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RESEARCH ARTICLE

Kleinian singularities and algebras generated by elements that have given spectra and satisfy a scalar sum relation

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Abstract. We consider the algebras $e_i\Pi^\lambda(Q)e_i$, where $\Pi^\lambda(Q)$ is the deformed preprojective algebra of weight λ and i is some vertex of Q, in the case where Q is an extended Dynkin diagram and λ lies on the hyperplane orthogonal to the minimal positive imaginary root δ. We prove that the center of $e_i\Pi^\lambda(Q)e_i$ is isomorphic to $\mathcal{O}^\lambda(Q)$, a deformation of the coordinate ring of the Kleinian singularity that corresponds to Q. We also find a minimal k for which a standard identity of degree k holds in $e_i\Pi^\lambda(Q)e_i$. We prove that the algebras $A_{P_1,\ldots,P_n;\mu} = \mathbb{C}\langle x_1,\ldots,x_n|P_i(x_i)=0, \sum_{i=1}^n x_i = \mu e\rangle$ make a special case of the algebras $e_c\Pi^\lambda(Q)e_c$ for star-like quivers Q with the origin c.

Introduction

Consider the problem of describing n-tuples of Hermitian operators $\{A_i\}$ on a Hilbert space satisfying given restrictions on the spectra $\sigma(A_i) \subset M_i$, with $M_i \subset \mathbb{R}$ being finite, and the relation $\sum_{i=1}^n A_i = \mu I$, where I is the identity operator and $\mu \in \mathbb{R}$. The study of such n-tuples is equivalent to a study of $*$-representations of a certain $*$-algebra. Dropping the involution, we arrive at the following class of algebras.

Definition 1. *Let P_1, $\ldots$, P_n be complex polynomials in one variable and $\mu \in \mathbb{C}$. We impose an inessential restriction that $P_i(0) = 0$. Define the following algebra by*

$$A_{P_1,\ldots,P_n;\mu} = \mathbb{C}\langle x_1,\ldots,x_n|P_i(x_i) = 0\,(i=1,\ldots,n), \sum_{i=1}^n x_i = \mu e\rangle.$$

In the joint work of the author with Yu. Samoilenko and M. Vlasenko (see [1]) we studied some properties of such algebras; we computed the growth of these algebras and proved existence of polynomial identities in certain cases (in fact, the finiteness over the center was proved).

These algebras are closely related to the deformed preprojective algebras of W. Crawley-Boevey and M.P. Holland ([2]). We briefly recall their definition. Let Q be a quiver with a set of vertices, I. Write $\bar{Q}$ for the double quiver of Q, that is, the quiver obtained by adding the reverse arrow $a^* : j \longrightarrow i$ for every arrow $a : i \longrightarrow j$, and write $\mathbb{C}\bar{Q}$ for its path algebra, which has the paths in $\bar{Q}$ as a basis, including the trivial paths e_i for each vertex i. If $\lambda = (\lambda_i) \in \mathbb{C}^I$, then the deformed preprojective algebra of weight λ is

$$\Pi^\lambda(Q) = \mathbb{C}\bar{Q}/(\sum_{a \in \mathrm{Arrows}(Q)} [a, a^*] - \lambda),$$

where $\mathrm{Arrows}(Q)$ denotes the set of arrows of Q, and λ is identified with the element $\sum_{i \in I} \lambda_i e_i$.

Let $A = A_{P_1,\dots,P_n;\mu}$. Consider a quiver $Q(A)$ with the vertices

$$I = \{(i,j) | i = 1,\dots,n,\ j = 1,\dots,\deg P_i - 1\} \cup \{c\}$$

and the arrows

$$\{a_{ij} : (i,j) \longrightarrow (i,j-1) | i = 1,\dots,n,\ j = 1,\dots,\deg P_i - 1\},$$

where $(i,0)$ is identified with c for $i = 1,\dots,n$.

