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Unitarizable and non-unitarizable represenations of algebras generated by idempotents

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Abstract. The problem of unitarization of representations of algebras generated by idempotents with linear relations is studied. Construction of non-unitarizable representations for some subintervals of continuous spectrum is presented. Unitarization of representations from discrete series is proven.

1. Introduction

Representations of algebras generated by projections and idempotents have deep applications in different areas of mathematics such as algebraic geometry, topology, analysis and mathematical physics and have been studied by many authors (see [5, 3, 7, 9, 8, 3] and bibliography therein). This article is a continuation of work begun in [10, 11, 12]. It deals with the algebras $\mathcal{P}_{n,\alpha}$, $\mathcal{P}_{n,abo,\alpha}$, $\alpha \in \mathbb{R}$, and their multi-parameter generalizations. Recall that the algebras $\mathcal{P}_{n,\alpha}$ and $\mathcal{P}_{n,abo,\alpha}$ are defined in terms of idempotent generators and relations as follows:

$$\mathcal{P}_{n,\alpha} = \mathbb{C}\langle q_1, \ldots, q_n | q_j^2 = q_j, \sum_{j=1}^n q_j = \alpha e \rangle, \tag{1}$$

$$\mathcal{P}_{n,abo,\alpha} = \mathbb{C}\langle p, p_1, \ldots, p_n | p^2 = p, p_i p_j = \delta_{ij} p_j, \sum_{j=1}^n p_j = e, p_j p p_j = \alpha p_j \rangle, \tag{2}$$

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where δ_{ij} is the Kronecker delta and e is the unit element. The main questions we study in this work are: for which values of parameters the algebras have finite-dimensional representations and whether they are unitarizable, i.e. equivalent to $*$-representations, where the images of idempotent generators are selfadjoint.

The paper is organized as follows: In the second section we construct certain homomorphisms between the algebras $\mathcal{P}_{n,\alpha}$, $\mathcal{P}_{n,abo,\tau}$ and matrix algebras over them which give us functors between the categories, $\operatorname{Rep}\mathcal{P}_{n,\alpha}$, $\operatorname{Rep}\mathcal{P}_{n,abo,\tau}$, of their representations for different parameters α, τ. Using this result we prove that finite-dimensional representations of $\mathcal{P}_{n,\alpha}$ exist only for $\alpha \in \Sigma_n \cap \mathbb{Q}$, where Σ_n is a union of so called discrete spectrum points and an interval of continuous spectrum points.

In Section 3 we study unitarizability of representations of $\mathcal{P}_{n,\alpha}$. By [16] any finite-dimensional representation of $\mathcal{P}_{n,\alpha}$ for discrete spectrum point α is unitarizable. Our conjecture is that unitarizability fails for rational points in continuous spectrum. We prove that this is in fact true for all rational points in a collection of subintervals of the interval of continuous spectrum. In Section 4 we discuss a generalization $\mathcal{P}_{(\alpha^{(1)},\dots,\alpha^{(t)}),\gamma}$ of $\mathcal{P}_{n,\alpha}$. There is a connection between finding parameters for which there are finite-dimensional representations of $\mathcal{P}_{(\alpha^{(1)},\dots,\alpha^{(t)}),\gamma}$, the famous Horn's problem and semi-stable representations of quivers (see [4, 7, 9, 11]). It allows us to describe generalized dimensions of irreducible representations in terms of roots of certain Kac-Moody algebras (Theorem 5). In Section 5 we study multi-dimensional version, $\mathcal{P}_{n,\alpha_1,\dots,\alpha_n}$ and $\mathcal{P}_{n,abo,\alpha_1,\dots,\alpha_n}$, of $\mathcal{P}_{n,\alpha}$ and $\mathcal{P}_{n,abo,\alpha}$, in particular, unitarizability and generalized dimensions of their finite-dimensional representations. Here we use several known results on diagonals of matrices.

2. Discrete Fourier transform and Coxeter Functors

The main objects of this section are the algebras $\mathcal{P}_{n,\alpha}$ and $\mathcal{P}_{n,abo,\alpha}$, $\alpha \in \mathbb{R}$, introduced in the previous section. We begin with finding a sequence of homomorphisms:

$$\mathcal{P}_{n,\tau n} \xrightarrow{\psi} \mathcal{P}_{n,abo,\tau} \xrightarrow{\phi} \mathcal{P}_{n,\tau n} \otimes M_n(\mathbb{C}).$$

Let ζ be a primitive root of unity of degree n. Further on all summation indices will run over $\{1,2,\dots,n\}$. Put $\psi(q_k)$ equal to $\sum_{i,j}\zeta^{(i-j)k}p_i p p_j$ and denote $\psi(q_k)$ by F_k to shorten notation. It is easy to see that $F_k^2 = F_k$ and $\sum_{k=1}^n F_k = n\tau$. Thus ψ extends to a homomorphism $\psi : \mathcal{P}_{n,\tau n} \to \mathcal{P}_{n,abo,\tau}$. Let $S = (\overline{\zeta}^{ij})_{ij}$ and $A = S^* \operatorname{diag}(F_1,\dots,F_n) S \in M_n(\mathbb{C}) \otimes$

$\mathcal{P}_{n,abo,\tau}$. Then

$$A_{ij}=\frac{1}{n}\sum_{k=1}^{n}\zeta^{ik}F_k\overline{\zeta}^{kj}=\frac{1}{n}\sum_{k=1}^{n}\zeta^{(i-j)k}\sum_{l,r}\zeta^{(l-r)k}p_rpp_l=\sum_{l=1}^{n}p_{i-j+l}pp_l$$

Set $g_{ij}=\frac{1}{\tau}p_ipp_j$. It is easy to see that the elements g_{ij} generate $\mathcal{P}_{n,abo,\tau}$. In these new generators we have $A_{ij}=\tau\sum_l g_{i-j+l,l}$.

Define a homomorphism $\phi:\mathcal{P}_{n,abo,\tau}\to\mathcal{P}_{n,n\tau}\otimes M_n(\mathbb{C})$ putting $\phi(p_j)=e_{jj}$ and $\phi(p)=S^*\operatorname{diag}(q_1,\dots,q_n)S$. Obviously, $\phi(p_j)^2=\phi(p_j)$, and it is a routine to check that $\phi(p)^2=\phi(p)$. We have also

$$\phi(p_j)\phi(p)\phi(p_j)=\frac{1}{n}\sum\zeta^{jk}q_k\otimes e_{jj}\overline{\zeta}^{kj}=\frac{1}{n}\sum_k q_ke_{jj}=\tau\phi(p_j),$$

so that ϕ extends to a homomorphism $\phi:\mathcal{P}_{n,abo,\tau}\to\mathcal{P}_{n,n\tau}\otimes M_n(\mathbb{C})$. One can check that $\phi(g_{ij})=\frac{1}{n\tau}\sum_k\zeta^{(i-j)k}q_k\otimes e_{ij}$.

