau_{N-1}}}, \tag{4}$$

where one can take any value of the square root. Note that $\det A$ is invariant under the cyclic shifts of numbers τ_i, so $\alpha(\tau)$ is invariant under such shifts as well.

Theorem 3. *If Γ is a cycle then the center $Z(R)$ of $R = R(\Gamma, \tau)$ is isomorphic to the principal ideal in $\mathbb{C}[x, x^{-1}]$ generated by either $x + \frac{1}{x} + \alpha(\tau)$ or $x + 1$. The second alternative holds if and only if $\alpha(\tau) = \pm 2$ and $rank A((-1)^{N-1} sign\ (\alpha(\tau))\sqrt{\tau_0 \dots \tau_{N-1}}) = N - 2$.*

Note that any $N-2$ consequtive columns of $A(x)$ are linearly independent for any x. Thus $rank A(x) \geq N - 2$.

In the first case the center of R, equipped with the unity $\mathbf{1}$, is isomorphic to the coordinate ring of the affine algebraic variety defined by

$$z^2 + y^2 + (\alpha - y)yz = 0, \tag{5}$$

where $\alpha = \alpha(\tau)$. One can find the proof in Section 4. In the second case the center, equipped with $\mathbf{1}$, is isomorphic to $\mathbb{C}[x, x^{-1}]$, i.e. to the coordinate ring of $yz = 1$.

The following is a consequence of Theorems 2 and 3, Amitsur-Levitsky Theorem (see [6]) and Propositions 9, 10 together.

Corollary 1. *If Γ is a cycle then*

(i) $R(\Gamma, \tau)$ is not isomorphic to a matrix algebra over a commutative algebra;

(ii) the following standard identity (S_{2N}-identity) holds in $R = R(\Gamma, \tau)$:

$$\sum_{\sigma \in S_{2N}} (-1)^{\deg \sigma} r_{\sigma(1)} \dots r_{\sigma(2N)} = 0$$

for any $r_1, \dots, r_{2N} \in R$. Moreover, $m = 2N$ is the minimal m, for which the S_m-identity in R holds;

(iii) the rings, corresponding to cycles of different length $N_1 \neq N_2$, are non-isomorphic;

(iv) the rings $R = R(\Gamma, \tau)$, corresponding to different maps τ, are non-isomorphic if the absolute values of the numbers $\alpha(\tau)$ are different;

(v) the rings $R = R(\Gamma, \tau)$ with $\alpha(\tau) = \pm 2$ and

$$rank A((-1)^{N-1} sign\ (\alpha(\tau))\sqrt{\tau_0 \dots \tau_{N-1}}) = N - 2$$

are non-isomorphic to the rings with

$$rank A((-1)^{N-1} sign\ (\alpha(\tau))\sqrt{\tau_0 \dots \tau_{N-1}}) > N - 2.$$

Note that in all our considerations $\mathbb{C}$ can be replaced by any algebraically closed field of characteristic 0.

1. An embedding into matrices

Recall that the graph Γ is a connected graph, $\tau : \Gamma_1 \longrightarrow \mathbb{C} \setminus \{0\}$, and $R = R(\Gamma, \tau)$. Let $N = |\Gamma_0|$ be the number of generating idempotents in the definition of R. One can check that the relations (1) and (2) together with $p_i^2 = p_i$ for all $i \in \Gamma_0$ form a Groebner basis, so the products $p_{i_1} \dots p_{i_m}$ along the pathes in Γ without loops of length 2 form a linear basis in R (see [3]). For details about the Groebner basis and the Diamond lemma see for example [5].

We are going to consider the subalgebra $K = p_i R p_i$ of R. Note that p_i is the unity in K.