$$\begin{tikzcd}
 & (1,1) \arrow[dl, "a_{11}"] & (1,2) \arrow[l, "a_{12}"'] & \cdots \arrow[l] & (1, \deg P_1 - 1) \arrow[l, "a_{1\,\deg P_1 - 1}"'] \\
c & (2,1) \arrow[l, "a_{21}"'] & (2,2) \arrow[l, "a_{22}"'] & \cdots \arrow[l] & (2, \deg P_2 - 1) \arrow[l, "a_{2\,\deg P_2 - 1}"'] \\
 & \cdots & \cdots & & \cdots \\
 & (n,1) \arrow[uul, "a_{n1}"] & (n,2) \arrow[l, "a_{n2}"'] & \cdots \arrow[l] & (n, \deg P_n - 1) \arrow[l, "a_{n\,\deg P_n - 1}"']
\end{tikzcd}$$

Note that the graph Q coincides with the graph of algebra A, considered in [1]. Below is an example of the quiver Q for the case $n = 3$, $\deg P_1 = 2$, $\deg P_2 = 3$, $\deg P_3 = 2$:

$$\begin{tikzcd}
e_{11} \arrow[r, "a_{11}"] & e_c & e_{21} \arrow[l, "a_{21}"'] & e_{22} \arrow[l, "a_{22}"'] \\
 & e_{31} \arrow[u, "a_{31}"'] & &
\end{tikzcd}$$

The first result establishes a connection between the algebras $A_{P_1,\dots,P_n;\mu}$ and the deformed preprojective algebras.

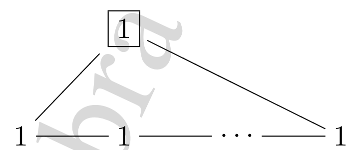
Theorem 1. *The algebra $A = A_{P_1,\dots,P_n;\mu}$ is isomorphic to $e_c\Pi^\lambda(Q)e_c$ under the isomorphism sending x_i to $a_{i1}a^*_{i1}$ for $Q = Q(A)$ and*

$$\lambda = \sum_{i=1}^{n} \sum_{j=1}^{\deg P_i - 1} (\alpha_{ij-1} - \alpha_{ij})e_{ij} + \mu e_c,$$

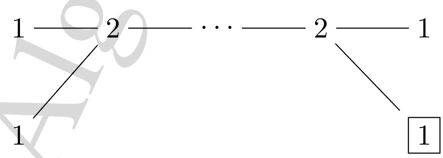
where $\alpha_{i0}, \alpha_{i1}, \dots, \alpha_{i\deg P_i-1}$ are all roots of the polynomial P_i taken with multiplicities in any order with $\alpha_{i0} = 0$.

Consider the case when the graph Q is an extended Dynkin diagram of type $\widetilde{A_n}$, $\widetilde{D_n}$, or $\widetilde{E_n}$. The following pictures show all such graphs together with coordinates of the so-called minimal imaginary root $\delta \in \mathbb{C}^I$. The boxed vertex is the extending vertex.

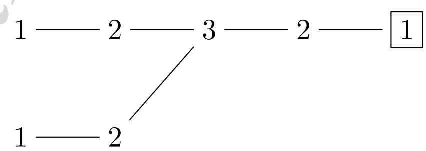
$\widetilde{A_n}$



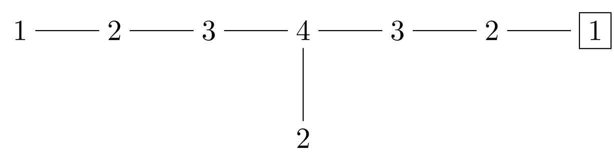
$\widetilde{D_n}$



$\widetilde{E_6}$



$\widetilde{E_7}$



$\widetilde{E_8}$

$$\begin{array}{ccccccccccccccc} 2 & - & 4 & - & 6 & - & 5 & - & 4 & - & 3 & - & 2 & - & \boxed{1} \\ & & & & | & & & & & & & & & & \\ & & & & 3 & & & & & & & & & & \end{array}$$

It was proved in [2] that $\Pi^\lambda(Q)$ is a PI-algebra (for PI algebras, see [3]) if and only if $\delta \cdot \lambda = 0$. The authors in [2] also studied the algebra $\mathcal{O}^\lambda(Q)$, which is $e_0\Pi^\lambda(Q)e_0$ where the 0-th vertex is the extending vertex of Q, and proved that this algebra is commutative if and only if $\delta \cdot \lambda = 0$. For $\lambda = 0$, the algebra $\mathcal{O}^0(Q)$ coincides with the coordinate ring of the corresponding Kleinian singularity.

