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RESEARCH ARTICLE

Representations of linear groups over $\widetilde{A}_2$-algebras

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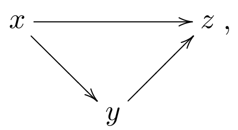
Abstract. In the space of irreducible unitary representations of a linear group over an algebra of type $\widetilde{A}_2$ an open dense subset of representations in the general position is singled out. This set is identified, up to simple direct factors, with the space of representations of a full linear group.

Let A be an algebra over the field $\mathbb{C}$ of complex numbers. A *linear group* over A is, by definition, the group $G(P, A)$ of automorphisms of a projective (finitely generated) A-module P. It is known (cf., e.g., [1]) that the classification of all unitary representations of linear groups over A is a *wild problem* provided A is not semisimple. On the other hand, in [1] for the so called *Dynkin algebras* it was shown that the dual space $\widehat{G}$, i.e. the space of irreducible unitary representations of the group $G = G(P, A)$ [3], contains an open dense subset isomorphic to $\widehat{\mathrm{GL}}(m, \mathbb{C})$ for some m. (It was called the set of "representations in general position".) For the Kronecker algebra $\widetilde{A}_1$, i.e. the path algebra of the quiver $x \rightrightarrows y$, a similar result was obtained in [5]. In this paper we consider the case of *algebras of type* $\widetilde{A}_2$, i.e. the path algebra of the quiver



which we denote by $\widetilde{A}_2$, the algebra $\widetilde{A}_2^{\tau}$ given by the quiver

$$x \underset{c}{\overset{b}{\rightrightarrows}} z \xrightarrow{a} y$$

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with relation $ac = 0$, which is tilted to the algebra $\widetilde{A}_2$ [4], and its opposite algebra $(\widetilde{A}_2^\tau)^{\mathrm{op}}$. We show that the situation is almost the same for linear groups over these algebras. It makes plausible that an analogous result is valid for *Euclidean* algebras, i.e. for path algebras of the Euclidean (or extended Dynkin) diagrams and for the algebras tilted to them.

Namely, denote by $\mathbb{C}^\times$ the multiplicative group of the field $\mathbb{C}$ and by Q_s the factor W_s/S_s, where $W_s \subset \mathbb{C}^s$ is the set $\{\,(\lambda_1,\dots,\lambda_s)\,|\,\lambda_i \neq \lambda_j \text{ if } i \neq j\,\}$ and S_s is the symmetric group acting on W_s by permutations. We shall prove the following theorem.

Theorem 1 (Main Theorem)**.** *Let $G = G(P,A)$ be a linear group over an algebra A of type $\widetilde{A}_2$. The space $\widehat{G}$ of irreducible unitary representations of G contains an open dense subset Γ isomorphic to $Q_s \times (\mathbb{C}^\times)^s \times \widehat{\mathrm{GL}}(t,\mathbb{C})$ for some s,t (possibly, $s = 0$, i.e. $\Gamma \simeq \widehat{\mathrm{GL}}(t,\mathbb{C})$, or $t = 0$, i.e. $\Gamma \simeq Q_s \times (\mathbb{C}^\times)^s$).*

As in [1], we call the representations from the set Γ the "*representations in general position.*"

Proof. Any projective module P over the algebra A, where $A = \widetilde{A}_2$ or $A = \widetilde{A}_2^\tau$, uniquely decomposes as $mA_x \oplus pA_y \oplus nA_z$, where $A_i = e_iA$ is the indecomposable projective A-module corresponding to the vertex i; e_i being the "empty" path at this vertex. We call the triple $\mathbf{d} = (m,p,n)$ the *vector dimension* of the projective module P and of the group G. Note also that a linear group over the opposite algebra A^{op} is an opposite group to G. Since any group is isomorphic to the opposite one, any linear group over $(\widetilde{A}_2^\tau)^{\mathrm{op}}$ is isomorphic to a linear group over $\widetilde{A}_2^\tau$. We set $|\mathbf{d}| = m+n+p$ and call $|\mathbf{d}|$ the *absolute dimension* of the group G. We shall prove the Main Theorem using induction by $|\mathbf{d}|$. Namely, we shall deduce it from the following lemma.