Proposition 1. *Subalgebras $p_i R p_i$ and $p_j R p_j$ are isomorphic for any i, j.*

Proof. Consider a path from i to j in Γ, let it go through the vertices $p_i = p_{i_0}, p_{i_1}, \dots, p_{i_m} = p_j$ and $\alpha_j \in \Gamma_1$ be the edge from i_{j-1} to i_j. Consider the elements

$$w = \frac{p_{i_0} p_{i_1} \dots p_{i_m}}{\sqrt{\tau(\alpha_1) \dots \tau(\alpha_m)}}, \quad v = \frac{p_{i_m} p_{i_{m-1}} \dots p_{i_0}}{\sqrt{\tau(\alpha_1) \dots \tau(\alpha_m)}}.$$

Then $wv = p_i = \mathbf{1}_{p_i R p_i}$, $vw = p_j = \mathbf{1}_{p_j R p_j}$, and the map from $p_i R p_i$ to $p_j R p_j$, sending x to vxw, is an isomorphism. $\square$

Proposition 2. *$K = \mathbb{C}$ if the graph Γ is a tree; $K = \mathbb{C}[x, x^{-1}]$ if Γ has exactly one cycle; K contains a free subalgebra of two generators otherwise.*

Proof. p_iRp_i is spanned by the products $p_ip_{j_1}\dots p_{j_m}p_i$ along the pathes in Γ without loops of length 2, starting and ending at i. In the case, when Γ is a tree, there is only one such path, and it is the trivial path at the vertex i. In the case, when Γ has exactly one cycle, we can take i to be a vertex of this cycle. Then p_iRp_i is spanned by p_i, powers of $w = p_ip_{j_1}\dots p_{j_m}p_i$, where the product is along the cycle, and powers of $v = p_ip_{j_m}\dots p_{j_1}p_i$, where the product is along the cycle but in the different direction. But $wv = \lambda p_i$ for some $\lambda \in \mathbb{C}$, which proves the statement in this case. For a graph with two or more cycles we can take such a products w_1 and w_2 along the pathes containing two different cycles, each of which doesn't contain another one, correspondently. Then for any word s in alphabet $\{w_1, w_2\}$ we can take the path in Γ which is a combination of copies of two pathes above in correspondent order. Let $\gamma(s)$ be the path obtained by consequtive removing of loops of length 2 from this path. The path $\gamma(s)$ goes through these two cycles in the same order as before removing of loops of length 2, so it uniquely depends on s. The product s is a multiple of the element of linear basis in R corresponding to the path $\gamma(s)$. Thus all such products s are linearly independent, which proves the statement in this case. □

We will need the following

Proposition 3. $\{r \in R; rRp_i = 0\} = \mathcal{A}nn_R R$ *for any* $i \in \Gamma_0$.

Proof. Denote the left hand side by $\mathcal{A}nn_R(Rp_i)$. Obviously, $\mathcal{A}nn_R R \subset \mathcal{A}nn_R(Rp_i)$. To show the reverse inclusion let $r \in \mathcal{A}nn_R(Rp_i)$. Then $0 = rRp_iR = rR$ because $p_j \in Rp_iR$ for any $j \in \Gamma_0$. □

Now, we construct an embedding of $R/\mathcal{A}nn_R R$ into $N \times N$ matrices over K.

Proposition 4. *There exist a homomorphism* $\phi : R \longrightarrow \mathcal{M}_N(K)$ *with* $\mathcal{K}er\phi = \mathcal{A}nn_R R$.

Proof. Consider any $i \in \Gamma_0$. Note that $Rp_i = \bigoplus_{j\in\Gamma_0} p_jRp_i$ in the category $\mathcal{M}od - p_iRp_i$ of right p_iRp_i modules. Each component of the direct sum above is a free p_iRp_i module of rank 1. Indeed, the products $p_jp_{k_1}\dots p_{k_m}p_i$ along the pathes from j to i span p_jRp_i. Consider any two such products: $s_1 = p_jp_{k_1}\dots p_{k_m}p_i$ and $s_2 = p_jp_{l_1}\dots p_{l_t}p_i$. Then

$$w = p_ip_{k_m}\dots p_{k_1}p_jp_{l_1}\dots p_{l_t}p_i$$

belongs to p_iRp_i, and $s_2 = \lambda s_1 w$ for some $\lambda \in \mathbb{C}$. Thus p_jRp_i is spanned by $s_1p_iRp_i$. To show it is free choose an arbitrary $s = p_jp_{k_1}\dots p_{k_m}p_i$

and consider the homomorphism $\psi : p_iRp_i \longrightarrow p_jRp_j$ of right p_iRp_i modules, sending arbitrary $w \in p_iRp_i$ to sw. Then $\mathcal{K}er\psi = \{0\}$ because $p_ip_{k_m} \ldots p_{k_1}p_j\psi(w) = \lambda w$ for some nonzero $\lambda \in \mathbb{C}$.