In this paper we consider the algebras $e_i\Pi^\lambda e_i$ for an arbitrary $i \in I$. For the case $\delta \cdot \lambda = 0$, we study the center of this algebra and find the minimal number k for which it has a standard identity of degree k, that is,

$$\sum_{\pi \in \mathcal{S}_k} \operatorname{sign}(\pi) \prod_{i=1}^{k} x_{\pi(i)} = 0.$$

We denote by $\mathcal{S}_k$ the group of permutations on k elements.

We will prove the following theorems.

Theorem 2. *If Q is an extended Dynkin diagram $\widetilde{A_n}$, $\widetilde{D_n}$ or $\widetilde{E_n}$, $\delta \cdot \lambda = 0$, and $i \in I$ is some vertex in Q, then the center of $e_i\Pi^\lambda(Q)e_i$ is isomorphic to $\mathcal{O}^\lambda(Q) = e_0\Pi^\lambda(Q)e_0$, where the 0-th vertex is the extending vertex of Q.*

Theorem 3. *If Q is an extended Dynkin diagram $\widetilde{A_n}$, $\widetilde{D_n}$ or $\widetilde{E_n}$, $\delta \cdot \lambda = 0$, and $i \in I$ is some vertex in Q, then $e_i\Pi^\lambda e_i$ possesses a standard identity of degree $2\delta_i$ and it is the minimal number with such a property.*

Theorem 3 was proved for the partial case of the diagram $\widetilde{D_4}$ in [4] using different approach.

1. Representations of groups

Let V be a two-dimensional complex vector space with a simplectic form ω. Let G be a finite subgroup of $SL(V)$. Let irreducible representations of G be precisely $\{V_i\}_{i \in I}$, where $I = \{0, 1, 2, \ldots, n\}$ with V_0 denoting the trivial representation. Suppose that

$$V \otimes V_i = \bigoplus_{j=1}^{n} m_{ij} V_j.$$

Then the McKay graph of G is defined to be a graph with the set of vertices I and the number of edges between the vertices i and j equal to m_{ij} (we always have $m_{ij} = m_{ji}$). According to J. McKay [5], the McKay graphs of finite subgroups of $SL(V)$ are extended Dynkin diagrams; $\widetilde{A_n}$ for cyclic groups, $\widetilde{D_n}$ for dihedral groups and $\widetilde{E_6}$, $\widetilde{E_7}$, $\widetilde{E_8}$ for binary tetrahedral, octahedral and icosahedral groups. Dimensions of irreducible representations are $\dim V_i = \delta_i$. Fixing an orientation of the McKay graph of G we obtain a quiver Q.

Let M be some $\mathbb{C}G$-module. Consider the vector space $F_0(M) = T(V^*) \otimes M$, where

$$T(V^*) = \bigoplus_{i=0}^{\infty} V^{*\otimes i}$$

is the tensor algebra of V^*.

Equip $F_0(M)$ with the componentwise action of G. Then we can consider the subspace of G-invariant vectors, $F(M) = F_0(M)^G$. Note that if M is an algebra and the multiplication respects the action of G, then both $F_0(M)$ and $F(M)$ become graded algebras with the grading $F_0(M)_i = V^{*\otimes i} \otimes M$ and $F(M)_i = (F_0(M)_i)^G$. Consider the algebra $F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))$, where V_Σ is the direct sum of all irreducible $\mathbb{C}G$-modules and G acts on $\mathrm{End}_{\mathbb{C}}(V_\Sigma)$ by conjugation. Clearly $F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_0$ coincides with $(\mathrm{End}_{\mathbb{C}}(V_\Sigma))^G$ which, in its turn, can be identified with $\mathbb{C}^I$. Our aim is to build a graded algebra isomorphism φ from $\mathbb{C}\bar{Q}$ to $F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))$ such that

$$\begin{gathered}\varphi \text{ is an identity on } \mathbb{C}^I, \\ \varphi\Big(\sum_{a\in \mathrm{Arrows}(Q)} [a, a^*]\Big) = \delta\omega.\end{gathered} \tag{*}$$

We accomplish this in two steps.