Lemma 2 (Main Lemma)**.** *Let $G = G(P,A)$, where $A = \widetilde{A}_2$ or $A = \widetilde{A}_2^\tau$. The dual space $\widehat{G}$ contains an open dense subset V isomorphic either to $Q_s \times (\mathbb{C}^\times)^s \times \widehat{\mathrm{GL}}(t,\mathbb{C})$ or to $\widehat{G'}$, where G' is a linear group of smaller absolute dimension over an algebra A' from the list $\{\widetilde{A}_2, \widetilde{A}_2^\tau, \widetilde{A}_1, A_2\}$.* (Here A_2 is the path algebra of the quiver $x \to y$.)

Proof. We follow the calculations from [1, 5]. First, let $A = \widetilde{A}_2$. The linear group G can be presented as the group of block matrices of the form

$$\begin{pmatrix} Z & B & C & D \\ 0 & Y & K & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & X \end{pmatrix},$$

where $X \in GL(m)$, $Y \in GL(p)$, $Z \in GL(n)$. We denote this group by $G(m,p,n)$. This group decomposes into the semidirect product $H \ltimes N$, where

$$N = \left\{ \begin{pmatrix} I & B & C & D \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \right\}, \quad H = \left\{ \begin{pmatrix} Z & 0 & 0 & 0 \\ 0 & Y & K & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & X \end{pmatrix} \right\}.$$

Obviously, N is an Abelian normal subgroup. Hence, we can apply the Mackey's "little" theorem [3] to calculate the representations of the group G. It gives a surjection $\pi : \widehat{G} \to \widehat{N}/H$ with slices $\pi^{-1}(\chi^H) \simeq \widehat{S}(\chi)$, where $\chi \in \widehat{N}$, $S(\chi)$ is the stabilizer of χ in H.

The space of characters $\widehat{N}$ can de identified with the dual vector space to H, which is isomorphic to the spaces of matrices of the form

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ B' & 0 & 0 & 0 \\ C' & 0 & 0 & 0 \\ D' & 0 & 0 & 0 \end{pmatrix}, \tag{1}$$

where B' is of the size $p \times n$, C' and D' are of the size $m \times n$. Namely, such a matrix F defines a character χ_F of N by the rule: $\chi_F(M) = \exp(i \operatorname{Re} \operatorname{tr}(FM))$ for $M \in N$. The action of the group H on $\widehat{N}$ correspond to its action on the matrices: if F is given by a triple (B', C', D') as in (1) and $h \in H$, then F^h is given by the triple

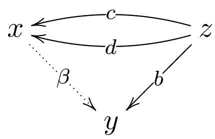
$$\big((YB' + KC')Z^{-1}, XC'Z^{-1}, XD'Z^{-1}\big). \tag{2}$$

To investigate the action (2), it is convenient to consider matrices F of the form (1) as matrices with coefficients from a bimodule U, like in [1]. Namely, U is the bimodule over the algebra $\Lambda \times \mathbb{C}$, where Λ is the set of 3×3 matrices over $\mathbb{C}$ of the form

$$\begin{pmatrix} y & k & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$$

and $U = \mathbb{C}^3$ with the natural action of Λ. Recall that a matrix with coefficients from U is, by definition, an element of $Q \otimes_\Lambda U \otimes L$, where Q is a projective Λ-module and L is a vector space. If $Q = p(e_1\Lambda) \oplus m(e_2\Lambda)$, where $e_1 = e_{11}, e_2 = e_{22} + e_{33}$ are primitive idempotents of Λ, and $\dim L = n$, we just get the matrices F from (1), and the action of $\operatorname{Aut} P \times \operatorname{Aut} L$ is then described by the formulas (2). The triple $\mathbf{dim}\, M = (m,p,n)$ will be called the *vector dimension* of the matrix F.