Denote by γ the homomorphism $(\psi\otimes id)\circ\phi:\mathcal{P}_{n,abo,\tau}\to\mathcal{P}_{n,\tau}\otimes M_n(\mathbb{C})$ and by ρ the homomorphism $\phi\circ\psi:\mathcal{P}_{n,n\tau}\to\mathcal{P}_{n,n\tau}\otimes M_n(\mathbb{C})$. Let $\hat{q}_t$ be the sum $\sum_k\zeta^{kt}q_k$. Formally, $(\hat{q}_1,\dots,\hat{q}_n)$ is the discrete Fourier transform of $(q_1,\dots,q_n)$. Then $\{\hat{q}_1,\dots,\hat{q}_n\}$ is again generators of $\mathcal{P}_{n,\tau}$ since $q_t=\frac{1}{n}\sum_k\overline{\zeta}^{kt}\hat{q}_k$. We have

$$\begin{aligned}\rho(\hat{q}_t)&=\sum_k\zeta^{kt}\phi(\sum_{i,j}\zeta^{(i-j)k}\tau g_{ij})=\sum_k\zeta^{kt}\tau\sum_{i,j}\zeta^{(i-j)k}\frac{1}{n\tau}\hat{q}_{i-j}\otimes e_{ij}\\&=\frac{1}{n}\sum_{k,i,j}\zeta^{(i-j+t)k}\hat{q}_{i-j}\otimes e_{ij}\end{aligned}$$

and the (i,j)-entry of $\rho(\hat{q}_t)$ is equal to $\hat{q}_{-t}$ for $j\equiv t+i\mod n$ and 0 otherwise. Thus $\rho(\hat{q}_t)=\hat{q}_{-t}\otimes U^t$ where $U\in M_n(\mathbb{C})$ is the cyclic permutation matrix $Ue_i=e_{i-1},Ue_1=e_n$ (e_j is the standard basis in $\mathbb{C}^n$). This will be used below to prove that ψ is an imbeding of $\mathcal{P}_{n,\tau n}$ into $\mathcal{P}_{n,abo,\tau}$. Here is some other formulas which can be useful later on. We have

$$\psi(\hat{q}_t)=\sum_k\zeta^{kt}\psi(q_k)=\sum_k\zeta^{kt}\sum_{i,j}\zeta^{(i-j)k}p_ipp_j=n\sum_i p_ipp_{i+t}.$$

Using this we obtain also

$$\begin{aligned}\gamma(g_{ij})&=(\psi\otimes id)(g_{ij})=(\psi\otimes id)(\frac{1}{n\tau}\sum_k\zeta^{(i-j)k}q_k\otimes e_{ij})\\&=\frac{1}{n\tau}\psi(\hat{q}_{i-j})\otimes e_{ij}=\frac{1}{n\tau}n\sum_s p_spp_{s+i-j}\otimes e_{ij}\\&=\sum_s g_{s,s+i-j}\otimes e_{ij}.\end{aligned}$$

Theorem 1. *The mapping $\psi : \mathcal{P}_{n,n\tau} \to \mathcal{P}_{n,abo,\tau}$ is injective.*

Proof. Since $\{\hat{q}_t\}_{t=1}^n$ generate $\mathcal{P}_{n,n\tau}$ there is a linear basis W of $\mathcal{P}_{n,n\tau}$ consisting of words $w = w(\hat{q}_1, \dots, \hat{q}_n)$. Since the mapping $\hat{q}_t \to \hat{q}_{-t}$ extends to an automorphism the set $W_- = \{w(\hat{q}_{-1}, \dots, \hat{q}_{-n}) | w \in W\}$ is also a basis. Thus the set $\{w \otimes U^t | w \in W_-, 0 \le t \le n-1\}$ is linearly independent. Clearly $\rho(\hat{q}_{i_1} \dots \hat{q}_{i_r}) = \hat{q}_{-i_1} \dots \hat{q}_{-i_r} \otimes U^{i_1+\dots+i_r}$ for any n-tuple of non-negative integers $i_1, \dots, i_r$. Hence $\{\rho(w)\}_{w \in W}$ is linearly independent and therefore ρ is injective. The injectivity of ψ follows from the relation $\rho = \phi \circ \psi$. $\square$

Let $\operatorname{Rep} \mathcal{P}_{n,\alpha}$ and $\operatorname{Rep} \mathcal{P}_{n,abo,\tau}$ ($\operatorname{Rep}_{fd} \mathcal{P}_{n,\alpha}$, and $\operatorname{Rep}_{fd} \mathcal{P}_{n,abo,\tau}$) be the categories of bounded (finite-dimensional) representations of $\mathcal{P}_{n,\alpha}$ and of $\mathcal{P}_{n,\alpha}$ and $\mathcal{P}_{n,abo,\tau}$ respectively. In [16] two homomorphisms $\xi : \mathcal{P}_{n,abo,\tau} \to \mathcal{P}_{n,\frac{1}{\tau}} \otimes M_n(\mathbb{C})$ and $\phi_2 : \mathcal{P}_{n,\frac{1}{\tau}} \to \mathcal{P}_{n,abo,\tau}$ are defined. In our notations $\xi(p_j) = q_j \otimes e_{jj}$, $\xi(p) = \tau \sum_{i,j} q_i q_j \otimes e_{ij}$ and $\phi_2(q_k) = \frac{1}{\tau} p p_k p$. It was proved (see [16]) that these homomorphisms induce mutually inverse equivalences of the categories $\operatorname{Rep} \mathcal{P}_{n,abo,\tau}$ and $\operatorname{Rep} \mathcal{P}_{n,\frac{1}{\tau}}$. Taking the compositions $\xi \circ \psi : \mathcal{P}_{n,n\tau} \to \mathcal{P}_{n,\frac{1}{\tau}} \otimes M_n(\mathbb{C})$ and $(\psi \otimes id) \circ \xi : \mathcal{P}_{n,abo,\tau} \to \mathcal{P}_{n,abo,\frac{n}{\tau}}$ we obtain the induced functors

$$\Pi : \operatorname{Rep} \mathcal{P}_{n,\alpha} \to \operatorname{Rep} \mathcal{P}_{n,\frac{n}{\alpha}} \text{ and } \tilde{\Pi} : \operatorname{Rep} \mathcal{P}_{n,abo,\tau} \to \operatorname{Rep} \mathcal{P}_{n,abo,\frac{n}{\tau}}.$$

Other equivalence functors

$$T : \operatorname{Rep} \mathcal{P}_{n,n-\alpha} \to \operatorname{Rep} \mathcal{P}_{n,\alpha} \text{ and } \tilde{T} : \operatorname{Rep} \mathcal{P}_{n,abo,1-\tau} \to \operatorname{Rep} \mathcal{P}_{n,abo,\tau}$$

we get by considering homomorphisms $t : \mathcal{P}_{n,\alpha} \to \mathcal{P}_{n,n-\alpha}$ and $t_{abo} : \mathcal{P}_{n,abo,\tau} \to \mathcal{P}_{n,abo,1-\tau}$ given on the generators by $t(q_i) = 1 - q_i$, $t_{abo}(p_i) = p_i$ and $t_{abo}(p) = 1 - p$. Clearly all mentioned functors map finite-dimensional representations into finite-dimensional ones.