Lemma 1. *The natural homomorphism*

$$F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_i \otimes_{F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_0} F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_j \longrightarrow F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_{i+j}$$

is an isomorphism

Proof. We make some identifications,

$$\begin{aligned}F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_i = (V^{*\otimes i} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G &\cong \mathrm{Hom}_G(V_\Sigma, V^{*\otimes i} \otimes V_\Sigma) \\ &\cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}^{a_i}),\end{aligned}$$

where

$$V^{*\otimes i} \otimes V_\Sigma \cong \bigoplus_{i\in I} V_i^{\oplus a_i}.$$

$$F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_j = (V^{*\otimes j} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G \cong \mathrm{Hom}_G(V^{\otimes j} \otimes V_\Sigma, V_\Sigma) \cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{b_i}, \mathbb{C}),$$

where

$$V^{\otimes j} \otimes V_\Sigma \cong \bigoplus_{i\in I} V_i^{\oplus b_i}.$$

$$F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_{i+j} = (V^{*\otimes(i+j)} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G \cong \mathrm{Hom}_G(V^{\otimes j} \otimes V_\Sigma, V^{*\otimes i} \otimes V_\Sigma) \cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{b_i}, \mathbb{C}^{a_i}).$$

Recall that

$$F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_0 \cong \bigoplus_{i\in I} \mathbb{C}.$$

Now the statement is clear. □

This lemma implies that the natural homomorphism from the tensor algebra of $F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_1$ over $F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_0$ to $F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))$ is an isomorphism. The graded algebra $\mathbb{C}\bar{Q}$ possesses the same property, so it is clear that for constructing an isomorphism of the graded algebras which would be an identity on $\mathbb{C}^I$, it is necessary and sufficient to establish an isomorphism of subbimodules of degree 1. Decompose $F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_1$ with respect to the primitive idempotents of $\mathbb{C}^I$,

$$F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))_1 = (V^* \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G \cong \bigoplus_{i,j\in I} \mathrm{Hom}_G(V \otimes V_i, V_j).$$

Clearly $\mathrm{Hom}_G(V \otimes V_i, V_j)$ is zero if there are no arrows from i to j in $\bar{Q}$ and is one dimensional if there is an arrow from i to j in $\bar{Q}$. A subbimodule of $\mathbb{C}^I$ of degree 1 has a similar decomposition. So any assignment $a \longrightarrow \varphi(a) \in \mathrm{Hom}_G(V \otimes V_i, V_j)$, $\varphi(a) \neq 0$ for $a \in \mathrm{Arrows}(\bar{Q})$, induces some isomorphism of the graded algebras, $\varphi : \mathbb{C}\bar{Q} \longrightarrow F(\mathrm{End}_{\mathbb{C}}(V_\Sigma))$.

Proposition 1. *For every arrow $a : i \longrightarrow j$ of Q choose any nonzero representative $\varphi(a) \in \mathrm{Hom}_G(V \otimes V_i, V_j)$. It is possible to choose $\varphi(a^*) \in \mathrm{Hom}_G(V_j, V^* \otimes V_i)$ such that*

$$\mathrm{tr}((\iota \otimes \mathrm{Id}_{V_i})\varphi(a^*)\varphi(a)) = \dim V_i \dim V_j,$$

where $\iota : V^* \longrightarrow V$ is such that

$$f(x) = \omega(\iota(f), x) \text{ for } f \in V^* \text{ and } x \in V.$$

This induces an isomorphism of the algebras which satisfies property (*).

Proof. Let us first consider the possibility of choosing the above $\varphi(a^*)$. In the decomposition of $V \otimes V_i$ into the direct sum of indecomposable $\mathbb{C}G$-modules, V_j occurs exactly once so, if we choose any nonzero $\varphi(a^*) \in \mathrm{Hom}_G(V \otimes V_i, V_j)$, we obtain that

$$(\iota \otimes \mathrm{Id}_{V_i})\varphi(a^*)\varphi(a)$$

is a projection on V_j in $V \otimes V_i$ multiplied by some complex constant, so its trace is nonzero and multiplying by a factor it is possible to make the trace to take any complex value. We only needed check that

$$\sum_{a \in \mathrm{Arrows}(Q)} [\varphi(a), \varphi(a^*)] = \delta\omega.$$

Choose some vertex i and multiply both sides by e_i to get

$$\sum_{\substack{j \in I,\, a: j \longrightarrow i,\\ a \in \mathrm{Arrows}(Q)}} \varphi(a)\varphi(a^*) - \sum_{\substack{j \in I,\, a: i \longrightarrow j,\\ a \in \mathrm{Arrows}(Q)}} \varphi(a^*)\varphi(a) = \delta_i \omega e_i. \tag{1}$$