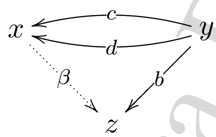
We shall describe the matrices from the bimodule U and other bimodules, which arise in the calculations, using *bigraphs* with relations. So, the bimodule U is described by the picture



$b = \beta c.$

Note that we show *all non-zero relations* near the bigraph. For instance, in the considered situation $\beta d = 0$. Here the solid arrows b, c, d describe a basis of U, while the dotted arrows describe a basis of the radical of the algebra Λ. For the matrices, it means that, except usual transformations of these matrices, corresponding to a base change, we can add any multiple of the matrix C' to the matrix B'. As a rule, we need to precise the vector dimension $\mathbf{dim}\, M = (m, p, n)$; then we shall write as follows

$\begin{pmatrix} m & & n \\ & p & \end{pmatrix}$:



$b = \beta c.$ (3)

Since we are interested in a "good" open dense subspace in the space of matrices F, we use the algorithm of *small reduction*, as in [1, 5]. It means that we reduce matrices to a normal form supposing, at every step, that the reduced matrix is of maximal possible rank. For details, as well as for the interpretation in terms of *boxes*, we refer to [1]. The result depends on the correlation between the dimensions m, n, p. We always start from the arrow c.

Case 1: $m = n$**.**

The small reduction of c glues the points x, z and kills the arrows c, b and β. So we get the picture

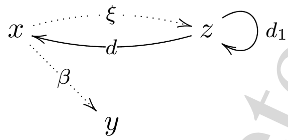
$$(m \quad p): \qquad d \circlearrowright x \qquad y\,. \tag{4}$$

The matrices over the obtained bimodule are given by two vector spaces, of dimensions m and p, and a linear operator D in the first of them. We can consider the (open dense) set W of such matrices that the operator D has m different eigenvalues. Then H-orbits from W are parameterised by the elements of Q_m (the sets of eigenvalues up to a permutation). The stabilizer of such an orbit is isomorphic to $(\mathbb{C}^\times)^m \times \mathrm{GL}(p)$. The first factor corresponds to the stabilizer of the operator D, which is the group of diagonal matrices, i.e. $(C^\times)^m$, while the second factor corresponds to the "isolated" point y of the picture (4). Thus, taking for V the preimage of W in $\widehat{G}$, we get that $V \simeq Q_m \times (\mathbb{C}^\times)^m \times \widehat{\mathrm{GL}}(p)$ (recall that $\widehat{\mathbb{C}}^\times \simeq \mathbb{C}^\times$).

Case 2: $m > n$**.**

The small reduction of c gives the picture

$\begin{pmatrix} m-n & n \\ & p & \end{pmatrix}$:



$$d_1 = \xi d. \qquad (5)$$

Now we have to reduce the arrow d. Again we have several cases.

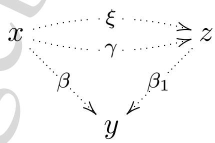
A) Let $m - n = n$. The small reduction of d glue x and z, kills d_1 and ξ, so gives the picture

$$(n \quad p) : \qquad x \overset{\beta}{\dashrightarrow} y\ .$$

Since there are no solid arrows, there is a unique matrix, and its stabilizer is described by the broken arrows. Thus it coincides with the linear group $G_0 = G_0(n, p)$ over the algebra of type A_2, and we can take for V the orbit of this matrix.

B) Let $m - n > n$. After the small reduction of the arrow d (from the vertex x to the vertex z) we obtain the picture (again with no solid arrows)

$\begin{pmatrix} m-2n & n \\ & p & \end{pmatrix}$:

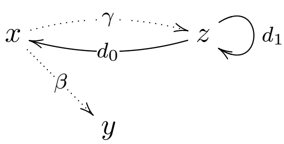


$$\beta = \beta_1 \gamma.$$

Note that, according to our agreement, it means that $\beta_1 \xi = 0$. Thus, in this case the stabilizer of the unique matrix is isomorphic to the linear group G' of vector dimension $(n, p, m - 2n)$ over the algebra $\widetilde{A}_2^\tau$.