So, Rp_i is a free K-module of rank N.

As Rp_i belongs to the category of bimodules $R - \mathcal{M}od - p_iRp_i$, we have $R/\mathcal{A}nn_R(Rp_i) \subset \mathcal{E}nd(Rp_i)_{p_iRp_i} \cong \mathcal{M}_N(K)$. Proof now follows from Proposition 3. □

An algebra is called a PI-algebra if some polynomial identity holds in it, see for example [6, 7].

Corollary 2. *R is a PI-algebra if and only if Γ contains not more than one cycle.*

Proof. If Γ contains more than one cycle, the statement follows from Proposition 2. Otherwise K is commutative and the S_{2N} identity holds in $\mathcal{M}_N(K)$ due to teh Amitsur-Levitsky Theorem ([6]). Then, for every $r_1, \ldots, r_{2N} \in R$, we have $S_{2N}(r_1, \ldots, r_{2N}) \in \mathcal{A}nn_R R$ due to Proposition 4, so the identity

$$S_{2N}(r_1, \ldots, r_{2N})r_1 = 0$$

holds in R. □

The following proposition describes the image of the homomorphism ϕ from Proposition 4.

Proposition 5. *(i)The centralizer of $\phi(R)$ in $\mathcal{M}_N(K)$ is included into the diagonal matrices with all diagonal elements conjugate in K.*

(ii) $R/\mathcal{A}nn_R R \cong \mathcal{M}_N(K)\phi(\sum_{i\in\Gamma_0} p_i)$.

Proof. Choose an arbitrary vertex in Γ_0, which we can assume is 0. We can identify $K \cong p_0Rp_0$, and then Rp_0 is a free right K-module of rank N, as it was shown in proposition 4. Choose a K-basis $\{e_i, i \in \Gamma_0\}$ in Rp_0, where $e_i = p_ip_{i_1} \ldots p_{i_m}p_0$ is a product along some path, connecting i with 0. In this basis $\phi(p_j)$ is a matrix with some elements $\{w_{j1}, \ldots, w_{jN}\}$ of K in the j-th row and all other rows equal to zero. Note that $w_{jj} = 1$, w_{ji} is an invertible element of K if there is an edge between j and i in Γ, and $w_{ji} = 0$ otherwise. Now (i) follows. Indeed, consider $B \in \mathcal{M}_N(K)$ commuting with $\phi(R)$. If $B = (b_{ij})$ commutes with $\phi(p_k)$, we have that all elements in the k-th column of B, except the diagonal ones, vanish, and $b_{jj} = w_{ji}b_{ii}w_{ji}^{-1}$ for all vertices i connected to j by an edge in Γ.

To prove (ii), consider the following matrices $P_i^j \in \mathcal{M}_N(K)$:

$$P_i^j = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ w_{i1} & w_{i2} & \dots & w_{iN} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \; j, i \in \Gamma_0,$$

where the only nonzero row in P_i^j is the j-th row. So, $\phi(p_j) = P_j^j$. It suffices to show that $\phi(p_j R p_i) = K P_i^j$. Consider any product $p_{i_1} \dots p_{i_m}$. One can prove by induction on m, that

$$\phi(p_{i_1} \dots p_{i_m}) = w_{i_1 i_2} \dots w_{i_{m-1} i_m} P_{i_m}^{i_1}.$$

So, $\phi(p_j R p_i) \subset K P_i^j$. For each $e_i = p_i p_{i_1} \dots p_{i_m} p_0$ from the basis consider the element $r_i = p_0 p_{i_m} \dots p_{i_1} p_i$. Then $r_i e_i = \lambda_i p_0$ for some nonzero $\lambda_i \in \mathbb{C}$. Take an arbitrary $\beta \in p_0 R p_0$, and let $v = \frac{e_j \beta r_i}{\lambda_i}$. Then $v \in p_j R p_i$, and $v e_i = e_j \beta$. Thus the matrix $\phi(v)$ has an element β on the intersection of the j-th row and the $i-th$ column, which implies $\phi(v) = \beta P_i^j$. $\beta \in K$ was arbitrary, and this completes the proof. □

The following proposition gives a simple necessary and sufficient condition for R to be a matrix algebra over K.