Both sides belong to $\mathrm{Hom}_G(V \otimes V \otimes V_i, V_i)$, which can be identified with $\mathrm{Hom}_G(V \otimes V_i, V^* \otimes V_i)$ by "lifting" the first element of the tensor product. Apply $\iota \otimes \mathrm{Id}_{V_i}$ to both sides. Since $(\omega(x))(y) = \omega(y, x)$ and $(\omega(x))(y) = \omega(\iota(\omega(x)), y)$, we have that $\iota(\omega(x)) = -x$ and

$$(\iota \otimes \mathrm{Id}_{V_i})\delta_i \omega e_i = -\delta_i \,\mathrm{Id}_{V \otimes V_i}.$$

Recall that each $(\iota \otimes \mathrm{Id}_{V_i})\varphi(a)\varphi(a^*)$ and $(\iota \otimes \mathrm{Id}_{V_i})\varphi(a^*)\varphi(a)$, which occur in (1), is a projection onto the summand V_j multiplied by some complex number where j is another endpoint of a distinct from i. Denote this projection by p_j. Then

$$(\iota \otimes \mathrm{Id}_{V_i})\varphi(a)\varphi(a^*) = \frac{\mathrm{tr}((\iota \otimes \mathrm{Id}_{V_i})\varphi(a)\varphi(a^*))}{\dim V_j} p_j \quad \text{and}$$

$$-(\iota \otimes \mathrm{Id}_{V_i})\varphi(a^*)\varphi(a) = \frac{-\,\mathrm{tr}((\iota \otimes \mathrm{Id}_{V_i})\varphi(a^*)\varphi(a))}{\dim V_j} p_j.$$

By definition,

$$\mathrm{tr}((\iota \otimes \mathrm{Id}_{V_i})\varphi(a^*)\varphi(a)) = \dim V_i \dim V_j.$$

There is an identity,

$$\mathrm{tr}((\iota \otimes \mathrm{Id}_{V_i})xy) = -\,\mathrm{tr}((\iota \otimes \mathrm{Id}_{V_j})yx),$$

which holds for every $x \in \mathrm{Hom}_{\mathbb{C}}(V \otimes V_j, V_i)$ and $y \in \mathrm{Hom}_{\mathbb{C}}(V \otimes V_i, V_j)$. It is enough to check this identity for $x = f_1 \otimes x_0$ and $y = f_2 \otimes y_0$, where $f_1, f_2 \in V^*$, $x_0 \in \mathrm{Hom}_{\mathbb{C}}(V_j, V_i)$, and $y_0 \in \mathrm{Hom}_{\mathbb{C}}(V_i, V_j)$,

$$\begin{aligned}\mathrm{tr}((\iota \otimes \mathrm{Id}_{V_i})xy) &= \mathrm{tr}(\iota(f_1)f_2 \otimes x_0y_0) = f_2(\iota(f_1))\,\mathrm{tr}(x_0y_0)\\ &= \omega(\iota(f_2), \iota(f_1))\,\mathrm{tr}(x_0y_0) = -\omega(\iota(f_1), \iota(f_2))\,\mathrm{tr}(y_0x_0)\\ &= -f_1(\iota(f_2))\,\mathrm{tr}(y_0x_0) = -\,\mathrm{tr}(\iota(f_2)f_1 \otimes y_0x_0) = -\,\mathrm{tr}((\iota \otimes \mathrm{Id}_{V_j})yx).\end{aligned}$$

Using this identity we have

$$\mathrm{tr}((\iota \otimes \mathrm{Id}_{V_i})\varphi(a)\varphi(a^*)) = -\,\mathrm{tr}((\iota \otimes \mathrm{Id}_{V_j})\varphi(a^*)\varphi(a)) = -\dim V_i \dim V_j.$$

It follows that $\iota \otimes \mathrm{Id}_{V_i}$, applied to left-hand side of (1), equals

$$-\dim V_i \sum_{\substack{j \in I,\, a: i \longrightarrow j,\\ a \in \mathrm{Arrows}(\bar{Q})}} p_j = -\dim V_i\, \mathrm{Id}_{V \otimes V_i},$$

and recalling that $\delta_i = \dim V_i$ we finish the proof. □

The next corollary summarizes the results obtained in this section.