C) Let $m - n < n$. Now the small reduction of d (from z to x) gives

$\begin{pmatrix} m-n & 2n-m \\ & p & \end{pmatrix}$:



$$d_1 = \gamma d_0.$$

It is the same as the picture (5), but with smaller dimensions. So we can repeat the same reductions, which gives the following result.

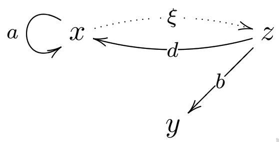
Proposition 3. *Suppose that* $m > n$*. Choose* $k \geq 1$ *so that* $(k-1)m < kn$ *and* $km \geq (k+1)n$*.* (Note that such k always exists and is unique.) *Set* $m' = kn - (k-1)m$*,* $n' = km - (k+1)n$*. Then* $\widehat{G}(m, p, n)$ *contains an open dense subset* V *isomorphic to* $\widehat{G'}$*, where* G' *is*

- *the linear group of vector dimension (m', p, n') over the algebra $\widetilde{A}_2^\tau$, if $n' \neq 0$.*
- *the linear group $G_0(m-n, p)$ of vector dimension $(m-n, p)$ over the algebra A_2, if $n' = 0$ (then $m' = m - n$).*

Case 3: $m < n$**.**

The small reduction of c gives the picture

$$\begin{pmatrix} m & n-m \\ & p \end{pmatrix} :$$

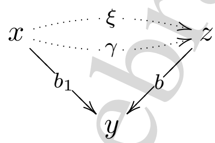


$$d_1 = d\xi$$

Calculations, quite analogous to those of Case 2, give the following result.

Proposition 4. *Suppose that $m < n$. Choose k so that $(k-1)n < km$ and $kn \geq (k+1)m$. Set $n' = km - (k-1)n$, $m' = kn - (k+1)m$. If $m' \neq 0$, $\widehat{G}(m, p, n)$ contains an open dense subset isomorphic to the set of matrices over the bimodule described by the picture*

$$\begin{pmatrix} m' & n' \\ & p \end{pmatrix} :$$

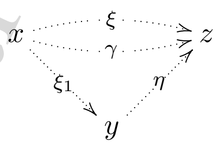


$$b_1 = b\gamma. \qquad (6)$$

If $m' = 0$ (then $n' = n - m$), $\widehat{G}(m, p, n)$ contains an open dense subset isomorphic to the linear group $G_0(n-m, p)$ over the algebra A_2.

Thus we have to consider the bimodule given by the picture (6). Here we reduce the arrow b. If $n' \geq p$, we get

$$\begin{pmatrix} m' & n'-p \\ & p \end{pmatrix} :$$

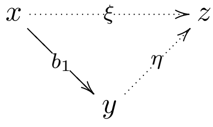


$$. \qquad \xi = \eta\xi_1.$$

Therefore the stabilizer is isomorphic to $G(m', p, n' - p)$. Especially, if $n' = p$, the vertex z vanishes and we get the group $G_0(m', p)$.

If $n' < p$, we get, setting $p' = p - n'$,

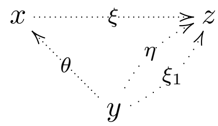
$$\begin{pmatrix} m' & n' \\ & p' \end{pmatrix} :$$



$$. \qquad (7)$$

Now, if $m' \geq p'$, the small reduction of b_1 gives

$$\begin{pmatrix} m'' & n' \\ & p' \end{pmatrix} :$$

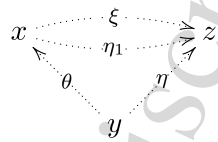


$$\xi_1 = \xi\theta,$$

where $m'' = m' + n' - p$. Therefore, the stabilizer is isomorphic to $G(p', m'', n')$. Especially, if $m' = p'$, the vertex x vanishes, so we get the linear group $G_1(p', n')$ of vector dimension (p', n') over the Kronecker algebra.