Let Λ_n, $\Lambda_{n,fd}$, $\widetilde{\Lambda}_n$, and $\widetilde{\Lambda}_{n,fd}$ be the set of all α for which $\operatorname{Rep} \mathcal{P}_{n,\alpha}$, $\operatorname{Rep}_{fd} \mathcal{P}_{n,\alpha}$, $\operatorname{Rep} \mathcal{P}_{n,abo,\alpha}$ and $\operatorname{Rep}_{fd} \mathcal{P}_{n,abo,\alpha}$ respectively are non-empty. Then the functors Π and $\tilde{\Pi}$ generate the dynamical system $x \to \dfrac{n}{x}$ on Λ_n, $\Lambda_{n,fd}$, $\tilde{\Lambda}_n$ and $\tilde{\Lambda}_{n,fd}$. Considering the compositions $(T\Pi)^2$ and $(\Pi T)^2$ we obtain also the following maps on Λ_n and $\Lambda_{n,fd}$: $\Phi^-(\alpha) = n - 1 - \frac{1}{\alpha - 1}$ and $\Phi^+(\alpha) = 1 + \frac{1}{n-1-\alpha}$.

In order to formulate our next statement we need to recall some facts from [10] on $*$-representations of the $*$-algebras $\mathcal{P}_{n,\alpha}$ with involution defined by $q_i^* = q_i$. If no confusion arise we write also $\operatorname{Rep} \mathcal{P}_{n,\alpha}$ for the category of $*$-representations $\pi : \mathcal{P}_{n,\alpha} \to B(H)$, $B(H)$ being the $*$-algebra of bounded operators on a Hilbert space H.

Let Σ_n be the set of all α for which there exist $*$-representations in $B(H)$ of the $*$-algebra $\mathcal{P}_{n,\alpha}$. The complete description of Σ_n was obtained in [10] using the Coxeter functors on the categories $\operatorname{Rep}\mathcal{P}_{n,\alpha}$ of $*$-representations of $\mathcal{P}_{n,\alpha}$, the one similar to $(T\Pi)^2$ and $(\Pi T)^2$. They also generate the same dynamical systems on Σ_n given by the mapping Φ^+ and Φ^-. The functor T which maps $*$-representations into $*$-representations gives the mapping $\Psi(\alpha) = n-\alpha$ on Σ_n. Let Φ^{+k} denote the k-th iteration of Φ^+. In [10] it was proved that

$$\Sigma_n = \Delta_n \cup (n - \Delta_n) \cup [\beta_n, n - \beta_n],$$

where $\Delta_n = \{\Phi^{+k}(0), \Phi^{+k}(1)\}_{k\geq 0}$ and $\beta_n = \frac{n-\sqrt{n^2-4n}}{2}$. The points of Δ_n and $n - \Delta_n$ are called the discrete spectrum points, and the points of $[\beta_n, n - \beta_n]$ are called the continuous spectrum points. We denote by $\Sigma_{n,fd}$ the set of those α for which there exist finite-dimensional $*$-representations of $\mathcal{P}_{n,\alpha}$. The following theorem was announced in [16].

Theorem 2. $\Lambda_{n,fd} = \Sigma_{n,fd} = (\Delta_n \cup (n - \Delta_n) \cup [\beta_n, n - \beta_n]) \cap \mathbb{Q}$.

Proof. The last equality was proved in [10]. To show the equality of the first and the third sets we claim first that $\Lambda_{n,fd} \cap (0, 1) = \emptyset$. In fact, it is known that $trQ = \dim ImQ$ for any idempotent Q and therefore taking the trace from the both hand sides of the equality $Q_1 + \ldots + Q_n = \alpha I$ we obtain

$$\dim ImQ_1 + \ldots + \dim ImQ_n = \alpha \cdot \dim H < \dim H$$

if $0 < \alpha < 1$. This implies that $ImQ_1 + \ldots + ImQ_n$ is a proper subspace of H. On the other hand,

$$v = \frac{1}{\alpha}(Q_1 + \ldots + Q_n)v \in ImQ_1 + \ldots + ImQ_1,$$

for any $v \in H$, a contradiction.

Applying now first our functor T and then the functor Π, we obtain $\Lambda_{n,fd} \cap (n-1, n) = \emptyset$ and $\Lambda_{n,fd} \cap (1, 1 + \frac{1}{n-1}) = \emptyset$, which was a hard part of the analogous result for Σ_n. One can easily show the following inequalities:

$$0 < 1 < 1 + \frac{1}{n-1} = \Phi^+(0) < \Phi^+(1) < \Phi^+(\Phi^+(0)) < \Phi^+(\Phi^+(1)) < \ldots$$

Since there is no finite-dimensional representation for α from $(0, 1)$ and $(1, \Phi^+(0))$, applying the functor $(T\Pi)^2 : \operatorname{Rep}_{fd}\mathcal{P}_{n,\alpha} \to \operatorname{Rep}_{fd}\mathcal{P}_{n,\Phi^+(\alpha)}$ we conclude that there is no finite-dimensional representation for α from the union $\cup_{k\geq 0}(\Phi^{+k}(0), \Phi^{+k}(1)) \bigcup \cup_{k\geq 0}(\Phi^{+k}(1), \Phi^{+(k+1)}(0))$. Since, clearly, $\{0,1\} \subset \Lambda_{n,fd}$, the application of the functor $(T\Pi)^2$ gives $\Delta_n \subset \Lambda_{n,fd}$. Noting now that $\Phi^{+k}(0)$ and $\Phi^{+k}(1)$ has the limit point β_n we get $\Lambda_{n,fd} \cap [0, \beta_n) = \Delta_n$. Similar argument shows that $\Lambda_{n,fd} \cap (n-\beta_n, n] = n - \Delta_n$.

By [10, Theorem 6], $[\beta_n, n-\beta_n] \cap \mathbb{Q} \subset \Sigma_{n,fd}$ implying $[\beta_n, n-\beta_n] \cap \mathbb{Q} \subset \Lambda_{n,fd}$. To finish the proof we just note that $\Lambda_{n,fd}$ can not contain irrational points. This can be seen by taking the trace from the both hand sides of the equality $Q_1 + Q_2 + \ldots + Q_n = \alpha I$. $\square$

The question when the sum of idempotent bounded operators on a Hilbert space (not necessarily finite-dimensional) is a multiple of the identity operator, was studied in [15]: for each $\alpha \in \mathbb{C}$ there exist at most five idempotents whose sum is αI, i.e. $\Lambda_n = \mathbb{C}$ if $n \geq 5$ while sum of one, two, three and four idempotents can be αI if and only if $\alpha \in \Sigma_1$, Σ_2, Σ_3 and Σ_4 respectively.