Proposition 6. *$R \cong \mathcal{M}_N(K)$ if and only if R contains a unity.*

Proof. $\mathcal{M}_N(K)$ contains a unity as K contains a unity. Thus if $R \cong \mathcal{M}_N(K)$ then R contains a unity.

If R contains $\mathbf{1}$ then $\mathcal{A}nn_R R = \{0\}$, so we have a monomorphism $\phi : R \longrightarrow \mathcal{E}nd(Rp_i)_{p_i R p_i}$.

Consider the homomorphism $\psi : \mathcal{E}nd(Rp_i)_{p_i R p_i} \longrightarrow \mathcal{E}nd(Rp_i R)_R$ given by $\psi(\eta)(a p_i b) = \eta(a p_i) b$ for $\eta \in \mathcal{E}nd(Rp_i)_{p_i R p_i}$ and $a, b \in R$. Let us show that it is correctly defined. Consider some $\sum_k a_k p_i b_k = 0$, where $a_k, b_k \in R$. We need $\sum_k \eta(a_k p_i) b_k = 0$. Note that $R = \bigoplus_j R p_j$, thus we can consider only the case $b_k \in R p_j$ for each k. So, $b_k = b_k p_j$. Consider $s_1 = p_j p_{k_1} \dots p_{k_m} p_i$, $s_2 = p_i p_{k_m} \dots p_{k_1} p_j$, then $s_1 s_2 = \lambda p_j$ for some nonzero $\lambda \in \mathbb{C}$. Then $\lambda \sum_k \eta(a_k p_i) b_k = \sum_k \eta(a_k p_i) b_k s_1 s_2 = \eta((\sum_k a_k p_i b_k) s_1) s_2 = 0$.

Now we show that ψ is a monomorphism. Indeed, if $\eta(a p_i) \neq 0$, then $\psi(\eta)(a p_i) = \eta(a p_i) \neq 0$. Note that $R p_i R = R$ because $p_j \in R p_i R$ for each $j \in \Gamma_0$. There is a natural isomorphism $\xi : \mathcal{E}nd R_R \longrightarrow R$ given by $\xi(\eta) = \eta(\mathbf{1})$.

Note that $\xi \circ \psi \circ \phi$ is an identity on R. So, ϕ is an isomorphism. Recall that $\mathcal{E}nd(Rp_i)_{p_i R p_i} \cong \mathcal{M}_N(K)$ from the proof of Proposition 4. □

In Sections 2-4 we consider examples in which K is commutative, especially when the graph Γ is a tree and when Γ contains exactly one cycle.

Proposition 7. *If K is commutative, then the center $Z = Z(R/\mathcal{A}nn_R R)$ is isomorphic to an ideal in K, and there is an inclusion*

$$\mathcal{M}_N(Z) \subset R/\mathcal{A}nn_R R.$$

Moreover, either the inclusion above is proper or $Z = K$ and $R/\mathcal{A}nn_R R \cong \mathcal{M}_N(K)$.

Proof. As K is commutative, the conjugate elements in it are equal. Thus, from part (i) of Proposition 5 we have that the centralizer of $\phi(R)$ coincides with $Z(\mathcal{M}_N(K)) = K$. So, $Z \cong \phi(R) \cap K$ as $R/\mathcal{A}nn_R R \cong \phi(R)$. $\phi(R) \cap K$ is an ideal in K and $\mathcal{M}_N(\phi(R) \cap K) \subset \phi(R)$ due to the part (ii), because $\phi(R)$ is right ideal in $\mathcal{M}_N(K)$.

If $\mathcal{M}_N(\phi(R) \cap K) = \phi(R)$, then $1 \in \phi(R) \cap K$. Indeed, the matrix $\phi(p_i)$ contains 1 on the i-th position of diagonal. □

2. The finite-dimensional case

Consider the case when the graph Γ is a tree with N vertices.

Proof of theorem 1. Note that R has the linear dimension N^2. Indeed, R is spanned by the products of the generating idempotents along the pathes without loops, and there is exactly one path from i to j for each $i, j \in \Gamma_0$.