Corollary 1. *The algebra $\Pi^\lambda(Q)$ is isomorphic to the algebra*

$$(T(V^*) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G/(\delta\omega - \lambda).$$

Moreover, this is an isomorphism of filtered algebras with the filtrations induced by the gradings of $\mathbb{C}\bar{Q}$ and $T(V^*)$.

2. The case $\lambda = 0$

In this section we are going to prove Theorems 2 and 3 in the case where $\lambda = 0$. The key result is the following lemma.

Lemma 2. *The algebra $\Pi^0(Q)$ is isomorphic to the algebra of polynomial G-equivariant maps from V to $\mathrm{End}_{\mathbb{C}}(V_\Sigma)$, that is, to the algebra*

$$(\mathrm{Sym}(V^*) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G,$$

where $\mathrm{Sym}(V^*)$ is the algebra of symmetric tensors over V^*. Moreover, this is an isomorphism of graded algebras.

Proof. We already know that $\Pi^0(Q)$ is isomorphic to

$$(T(V^*) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G/(\delta\omega) = (T(V^*) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G/\omega.$$

Since

$$\mathrm{Sym}(V^*) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma) = \\ = (T(V^*)/w) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma) = (T(V^*) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))/w,$$

it is sufficient to prove that the idempotent

$$\varepsilon = \frac{1}{|G|} \sum_{g \in G} g$$

maps the ideal generated by ω in $T(V^*) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma)$ into the ideal generated by ω in $(T(V^*) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G$. To prove this, take some $f \in V^{*\otimes i} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma)$, $g \in V^{*\otimes j} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma)$, and consider $\varepsilon(f\omega g)$. Note that $f\omega g$ is antisymmetric in the $(i+1)$-th and $(i+2)$-th arguments. It follows that $\varepsilon(f\omega g)$ is antisymmetric in the $(i+1)$-th and $(i+2)$-th argument as well. Since

$$\varepsilon(f\omega g) \in (V^{*\otimes(i+j+2)} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G,$$

and we know from Lemma 1 that

$$(V^{*\otimes(i+j+2)} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G = (V^{*\otimes i} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G \\ \otimes_{\mathbb{C}G} (V^{*\otimes 2} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G \otimes_{\mathbb{C}G} (V^{*\otimes j} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G,$$

we can decompose

$$\varepsilon(f\omega g) = \sum_{k=1}^{K} f_k \omega_k g_k$$

with $f_k \in (V^{*\otimes i} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G$, $g_k \in (V^{*\otimes j} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G$ and $\omega_k \in (V^{*\otimes 2} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G$. Denote by τ the operator acting on elements of $(V^{*\otimes(i+j+2)} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G$ by interchanging the $(i+1)$-th and $(i+2)$-th arguments. Then

$$\tau\varepsilon(f\omega g) = \sum_{k=1}^{K} f_k \omega'_k g_k$$

with ω'_k obtained from ω_k by interchanging the first two arguments. Hence,

$$\varepsilon(f\omega g) = \frac{1}{2}(\varepsilon(f\omega g) - \tau\varepsilon(f\omega g)) = \frac{1}{2} \sum_{k=1}^{K} f_k(\omega_k - \omega'_k) g_k.$$

Since $\omega_k - \omega'_k \in \mathrm{Hom}_G(V \otimes V, \mathrm{End}_{\mathbb{C}}(V_\Sigma))$ is antisymmetric and V is two dimensional, it can be represented as ωx_k with $x_k \in \mathrm{End}_{\mathbb{C}}(V_\Sigma)^G$. Thus

$$\varepsilon(f\omega g) = \frac{1}{2} \sum_{k=1}^{K} f_k \omega x_k g_k$$

with f_k, x_k and g_k from $(T(V^*) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G$. This completes the proof. $\square$

The next propositions follow immediately.