3. Non unitarizable representations on continuous spectrum

In this section we will investigate the question whether each n-tuple of idempotents $(Q_1, \ldots, Q_n)$ on a finite-dimensional space with the sum equal to αI is similar to an n-tuple of projections $(P_1, \ldots, P_n)$, i.e., whether there exists an invertible matrix S such that

$$S^{-1} Q_i S = P_i, \qquad i = 1, \ldots, k.$$

This is equivalent to the question whether there exists an equivalent Hermitian form on the finite-dimensional space such that the representation π of $\mathcal{P}_{n,\alpha}$, $\pi(q_i) = Q_i$, $1 \leq i \leq n$, is a $*$-representation with respect to this Hermitian form, in this case we say that the representation π is unitarizable. As it follows from Theorem 2 finite-dimensional representations of $\mathcal{P}_{n,\alpha}$ exist only for those α for which there exists a finite-dimensional $*$-representation of $\mathcal{P}_{n,\alpha}$. The following theorem was proved in [16].

Theorem 3. *If $\alpha \in \Sigma_n \cap \mathbb{Q}$ for $n = 1, 2, 3$ and $\alpha \in (\Delta_n \cup (n - \Delta_n)) \cap \mathbb{Q}$ for $n \geq 4$, then any finite-dimensional representation of the algebra $\mathcal{P}_{n,\alpha}$ is unitarizable.*

For $\alpha = 2$, $k = 4$ it follows from [2] that in each space of dimension $7k$, $k \in \mathbb{N}$, there exist indecomposable n-tuples of idempotents with sum equal to αI while any indecomposable (=irreducible) n-tuple of such projections act either in one- or two-dimensional spaces (see [17]). Therefore, there exist non-unitarizable finite-dimensional representations of the algebra $\mathcal{P}_{4,2}$.

Conjecture. For any $\alpha \in [\beta_n, n-\beta_n] \cap \mathbb{Q}$ there exist idempotent matrices $Q_1, \ldots, Q_n$ which are not similar to any n-tuple of projection matrices and such that $Q_1 + \ldots + Q_n = \alpha I$.

In support of this conjecture we have the following

Proposition 1. *Let $I_n = [2+\beta_{n-2}/2, n-2-\beta_{n-2}/2]$, $\Psi(\alpha) = n-\alpha$. Then for any $n \geq 9$ and any $\alpha \in (\cup_{k\geq 0}\Phi^{+k}(I_n) \cup \Psi(\Phi^{+k}(I_n))) \cap \mathbb{Q}$ there exists a non-unitarizable representation of the algebra $\mathcal{P}_{n,\alpha}$.*

Proof. We will use the fact that there exist non-unitarizable quadruple of idempotents Q_1, Q_2,Q_3, Q_4 in $\mathbb{C}^{7k}$, $k \in \mathbb{N}$ such that $Q_1+Q_2+Q_3+Q_4 = 2I$. Clearly, for each $\delta \in \Sigma_{n-4} \cap \mathbb{Q}$ we can find $n-4$ projections P_k with the sum equal to δI in one of the spaces $\mathbb{C}^{7k}$ by taking if necessary a direct sum of the same projections. Then the n-tuple $\{Q_i,\ 1 \leq i \leq 4,\ P_i,\ 1 \leq i \leq n-4\}$ is non-unitarizable and $\sum_{i=1}^{4} Q_i + \sum_{i=1}^{n-4} P_i = (\delta+2)I$. We have therefore non-unitarizable representations of $\mathcal{P}_{n,\alpha}$ for $\alpha \in [2+\beta_{n-4}, n-2-\beta_{n-4}]$.

Next we search for idempotents P_i, $1 \leq i \leq n$, in $\mathbb{C}^l \oplus \mathbb{C}^l$, $l = 7k$, in the form $P_i = Q_i \oplus R_i$, $1 \leq i \leq 4$, $P_i = 0_l \oplus R_i$, $5 \leq i \leq n-2$ and

$$P_{n-1} = \begin{pmatrix} \tau I_l & \sqrt{\tau(1-\tau)} I_l \\ \sqrt{\tau(1-\tau)} I_l & (1-\tau) I_l \end{pmatrix},$$

$$P_n = \begin{pmatrix} \tau I_l & -\sqrt{\tau(1-\tau)} I_l \\ -\sqrt{\tau(1-\tau)} I_l & (1-\tau) I_l \end{pmatrix},$$

such that $\{Q_i\}_{i=1}^4$ is a non-unitarizable quadruple of idempotents with the sum equal to $2I_l$ and $\sum_{i=1}^{n} P_i = \alpha I_{2l}$. This forces $2\tau = \alpha - 2$ and R_i, $1 \leq i \leq n-2$, to be idempotents in $\mathbb{C}^l$ satisfying

$$\sum_{i=1}^{n-2} R_i = (\alpha - 2(1-\tau)) I_l = (2\alpha - 4) I_l.$$

Such idempotent matrices R_i exist for some $k \in \mathbb{N}$ if $2\alpha - 4 \in [\beta_{n-2}, n-2-\beta_{n-2}]$, i.e, $\alpha \in [2+\beta_{n-2}/2, 1+(n-\beta_{n-2})/2]$. Then for P_{n-1}, P_n to be idempotents we have $\tau \in [0,1]$ implying also $\alpha \in [2,4]$.

We claim now that for $\alpha \in [2+\beta_{n-2}/2, 1+(n-\beta_{n-2})/2] \cap [2,4] \setminus \{3\} = [2+\beta_{n-2}/2, 4] \setminus \{3\}$ the corresponding n-tuple $\{P_i\}_{i=1}^n$ is not equivalent to an n-tuple of projections. In fact, assuming that there exists an invertible matrix C such that $(CP_iC^{-1})^* = CP_iC^{-1}$, $1 \leq i \leq n$, we obtain from the equality for $i = n-1$, n that $C^*C = C_1 \oplus C_2$. Then the equality for $1 \leq i \leq 4$ gives $Q_i^* C_1 = C_1 Q_i$ and $(\sqrt{C_1} Q_i \sqrt{C_1}^{-1})^* = \sqrt{C_1} Q_i \sqrt{C_1}^{-1}$, a contradiction. For $\alpha = 3$ we construct a non-unitarizable n-tuple by taking $P_i = Q_i$, $1 \le i \le 4$, $P_4 = I$ and $P_i = 0$ for $6 \le i \le n$. Thus we have non-unitarizable representations of $\mathcal{P}_{n,\alpha}$ for

$$\alpha \in [2+\beta_{n-2}/2, 4] \cup [2+\beta_{n-4}, n-2-\beta_{n-4}] = [2+\beta_{n-2}/2, n-2-\beta_{n-4}].$$

Applying now the Coxeter functors corresponding to Φ^+, Φ^-, Ψ which map non-unitarizable representations into non-unitarizable (see [16, Theorem 6]), we obtain the statement. □

We say that an interval $I_1 = [a, b]$ is less than an interval $I_2 = [c, d]$ and write $I_1 < I_2$ if $b < c$. A simple calculation shows that the intervals from the proposition above satisfy $\Phi^{+(k+1)}(I_n) < \Phi^{+k}(I_n)$ and $\Psi(\Phi^{+k})(I_n) < \Psi(\Phi^{+(k+1)})(I_n)$. There are still a lot of points left where we do not know the non-unitarizability, e.g. the length of the interval between $\Phi^+(I_n)$ and I_n is greater than 5/6 and goes to 5/6 as $n \to \infty$.