Now $K = \mathbb{C}$ due to Proposition 2. As $Z(R/\mathcal{A}nn_R R)$ is isomorphic to an ideal in $\mathbb{C}$ by Proposition 7, we have two possibilities: $Z(R/\mathcal{A}nn_R R) = \{0\}$ or $Z(R/\mathcal{A}nn_R R) = \mathbb{C}$.

In the first case we have $\mathcal{A}nn_R R \neq \{0\}$. Indeed, $\dim R = N^2 = \dim \mathcal{M}_N(\mathbb{C})$ so if $\mathcal{A}nn_R R = \{0\}$ then $R = \mathcal{M}_N(\mathbb{C})$ and $Z(R) = \mathbb{C}$. Thus R is non semisimple. $Z(R) \subset \mathcal{A}nn_R R$ in this case as well, thus $Z(R) = \{r \in R; Rr = rR = 0\}$.

In the second case $R = \mathcal{M}_N(\mathbb{C})$ due to the equality of their dimensions and Proposition 7.

Note that, due to Proposition 5, the first case is possible if and only if $\det \phi(\sum_{i \in \Gamma_0} p_i) = 0$. Fixing some vertex $0 \in \Gamma_0$ and taking the same basis $\{e_i; i \in \Gamma_0\}$ in Rp_0 as in the proof of Proposition 5, we will get that all entries of the matrix $\phi(\sum_{i \in \Gamma_0} p_i)$ are either equal to 0 or to 1 or to $\tau(\gamma)$ for some $\gamma \in \Gamma_1$. So we can put $Q(\tau(\Gamma_1)) = \det \phi(\sum_{i \in \Gamma_0} p_i)$ as it is a polynomial in $\{\tau(\gamma); \gamma \in \Gamma_1\}$. □

Consider a simple example: let Γ consist of two vertices 1 and 2, connected by an edge γ and $\tau(\gamma) = \tau$ on it. Now $R = R(\Gamma, \tau) = \mathbb{C}\langle p_1, p_2 | p_1^2 = p_1, p_2^2 = p_2, p_1p_2p_1 = \tau p_1, p_2p_1p_2 = \tau p_2\rangle$ with the linear basis $\{p_1, p_2, p_1p_2, p_2p_1\}$. Consider the basis $e_1 = p_1$ and $e_2 = p_2p_1$ in Rp_1. Then

$$\phi(p_1) = \begin{pmatrix} 1 & \tau \\ 0 & 0 \end{pmatrix}, \phi(p_2) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

$\det \phi(p_1 + p_2) = 1 - \tau$, so either $\tau = 1$ or

$$\phi(p_1 + p_2)^{-1} = \frac{1}{1-\tau} \begin{pmatrix} 1 & -\tau \\ -1 & 1 \end{pmatrix} = \frac{1}{1-\tau}\phi(2 - p_1 - p_2).$$

Thus, if $\tau \neq 1$ then there is unity $\mathbf{1}$ in R given by

$$\mathbf{1} = \frac{(2 - p_1 - p_2)(p_1 + p_2)}{1-\tau} = \frac{p_1 + p_2 - p_1p_2 - p_2p_1}{1-\tau},$$

and $R \cong \mathcal{M}_2(\mathbb{C})$. If $\tau = 1$, then one can directly check that

$$Z(R) = \mathbb{C}(p_1 + p_2 - p_1p_2 - p_2p_1),$$

$$\mathcal{A}nn_R R = \mathbb{C}(p_1 - p_1p_2) + \mathbb{C}(p_2 - p_2p_1),$$

$$R/\mathcal{A}nn_R R \cong \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix}; a, b \in \mathbb{C} \right\} \subset \mathcal{M}_2(\mathbb{C}).$$

The Jacobson radical $\mathcal{R}ad(S)$ of $S = R/\mathcal{A}nn_R R$ is $\mathbb{C}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, and $S/\mathcal{R}ad(S) \cong \mathbb{C}$.

3. Description of the center for a graph with one cycle

Consider the case when Γ contains exactly one cycle. Then due to proposition 2 we have $K = \mathbb{C}[x, x^{-1}]$, so it is an Euclidean domain. Then the greatest common divisor ($g.c.d.$) is defined for any finite set of elements of K (up to the multiplication by an invertible element). We write $a \sim b$ if $a = cb$ for some invertible element c. Recall that all invertible elements are of the form λx^n for $0 \neq \lambda \in \mathbb{C}$ and some integer n.