At last, if $m' < p'$, the small reduction of b_1 gives

$$\begin{pmatrix} m' & n' \\ & p' - m' \end{pmatrix} :$$



$$\eta = \eta_1\theta.$$

It describes the algebra $(\widetilde{A}_2^\tau)^{\mathrm{op}}$; thus the stabilizer is the linear group of the "opposite" vector dimension $(n', m', p' - m')$ over the algebra $\widetilde{A}_2^\tau$.

Finally, we consider representations of the linear group of vector dimension (m, p, n) over the algebra $\widetilde{A}_2^\tau$, which may be represented as a group of block matrices of the following form:

$$\begin{pmatrix} Y & 0 & K & D \\ 0 & Z & 0 & C \\ 0 & 0 & Z & B \\ 0 & 0 & 0 & X \end{pmatrix}$$

where $X \in GL(m)$, $Y \in GL(p)$, $Z \in GL(n)$. Now $G = N \ltimes H$, where

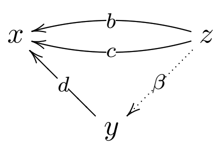
$$N = \left\{ \begin{pmatrix} I & 0 & 0 & D \\ 0 & I & 0 & C \\ 0 & 0 & I & B \\ 0 & 0 & 0 & I \end{pmatrix} \right\}, \quad H = \left\{ \begin{pmatrix} Z & 0 & K & 0 \\ 0 & Y & 0 & 0 \\ 0 & 0 & Y & 0 \\ 0 & 0 & 0 & X \end{pmatrix} \right\},$$

so the space of characters $\widehat{N}$ coincide with that of the matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ D' & C' & B' & 0 \end{pmatrix}.$$

It can be treated as matrices over the bimodule given by the picture

$$\begin{pmatrix} m & n \\ & p \end{pmatrix} :$$



$$b = d\beta.$$

We start with the small reduction of the arrow d.

Case 1′: $m = p$.

The reduction glues y with z and kills b, c and β, so that

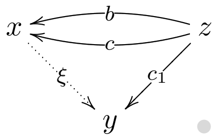
$$(m \quad n) : \qquad x \xleftarrow{\;\; c \;\;} z$$

remains, describing the dual space of the group $G_0(n, m)$.

Case 2′: $m > p$.

The reduction gives the picture

$$\begin{pmatrix} m-p & n \\ & p \end{pmatrix} :$$



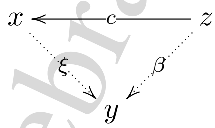
$$c_1 = \xi c,$$

which coincides with (3) up to notations.

Case 3′: $n < p$.

The reduction gives the picture

$$\begin{pmatrix} m & n \\ p-m & \end{pmatrix} :$$



,

which coincides with (7) up to notations.

It accomplishes the proof of the Main Lemma. □

Now the proof of the Main Theorem is obvious. Namely, if the algebra A' from the Main Lemma is either $\widetilde{A}_2$ or $\widetilde{A}_2^{\tau}$, we can use the induction hypothesis, which shows that V contains an open dense subset Γ of the necessary form. If $A' = \widetilde{A}_1$, we know the same from [5]. At last, if $A' = A_2$, it is known that $\widehat{G'}$ contains an open dense subset $\Gamma \simeq \widehat{\mathrm{GL}}(t)$ (cf. e.g. [1]). □

References

[1] Y. A. Drozd. *Matrix problems, small reduction and representations of a class of mixed Lie groups.* In: *Representations of Algebras and Related Topics*, Cambridge Univ. Press, 1992, pp. 225–249.

[2] Y. A. Drozd. *Reduction algorithm and representations of boxes and algebras.* Comtes Rendue Math. Acad. Sci. Canada 23 (2001) 97-125.

[3] A. A. Kirillov. *Elements of Representation Theory.* Nauka, Moscow, 1978.

[4] C. M. Ringel. *Tame Algebras and Integral Quadratic Forms.* Lecture Notes in Math. 1099, Springer-Verlag, 1984.

[5] A. S. Timoshin. *Representations of liner groups over Kronecker algebra.* Visnyk (Bulletin) of Kiev University, No. 3 (2002), 60–64.

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