Theorem 4. *The functor* $\Pi : \operatorname{Rep} \mathcal{P}_{n,\alpha} \to \operatorname{Rep} \mathcal{P}_{n,\frac{n}{\alpha}}$ *maps unitarizable representations into unitarizable one.*

Proof. It follows from the fact that the functor Π maps $*$-representations of $\mathcal{P}_{n,\alpha}$ into $*$-representations of $\mathcal{P}_{n,\frac{n}{\alpha}}$. □

Note that if we knew that the functor Π maps non-unitarizable representations into non-unitarizable ones, we could prove the existence of non-unitarizable representations for each continuous spectrum point α if $n \ge 9$.

4. Algebras generated by idempotents with orthogonality condition

In this section we consider a generalization of the algebras considered in the previous sections. The famous Horn's problem and its variations (see[4, 7, 9, 11]) can be stated in terms of representations of $*$-algebras. Let $\alpha^{(s)} = (\alpha_1^{(s)}, \alpha_2^{(s)}, \ldots, \alpha_{d_s}^{(s)})$, $1 \le s \le t$, be vectors with real strictly increasing coefficients. Let us define the following algebra (see [16]):

$$\mathcal{P}_{(\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(t)}),\gamma} = \\ = \mathbb{C}\langle \{p_j^{(s)}\}_{1\le s\le t, 1\le j\le d_s} | p_i^{(s)} p_j^{(s)} = \delta_{ij} p_i^{(s)}, \sum_{s=1}^{t} \sum_{j=1}^{d_s} \alpha_j^{(s)} p_j^{(s)} = \gamma e\rangle.$$

This is a $*$-algebra if we require all generators to be self-adjoint. We use $\mathcal{P}_{(\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(t)})}$ as a shorthand of $\mathcal{P}_{(\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(t)}),1}$. In [11] the following sets are considered $T_{(d_1,\dots,d_t)} = \{(\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(t)}, \gamma) | \alpha^{(s)} \in \mathbb{R}^{d_s}, \gamma \in \mathbb{R}\}$ and

$$W_{(d_1,\dots,d_t)} = \{((\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(t)}, \gamma) \in T_{(d_1,\dots,d_t)} | \\ \textit{there is a representation of } \mathcal{P}_{(\alpha^{(1)},\alpha^{(2)},\dots,\alpha^{(t)}),\gamma}\}$$

dependent only on $(d_1, \dots, d_t)$. With an integer vector $(d_1, \dots, d_t)$ we associate a non-oriented tree with t rays of length $d_1, \dots, d_t$ correspondingly coming from a single root. Further on we will denote W by $W(G)$ and $T_{(d_1,\dots,d_t)}$ by $T(G)$, where G is such a tree. In [11, 14] an equivalence of the category of $*$-representations of $\mathcal{P}_{(\alpha^{(1)},\alpha^{(2)},\dots,\alpha^{(t)}),1}$ and a subcategory of locally-scalar graph representations (see [13]) has been established and in the case of a Dynkin graph G a complete description of $W(G)$ as well as an algorithm for writing out all irreducible $*$- representations were obtained.

In [4] a connection of Horn's problem with semi-stable quiver representations has been established. It allows to reformulate the problem of finding parameters, for which there are representations (not necessarily $*$-representations) of $\mathcal{P}_{(\alpha^{(1)},\alpha^{(2)},\dots,\alpha^{(t)})}$, in terms of root systems of Kac-Moody Lie algebras. Here we will describe a connection between representations of the algebra $\mathcal{P}_{(\alpha^{(1)},\alpha^{(2)},\dots,\alpha^{(t)}),\gamma}$ and semi-stable quiver representations in a way convenient for our application.

We denote by G also a quiver obtained from the graph G by orienting all arrows toward the root. Recall that a semi-stable representation of the quiver G is given by vector spaces H_v and complex numbers λ_v for each vertex v, and by linear maps $a : H_v \to H_w$ and $a^* : H_w \to H_v$ for each arrow $a : v \to w$ in G such that the linear maps satisfy

$$\sum_{a\in G,\ h(a)=v} aa^* - \sum_{a\in G,\ t(a)=v} a^*a = \lambda_v I$$

for each vertex v. Here $t(v)$ and $h(v)$ denote the tail and the head vertices of the arrow a.

Let π be a $*$-representation of the algebra $\mathcal{P}_{(\nu^{(1)},\dots,\nu^{(t)}),\sigma}$ in a space H of dimension n. Let $P_j^{(s)} = \pi(p_j^{(s)})$. Without loss of generality we can assume that $\nu_{d_s}^{(s)} = 0$ for all $1 \le s \le t$. Let $R_j^{(s)} = P_1^{(s)} + \dots + P_j^{(s)}$ (a projection), $H_j^{(s)} = \operatorname{Im} R_j^{(s)}$ and let $\Gamma_j^{(s)} : H_j^{(s)} \to H$ be the natural isometries. Then, in particular, $\Gamma_j^{(s)*}\Gamma_j^{(s)} = I_{H_j^{(s)}}$, and $\Gamma_j^{(s)}\Gamma_j^{(s)*} = R_j^{(s)}$. Denote by $V_j^{(s)}$ the operator $\Gamma_{j+1}^{(s)*}(\sum_{i=1}^{j} \sqrt{\nu_i^{(s)} - \nu_{j+1}^{(s)}} P_i^{(s)})\Gamma_j^{(s)} : H_j^{(s)} \to$

$H^{(s)}_{j+1}$, where $1 \leq j \leq d_s - 1$. Then

$$V_j^{(s)*}V_j^{(s)} - V_{j-1}^{(s)}V_{j-1}^{(s)*} = (\nu_j^{(s)} - \nu_{j+1}^{(s)})I_{H_j^{(s)}},$$
$$V_1^{(s)*}V_1^{(s)} = (\nu_1^{(s)} - \nu_2^{(s)})I_{H_1^{(s)}},$$
$$V_{d_s-1}^{(s)}V_{d_s-1}^{(s)*} = A^{(s)}.$$

Here $A^{(s)} : H \to H$ is a self-adjoint operator $A^{(s)} = \sum_{i=1}^{d_s} \nu_i^{(s)} P_i^{(s)}$. Moreover, $A^{(1)} + \ldots + A^{(t)} = \sigma I_H$. The operators $V_j^{(s)}$ together with their conjugate give rise to a semi-stable representation of the quiver G.

This correspondence is in fact an equivalence functor between the category of $*$-representations of algebra $\mathcal{P}_{(\nu^{(1)},\ldots,\nu^{(t)}),\sigma}$ and the subcategory of semi-stable $*$-representations of the corresponding quiver (the proof is similar to one in [5]).