Consider the matrix $A = \phi(\sum_{i\in\Gamma_0} p_i)$, where ϕ is the homomorphism from Proposition 4. Let $A_{ij}^{\vee}$ be the $(N-1)\times(N-1)$ matrix, obtained from A by deleting its i-th row and j-th column. Let $A_{ij}^{-} = (-1)^{i+j}\det(A_{ji}^{\vee})$ and $A^{-} = (A_{ij}^{-})$ be a matrix from $\mathcal{M}_N(K)$. Consider $a \sim g.c.d.(A_{ij}^{-})$ if some $A_{ij}^{-} \neq 0$, and take any nonzero element a otherwise. Then a divides $\det A$ as $A^{-}A = (\det A)\mathbf{1}_{\mathcal{M}_N(K)}$, where the left hand side is divisible by a.

We will prove that the center of $\phi(R)$ is the principal ideal in $\mathbb{C}[x, x^{-1}]$ generated by $\frac{\det A}{a}$. Then we can take a polynomial $P \sim \frac{\det A}{a}$ in the statement of theorem 2, and

Proof of Theorem 2. The second part of the statement is a consequence of the first part together with propositions 7 and 4. So it remains to prove that the center of $\phi(R)$ is a principal ideal in $\mathbb{C}[x, x^{-1}]$ generated by $\frac{\det A}{a}$, and take a polynomial $P \sim \frac{\det A}{a}$ defined above.

$Z(\phi(R))$ is isomorphic to an ideal in K due to Proposition 7, and $\phi(R)$ is right ideal in $\mathcal{M}_N(K)$, generated by the matrix A defined above. Let $\beta \in K$ be such that $\beta \mathbf{1}_{\mathcal{M}_N(K)}$ lies in $\phi(R)$. Then there exists matrix $B \in \mathcal{M}_N(K)$, such that $BA = \beta \mathbf{1}_{\mathcal{M}_N(K)}$. If $\beta \neq 0$, then for all x such that $\beta(x) \neq 0$ we have $\frac{1}{\beta(x)} B(x)A(x) = \mathbf{1}_{\mathcal{M}_N(\mathbb{C})}$, so $\det A(x) = (\det A)(x) \neq 0$. Thus if $\det A = 0$ then $Z(\phi(R)) = \{0\}$ which proves the theorem in this case. Let $\det A \neq 0$. Consider F — the field of fractions of K, i.e. the field of rational functions in x. If two such functions coincide in an infinite number of points they are equal. Consider the matrices $\frac{1}{\det A} A^-$ and $\frac{1}{\beta} B$ from $\mathcal{M}_N(F)$. They are equal in $\mathcal{M}_N(\mathbb{C})$ for all nonzero values of x except may be the roots of $\det A$ and β. Thus $B = \frac{\beta}{\det A} A^-$ in $\mathcal{M}_N(F)$. But $B \in \mathcal{M}_N(K)$. So, we have that $\frac{\det A}{a}$ divides β.

Conversely, if some $\beta = c\frac{\det A}{a}$, then $\beta \mathbf{1}_{\mathcal{M}_N(K)} = \frac{c}{a} A^- A \in \phi(R)$. $\square$

4. An example with a cyclic graph

Consider Γ to be a cycle with N vertices $0, \ldots, N-1$, and the number τ_i on the edge between i and $i+1$ (where N means 0). The following proposition can be applied in this case.

Proposition 8. *If every vertex in Γ has degree greater than 1, than $\mathcal{A}nn_R R = \{0\}$.*

Proof. Suppose that this is not the case. Let $y \in \mathcal{A}nn_R R$ be nonzero. Consider the following order on the idempotents generating R: $p_0 < p_1 < \cdots < p_{N-1}$. Then the homogenous lexicographical order on the monomials is defined. Let $p_{i_1} \ldots p_{i_m}$ be the highest monomial in y. As $i_m \in \Gamma_0$ has degree greater than 1, there exists a vertex $k \in \Gamma_0$, $k \neq p_{i_{m-1}}$ with the edge between k and p_{i_m}. Then $z = yp_k$ is again nonzero, because the highest monomial of z will be $p_{i_1} \ldots p_{i_m} p_k$. $\square$

Recall the matrix A, defined by (3), and the number $\alpha(\tau)$, defined by

(4), in the introduction. Denote

$$F_m(x_1,\dots,x_m)=\det\begin{pmatrix}1 & x_1 & 0 & \dots & 0\\ 1 & 1 & x_2 & \dots & 0\\ 0 & 1 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & x_m\\ 0 & 0 & 0 & \dots & 1\end{pmatrix}.$$

Proof of the Theorem 3. We use notation of the previous sections. Put $\tau_0\dots\tau_{N-1} = T$. We construct a homomorphism $\phi : R \longrightarrow \mathcal{M}_N(\mathbb{C}[x,x^{-1}])$ taking $e_i = p_ip_{i-1}\dots p_0$ as a basis in Rp_0 over p_0Rp_0 and $x = p_0p_{n-1}\dots p_1p_0 \in p_0Rp_0$. ϕ is now a monomorphism due to Proposition 8. One can calculate that $\phi(\sum_{i\in\Gamma_0} p_i)$ equals

$$A=\begin{pmatrix}1 & \tau_0 & 0 & \dots & x\\ 1 & 1 & \tau_1 & \dots & 0\\ 0 & 1 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \tau_{N-2}\\ \frac{\tau_{N-1}}{x} & 0 & 0 & \dots & 1\end{pmatrix},$$

defined by (3). Then

$$\det A = F_{N-2}(\tau_1,\dots,\tau_{N-2})-\tau_0F_{N-3}(\tau_2,\dots,\tau_{N-2})-\\ -\tau_{N-1}F_{N-3}(\tau_1,\dots,\tau_{N-3})+(-1)^{N-1}(x+Tx^{-1}).$$

So $\det A$ is nonzero and $\det A\sim x^2+\gamma x+T$ where $\gamma=(-1)^N\det A(1)+T+1$. The center of $\phi(R)$ is isomorphic to the ideal in $\mathbb{C}[x,x^{-1}]$ generated by $\frac{\det A}{a}$ for some $a\sim g.c.d.(A^-_{ij})$ by Theorem 2. As

$$A^-_{N1}=(-1)^{N-1}-\tau_{N-1}F_{N-3}(\tau_1,\dots,\tau_{N-3})\frac{1}{x}$$

and

$$A^-_{1N}=(-1)^{N-1}\frac{T}{\tau_{N-1}}-F_{N-3}(\tau_1,\dots,\tau_{N-3})x,$$

so either $F_{N-3}(\tau_1,\dots,\tau_{N-3})=0$ and $a\sim 1$, or

$$\frac{\tau_{N-1}A^-_{1N}-TA^-_{N1}}{\tau_{N-1}F_{N-3}(\tau_1,\dots,\tau_{N-3})}=\frac{T}{x}-x$$

is divisible by a. Consider the second case. We have that a divides both $x^2+\gamma x+T\sim\det A$ and $x^2-T\sim\frac{T}{x}-x$. Let us fix a value of $\sqrt{T}$ the same as the one from (4). One can see that either $a \sim 1$ or $a \sim x - \sqrt{T}$ or $a \sim x + \sqrt{T}$ as T is nonzero.

In cases $a \sim 1$ we have that $x + \frac{T}{x} + \gamma$ generates $Z(\phi(R))$ in $\mathbb{C}[x, x^{-1}]$, and, taking automorphism of $\mathbb{C}[x, x^{-1}]$ sending x to $(-1)^N \sqrt{T}x$, we get an isomorphic ideal, generated by $x + \frac{1}{x} + \alpha(\tau)$ where $\alpha(\tau) = (-1)^N \frac{\gamma}{\sqrt{T}}$.