Below we show how to obtain a description of all possible generalized dimensions of representations of the algebra $\mathcal{P}_{(\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(t)}),\gamma}$ using the results from [5]. Note that to single out which of them occur as dimensions of $*$-representations is another problem(see [16, 11]).

An n-tuple of diagonalizable matrices A_j such that $A_1 + \ldots + A_n = 0$ gives rise to a representation of the algebra $\mathcal{P}_{(\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(t)}),0}$, where $\alpha^{(j)}$ is the spectrum of A_j counted with multiplicities. Indeed, $A_j = \sum_k \alpha_k^{(j)} P_k^{(j)}$ where $I = P_1^{(j)} + \ldots + P_{d_j}^{(j)}$ is a decomposition of the identity into pairwise orthogonal idempotents. These idempotents form a representation of $\mathcal{P}_{(\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(t)}),0}$. Conversely, having a representation $P_k^{(j)} = \pi(p_k^{(j)})$ of the algebra $\mathcal{P}_{(\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(t)}),\gamma}$ we can form diagonalizable operators summing up to zero as follows $A_1 = \sum_{k=1}^{d_1} \alpha_k^{(1)} P_k^{(1)} - \gamma I$, $A_s = \sum_{k=1}^{d_s} \alpha_k^{(s)} P_k^{(s)}$ for $1 < j \leq t$. Theorem 1 [5] describes possible spectra of diagonalizable matrices summing up to zero. To formulate next result we need to recall some notations from [5].

Consider a graph G with vertices $I = \{0\} \cup \{[i,j] | 1 \leq i \leq t; 1 \leq j \leq d_i\}$. Let C be a generalized Cartan matrix with columns indexed by elements of I such that $C_{v,v} = 2$ for $v \in I$ and $C_{u,v} = -1$ for $u \neq v$ if u and v are connected by an edge. Let R be the root system of Kac-Moody Lie algebra with Cartan matrix C. If $\lambda \in \mathbb{C}^I$, then R_λ^+ will denote the set of positive roots $\alpha \in \mathbb{N}^I$ such that $\lambda\alpha = \sum_{v \in I} \lambda_v \alpha_v = 0$. Set $p(\alpha) = 1 - (1/2)\alpha^T C \alpha \in \mathbb{Z}$. Remark that $p(\alpha) = 0$, if α is a real root.

In [5] Σ_λ is defined as the set of all $\alpha \in R_\lambda^+$ such that $p(\alpha) > p(\beta^{(1)}) + p(\beta^{(2)}) + \ldots$, for any decomposition $\alpha = \beta^{(1)} + \beta^{(2)} + \ldots$ as a sum of two or more elements of R_λ^+. Put $d = \sum_{s=1}^t d_s$. If G is a Dynkin graph then all roots are real, so $p(\alpha) = 0$. Thus Σ_λ is the set of all $\alpha \in R_\lambda^+$ for which there is no decomposition $\alpha = \beta^{(1)} + \beta^{(2)} + \ldots$ as a sum of two or more elements of R_λ^+. Let $\overline{d} = (d_j^{(s)}) \in \mathbb{N}^d$ where $1 \le s \le t$; $1 \le j \le d_s$ be a vector such that $n := \sum_{j=1}^{d_s} d_j^{(s)}$ is independent of $1 \le s \le t$. Put $\widetilde{\alpha}[s,r] = n - \sum_{j=1}^{r} d_j^{(s)}$, $1 \le r \le d_s - 1$, $\widetilde{\alpha}_0 = n$ and let $\lambda[s,j] = \alpha_j^{(s)} - \alpha_{j+1}^{(s)}, 1 \le j \le d_s - 1$, $\lambda_0 = \gamma - (\alpha_1^{(1)} + \alpha_1^{(2)} + \ldots + \alpha_1^{(t)})$. We will write $\widetilde{\alpha} = \widetilde{\alpha}(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(t)})$ and $\lambda = \lambda(d^{(1)}, \ldots, d^{(t)})$ to emphasize the dependence of the parameters. As a corollary of [5, Theorem 1] we obtain the following

Theorem 5. *Algebra $\mathcal{P}_{(\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(t)}),\gamma}$ has an irreducible representation in generalized dimension $\overline{d}$ if and only if $\widetilde{\alpha}(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(t)}) \in \Sigma_{\lambda(d^{(1)},\ldots,d^{(t)})}$.*

5. Multi-parameter algebras

One of the interesting subfamilies of the family of algebras considered in the previous section is obtained when $d_1 = d_2 = \ldots = d_t = 1$. $*$-Representations of these algebras has been investigated in [12]. The corresponding Coxeter functors are constructed in [12, 16]. They give rise to certain dynamical systems on the set of parameters $(\alpha_1, \ldots, \alpha_n)$. The corresponding mapping are called multi-parameter Coxeter maps. Unlike the one-parameter algebras neither the complete set of parameters for which representations exist no classification of such representations in a "discrete case" is obtained yet. Let us write these algebras and their "*abo*" analogs explicitly:

$$\mathcal{P}_{n,abo,(\alpha_1,\ldots,\alpha_n)} = \mathbb{C}\langle q_1, \ldots, q_n, q | q_j^2 = q_j, q^2 = q, \sum_{j=1}^n q_j = e, \\ q_i q_j = \delta_{ij} q_i, q_j q q_j = \alpha_j q_j \text{ for all } j = 1, \ldots, n\rangle.$$

and

$$\mathcal{P}_{n,(\alpha_1,\ldots,\alpha_n)} = \mathbb{C}\langle p_1, \ldots, p_n | \sum_{k=1}^n \alpha_k p_k = e, p_j^2 = p_j \text{ for all } j = 1, \ldots, n\rangle.$$

Further on we will assume that $\alpha_j \neq 0$ for all $j = 1, \ldots, n$. These are $*$-algebras if we assume all generators to be self-adjoint. If π is a representation of $\mathcal{P}_{n,(\alpha_1,\ldots,\alpha_n)}$ on a space H then the vector $(d; d_1, \ldots, d_n)$ where $d = \dim H$, $d_j = \operatorname{rank} \pi(p_j)$ will be called the generalized dimension of π. Similarly, if π is a representation of $\mathcal{P}_{n,abo,(\alpha_1,\ldots,\alpha_n)}$ on H then the vector

$(d; d_1, \ldots, d_n)$ where $d = \operatorname{rank} \pi(q)$, $d_j = \operatorname{rank} \pi(q_j)$ will be called the generalized dimension of π.

We are going to give an explicit construction of a non-unitarizable representations of $\mathcal{P}_{n,abo,(\tau_1,\ldots,\tau_n)}$ when $\sum_j \tau_j = n/2$ and n is even.