If $a \sim x + \sqrt{T}$ or $a \sim x - \sqrt{T}$ then $Z(\phi(R))$ is obviously isomorphic to an ideal in $\mathbb{C}[x, x^{-1}]$ generated by $x+1$. Let us establish some conditions under which $a \sim x - \epsilon\sqrt{T}$ for a given $\epsilon = \pm 1$. Then $\epsilon\sqrt{T}$ is a root of $x^2 + \gamma x + T$, so $\gamma = -\epsilon 2\sqrt{T}$ and $\alpha(\tau) = (-1)^{N+1}\epsilon 2$. If the latter holds then $A^-(\epsilon\sqrt{T}) = \mathbf{0}$ is a necessary and a sufficient condition for $a \sim x - \epsilon\sqrt{T}$. The condition $A^-(\epsilon\sqrt{T}) = \mathbf{0}$ is equivalent to $rank A(\epsilon\sqrt{T}) = N - 2$ because the columns $2, 3, \ldots, N-1$ of matrix $A(x)$ are linearly independent. $\square$

Denote by $I(\alpha)$ the principal ideal in $\mathbb{C}[x, x^{-1}]$ generated by $x + \frac{1}{x} + \alpha$.

Proposition 9. *$I(\alpha)$ and $I(\beta)$ are isomorphic algebras if and only if $\beta = \pm\alpha$.*

Proof. Consider $y = x + \frac{1}{x} + \alpha, z = x^2 + 1 + \alpha x \in I(\alpha)$. Then $z^2 + y^2 + (\alpha - y)yz = 0$ and $I(\alpha)$ is generated over $\mathbb{C}$ by y and z. Suppose that it is also generated over $\mathbb{C}$ by some elements $Y, Z \in I(\alpha)$, satisfying

$$Z^2 + Y^2 + (\beta - Y)YZ = 0. \tag{6}$$

Then $Y = P(x)x^n$, $Z = Q(x)x^m$, where n, m are integers, P, Q are polynomials in x with $P(0), Q(0) \neq 0$. Dividing (6) by $g.c.d.(P, Q)^2$, we conclude that $Q = \gamma P$ for some nonzero $\gamma \in \mathbb{C}$. Moreover, $P = \delta(x^2 + 1 + \alpha x)$ for some nonzero $\delta \in \mathbb{C}$, as Y, Z generate $I(\alpha)$ over $\mathbb{C}$. So, dividing (6) by $P^2 x^{m+n}$, we get

$$\gamma^2 x^{m-n} + x^{n-m} + (\beta - \delta(x^2 + \alpha x + 1)x^n)\gamma = 0,$$

or

$$\gamma^2 x^{m-n} + x^{n-m} + \gamma\beta = \gamma\delta x^{2+n} + \gamma\delta\alpha x^{1+n} + \gamma\delta x^n.$$

Hence, $n = -1$, $\gamma = \delta = \pm 1$, thus $\beta = \pm\alpha$.

$I(\alpha)$ is turned into $I(-\alpha)$ under the automorphism of $\mathbb{C}[x, x^{-1}]$ sending x into $-x$. So, these ideals are isomorphic as algebras. $\square$

Note that for $y = x + \frac{1}{x} + \alpha$ and $z = x^2 + 1 + \alpha x \in I(\alpha)$ as above the elments $\{y^n, zy^n; n \geq 0\}$ are linearly independent. Indeed, assume that $P(y) + zQ(y) = 0$ holds for some polynomials P and Q. Then

$$0 = P\left(y\left(\frac{1}{x}\right)\right) + z\left(\frac{1}{x}\right)Q\left(y\left(\frac{1}{x}\right)\right) = P(y(x)) + z\left(\frac{1}{x}\right)Q(y(x)),$$

so $(z(x)-z(\frac{1}{x}))Q(y(x))=0$. Hence $P=Q=0$. So, $I(\alpha)$, equipped with the unity $\mathbf{1}$, is isomorphic to the coordinate ring of the affine algebraic variety (5).

Denote by J the ideal in $\mathbb{C}[x,x^{-1}]$ generated by $x+1$.

Proposition 10. *J and $I(\alpha)$ are nonisomorphic for any α.*

Proof. $y=1+\frac{1}{x}$ and $z=1+x$ generate J over $\mathbb{C}$ and $yz=y+z$. One can check that there are no elements $Y,Z\in I(\alpha)$, which generate $I(\alpha)$ over $\mathbb{C}$ and satisfy $YZ=Y+Z$ in the same way as in Proposition 9. $\square$

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