For this consider for $2 \le k \le n/2$ the following vectors

$$v_k = (\frac{(\tau_1 - 1)(\tau_k - 1)}{\tau_1}, \tau_k - 1, \tau_k - 1, \ldots, \tau_k - 1, \tau_k, \tau_k - 1, \ldots, \tau_k - 1)$$

(here τ_k occurs on the k-th place). For $m \in \{1, n/2 + 2, n/2 + 3, \ldots, n\}$ set $v_m = \tau_m(1, 1, \ldots, 1)$ and

$$v_{n/2+1} = (y, \tau_{n/2+1} - 1, \ldots, \tau_{n/2+1} - 1, \tau_{n/2+1}, \ldots, \tau_{n/2+1})$$

(here $\tau_{n/2+1}$ for the first time occurs on the $n/2 + 1$ place), where

$$y = (1 - \tau_1) - \sum_{j=n/2+2}^{n} \tau_j + \frac{1 - \tau_1}{\tau_1}(1 - n/2 + \sum_{j=2}^{n/2} \tau_j).$$

Consider the matrix P with columns $v_1^T, \ldots, v_n^T$, i.e.

$$\begin{pmatrix}
\tau_1 & \frac{(\tau_1-1)(\tau_2-1)}{\tau_1} & \frac{(\tau_1-1)(\tau_3-1)}{\tau_1} & \ldots & \frac{(\tau_1-1)(\tau_{n/2}-1)}{\tau_1} & y & \tau_{n/2+2} & \ldots & \tau_n \\
\tau_1 & \tau_2 & \tau_3 - 1 & \ldots & \tau_{n/2} - 1 & \tau_{n/2+1} - 1 & \tau_{n/2+2} & \ldots & \tau_n \\
\tau_1 & \tau_2 - 1 & \tau_3 & \ldots & \tau_{n/2} - 1 & \tau_{n/2+1} - 1 & \tau_{n/2+2} & \ldots & \tau_n \\
 & & & & \ldots & & & & \\
\tau_1 & \tau_2 - 1 & \tau_3 - 1 & \ldots & \tau_{n/2} - 1 & \tau_{n/2+1} & \tau_{n/2+2} & \ldots & \tau_n
\end{pmatrix}$$

Proposition 2. *Let $\sum_j \tau_j = n/2$, $\tau_j \neq 1, \tau_j \neq 0$ $(1 \le j \le n)$ and let n be even. Then the mapping $p_j \to e_{jj}, p \to P$ extends to a homomorphism $\pi : \mathcal{P}_{n,abo,\tau_1,\ldots,\tau_n} \to M_n(\mathbb{C})$. Moreover, if $\langle \cdot, \cdot \rangle$ is a sesquilinear form such that $\langle \pi(x)u, v \rangle = \langle u, \pi(x^*)v \rangle$ for all $x \in \mathcal{P}_{n,abo,\tau_1,\ldots,\tau_n}$ and all vectors u, v then $\langle \cdot, \cdot \rangle$ is zero.*

Proof. It is a routine to check that P is an idempotent. Let C be the matrix of the sesquilinear form $\langle \cdot, \cdot \rangle$ in the standard basis then the condition $\langle \pi(p_j)u, v \rangle = \langle u, \pi(p_j)v \rangle$ (which is equivalent to $Ce_{ii} = e_{ii}^T C$ for all $1 \le i \le n$) implies that C is a diagonal matrix $\operatorname{diag}(c_1, c_2, \ldots, c_n)$. We are left with the condition $\langle \pi(p)u, v \rangle = \langle u, \pi(p)v \rangle$ which is equivalent to $CP = P^T C$. Considering the second rows of these two matrices we obtain

$$\begin{aligned}
c_3 &= \frac{c_2(\tau_3 - 1)}{\tau_2 - 1}, \ c_4 = \frac{c_2(\tau_4 - 1)}{\tau_2 - 1}, \\
c_5 &= \frac{c_2(\tau_5 - 1)}{\tau_2 - 1}, \ldots, c_{\frac{n}{2}+1} = \frac{c_2(\tau_{\frac{n}{2}+1} - 1)}{\tau_2 - 1}, \\
c_{\frac{n}{2}+2} &= \frac{c_2(\tau_{\frac{n}{2}+2})}{\tau_2 - 1}, \ldots, c_n = \frac{c_2 \tau_n}{\tau_2 - 1}.
\end{aligned} \tag{3}$$

Considering the $(\frac{n}{2}+1, n)$-entry of these matrices we obtain $c_n\tau_{\frac{n}{2}+1} = c_{\frac{n}{2}+1}\tau_n$ and using (3) we get $\frac{c_2\tau_n}{\tau_2-1}\tau_{\frac{n}{2}+1} = \frac{c_2(\tau_{\frac{n}{2}+1}-1)}{\tau_2-1}\tau_n$. This implies $c_2 = 0$. Equations (3) then show that $c_j = 0$ for all $2 \le j \le n$. Considering the $(1,2)$-entry we obtain $c_1\frac{(\tau_1-1)(\tau_2-1)}{\tau_1} = 0$ thus c_1 is also zero. So $C = 0$. □

Note that putting $\tau_j = 1/2$ we obtain a non-unitarizable representation of the algebra $\mathcal{P}_{2n,abo,1/2}$ and hence (applying the functor generated by the homomorphism ϕ_2 from Section 2) of the algebra $\mathcal{P}_{2n,2}$ in dimension n.

It is worth mentioning that the number $A = \sum_{i=1}^n \alpha_i$ changes under multi-parameter Coxeter map by the rule $A \to \Phi^{\pm}(A)$, where Φ^+ and Φ^- are the one-parameter Coxeter maps. This justifies the following definition:

Definition 1. *A point $\alpha \in \mathbb{C}\setminus\{0\}$ will be called n-universal if for each $\alpha_1, \dots \alpha_n \in \mathbb{C}\setminus\{0\}$ with $\sum_{j=1}^n \alpha_j = \alpha$ the algebra $\mathcal{P}_{n,(\alpha_1,\dots,\alpha_n)}$ has a finite dimensional representation.*

Theorem 6. *If $\alpha \in \mathbb{R}$ is n-universal then $\alpha \in \Sigma_n \cap \mathbb{Q}$.*

Proof. If $\alpha \in \mathbb{R}$ is n-universal then $\mathrm{Rep}_{fd}\,\mathcal{P}_{n,\{\frac{\alpha}{n},\dots,\frac{\alpha}{n}\}} \neq \emptyset$. Obviously, in this case $\mathrm{Rep}_{fd}\,\mathcal{P}_{n,\frac{n}{\alpha}} \neq \emptyset$. Since the categories $\mathrm{Rep}_{fd}\,\mathcal{P}_{n,\frac{n}{\alpha}}$ and $\mathrm{Rep}_{fd}\,\mathcal{P}_{n,abo,\frac{\alpha}{n}}$ are equivalent we have $\frac{\alpha}{n} \in \widetilde{\Lambda}_{n,fd} = \widetilde{\Sigma}_{n,fd} = \frac{1}{n}\Sigma_n \cap \mathbb{Q}$ (for the last two equalities see [16]). Thus $\alpha \in \Sigma_n \cap \mathbb{Q}$. □

Theorem 7. *If $\alpha \in \mathbb{C}\setminus\{0\}$ is such that there exist $\alpha_1, \dots \alpha_n \in \mathbb{C}\setminus\{0\}$ with $\sum_{j=1}^n \alpha_j = \alpha$ and $\mathcal{P}_{n,abo,\{\alpha_1,\dots\alpha_n\}}$ has a finite dimensional representation in generalized dimension $(1,1,\dots,1)$ then α is n-universal.*

Proof. From the theorem of Fillmore [6] we know that a matrix C is similar to a matrix with the main diagonal $(\beta_1, \dots \beta_n)$ if and only if $\sum_{j=1}^n \beta_j = \mathrm{Tr}\, C$. The condition that $\mathcal{P}_{n,abo,\{\alpha_1,\dots,\alpha_n\}}$ has a finite-dimensional representation in generalized dimension $(1,1,\dots,1)$ is equivalent to the existence of an idempotent $n\times n$ matrix Q with the main diagonal $(\alpha_1, \dots \alpha_n)$. By the mentioned above theorem for any $(\beta_1, \dots, \beta_n)$ with $\sum_{j=1}^n \beta_j = \alpha$ there is a matrix F similar to Q (hence an idempotent) with the main diagonal $(\beta_1, \dots, \beta_n)$. □

Introduced above notion of universal point is appropriate only for the category of all finite dimensional representations. If we restrict ourselves to $*$-representations then no reasonable candidate is known since the possible vectors of parameters are subject to special linear inequalities.

But in the case of generalized dimension $(1, 1, \ldots, 1)$ the answer follows from theorems of Schur and Horn (see [1],p.304). Let $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{R}^n$ and let $\mathcal{O}_\alpha$ denote the set of Hermitian matrices with eigenvalues α. Then the image of the map $\Phi : \mathcal{O}_\alpha \to \mathbb{R}^n$ that takes a matrix to its diagonal is a convex polyhedron whose vertices are $n!$ permutations of α. Let us denote this polyhedron by Π_α.

Theorem 8. *The $*$-algebra $\mathcal{P}_{n,(\alpha_1,\ldots,\alpha_n)}$ has a finite dimensional $*$-representation in generalized dimension $(r; 1, 1, \ldots, 1)$ iff*

$$(\alpha_1, \ldots, \alpha_n) \in \Pi_{(1,1,\ldots,1,0,0,\ldots,0)}$$

(the subscript vector consists of r 1's and $n - r$ 0's.)

Proof. Since the categories of $*$-representations of the algebras $\mathcal{P}_{n,(\alpha_1,\ldots,\alpha_n)}$ and $\mathcal{P}_{n,abo,(\alpha_1,\ldots,\alpha_n)}$ are equivalent (see [16]) and any representation of $\mathcal{P}_{(\alpha_1,\ldots,\alpha_n)}$ with generalized dimension vector $(r; 1, 1, \ldots, 1)$ under this equivalence goes to a representation of $\mathcal{P}_{n,abo,(\alpha_1,\ldots,\alpha_n)}$ with the same generalized dimension the $*$-algebra $\mathcal{P}_{n,(\alpha_1,\ldots,\alpha_n)}$ has a finite dimensional $*$-representation in generalized dimension $(r; 1, 1, \ldots, 1)$ if and only if there is an $n \times n$ projection matrix P of rank r with the main diagonal $(\alpha_1, \ldots, \alpha_n)$. This happens exactly when $(\alpha_1, \ldots, \alpha_n) \in \Pi_{(1,1,\ldots,1,0,0,\ldots,0)}$. □

References

[1] Rajendra Bhatia Linear Algebra to Quantum Cohomology: The Story of Alfred Horn's Inequalities. Amer. Math. Monthly 108 (2001), no. 4, 289–318.

[2] V.M. Bondarenko, On certain wild algebras generated by idempotents. Methods Funct. Anal. Topology 5 (1999), no. 3, 1–3.

[3] A. Böttcher, I. Gohberg, Yu. Karlovich, N. Krupnik, S. Roch, B.Silbermann, Banach algebras generated by N idempotents and application. *Operator theory: Adv. and Appl.*, **90**, Birkhäuser, Verlag, Basel, (1996), 19–54.

[4] W. Crawley-Boevey and Ch. Geiss, Horn's problem and semi-stability for quiver representations – Representations of Algebras.-V.I. Poc. IX Intern. Conf. Benjing 21 aug.-1 sept.2000, Red. T.Happel, V.B. Zhong Benjing, (2002), 40-48.

[5] W. Crawley-Boevey, On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero, Duke Math. J. **118** (2003), no.2, 339-352.

[6] P.A.Fillmore, On similarity and the diagonal of a matrix, Amer. Math. Monthly **76** (1969), 167-169.

[7] W.Fulton, Eigenvalues, invariant factors, highest weights and Schubert calculus, Bull. Amer. Math. Soc. **37** (2000), 209-249.

[8] V. Jones, Hecke algebra representations of braid groups and link polynomials, Annal. Math., **126** (1987) 335-388.

[9] A. A. Klyachko, Stable bundles, representation theory and Hermitian operators, Sel. Math., New ser. **4** (1998), 419 - 445.

[10] S.A.Kruglyak, V.I.Rabanovich, Yu.S.Samoĭlenko, On sums of projections, (Russian) Funktsional. Anal. i Prilozhen. 36 (2002), no. 3, 20–35, 96; translation in Funct. Anal. Appl. 36 (2002), no. 3, 182–195.

[11] S. Krugljak, S. Popovych, Yu. Samoĭlenko, Representations of *-algebras associated with Dynkin graphs and Horn's problem, Uchenye zapiski TNU (2003), seria matem., meh., kiber., **16**, no.2, 132-139.

[12] S.A. Kruglyak, Coxeter functors for a certain class of $*$-quivers and $*$-algebras, Methods of Funct. Annal. and Topology, **8** (2002), no.4, 49-58.

[13] S.A.Kruglyak, A.V.Roiter, Locally-scalar representations of graphs in the category of Hilbert spaces. Preprint.

[14] S. Kruglyak, S. Popovych, Yu. Samoĭlenko, Spectral Horn's problem, to appear.

[15] V.I. Rabanovich, Yu.S. Samoĭlenko, When the sum of idempotents or projectors is a multiple of unity. (Russian) Funktsional. Anal. i Prilozhen. **34** (2000), no. 4, 91–93; translation in Funct. Anal. Appl. **34** (2000), no. 4, 311–313.

[16] S.V.Popovych, Yu.S. Samoĭlenko, On homomorphisms of algebras generated by projections, and Coxeter functors, Ukr.Mat.Zh.**55** (2003), no.9, 1224-1237 (Russian).

[17] V.Ostrovskyi, Yu.Samoilenko, Introduction to the theory of representations of finitely presented *-algebras. I. Representations by bounded operators. Reviews in Mathematics and Mathematical Physics, 11, pt.1. Harwood Academic Publishers, Amsterdam, 1999. iv+261 pp.

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