Proposition 2. *The algebra $e_i\Pi^0(Q)e_i$ is isomorphic to the algebra of polynomial G-equivariant maps from V to $\mathrm{End}_{\mathbb{C}}(V_i)$ for any $i \in I$. In particular, $\mathcal{O}^0(Q) = e_0\Pi^0(Q)e_0$ is isomorphic to the algebra of invariants of G on V.*

Proposition 3. *The algebra $e_i\Pi^0(Q)e_i$ has a standard identity of degree $2\delta_i$ for any $i \in I$.*

Proposition 4. *There is a graded inclusion from $e_0\Pi^0(Q)e_0$ to the center of $\Pi^0(Q)$, and the graded inclusions from $e_0\Pi^0(Q)e_0$ to the center of $e_i\Pi^0(Q)e_i$, for $i \in I$, are induced by the inclusions $\mathbb{C} \subset \mathrm{End}_{\mathbb{C}}(V_\Sigma)$ and $\mathbb{C} \subset \mathrm{End}_{\mathbb{C}}(V_i)$, correspondingly.*

For any $i \in I$ and $x \in V$ denote by $\mu_i(x)$ the subset of $\mathrm{End}_{\mathbb{C}}(V_i)$ defined by

$$\mu_i(x) = \{f(x) | f \text{ is a polynomial } G\text{-equivariant map from } V \text{ to } \mathrm{End}_{\mathbb{C}}(V_i)\}.$$

In what follows we will need the following statement.

Lemma 3. *The set of $x \in V$ such that $\mu_i(x) = \mathrm{End}_{\mathbb{C}}(V_i)$ is algebraically dense for any $i \in I$.*

Proof. Suppose $f : V \longrightarrow \mathbb{C}$ is a non-constant G-invariant polynomial function. Then its differential df is a polynomial G-equivariant map from V to V^*. Denote by U the set of $x \in V$ for which $(df(x))(x) \neq 0$. Clearly U is open and U is not empty since $(df(x))(x) = 0$ implies that f is a constant. Denote by U' the subset of U of all x such that $f(x) \neq 0$. Since U' is open and not empty, it is dense. We will prove that every x from U' satisfies the required condition. So let $f(x) \neq 0$ and let $(df(x))(x) \neq 0$. Then $\iota df(x) \in V$ ($\iota : V^* \longrightarrow V$ is such that $\omega(\iota(y_1), y_2) = y_1(y_2)$ for every $y_1 \in V^*$ and $y_2 \in V$) is not a multiple of x because if $\iota df(x) = Cx$, $C \in \mathbb{C}$, then

$$(df(x))(x) = \omega(\iota df(x), x) = \omega(Cx, x) = 0.$$

It follows that $f(x)x$ and $\iota df(x)$ span V. Since $g_1 : V \longrightarrow V$ defined by $g_1(y) = f(y)y$ and $g_2 : V \longrightarrow V$ defined by $g_2(y) = df(y)$ are polynomial and G-equivariant we have that every element of V is a value in x of some polynomial G-equivariant map from V to V. It follows that for every k every element of $V^{\otimes k}$ is a value in x of some polynomial G-equivariant map from V to $V^{\otimes k}$. Since every finite dimensional $\mathbb{C}G$-module is a submodule of $V^{\otimes k}$ for some k, the statement holds for every finite dimensional $\mathbb{C}G$-module, in particular for $\mathrm{End}_{\mathbb{C}}(V_i)$. $\square$

This lemma implies that there is no $k < 2\delta_i$ such that $e_i\Pi^0(Q)e_i$ has a standard identity of degree k (since some factor of $e_i\Pi^0(Q)e_i$ is the algebra of $\delta_i \times \delta_i$-matrices). Moreover, this implies that every polynomial map from V to $\mathrm{End}_{\mathbb{C}}(V_i)$ commuting with all G-equivariant polynomial maps from V to $\mathrm{End}_{\mathbb{C}}(V_i)$ takes only scalar values, and thus the inclusion in Proposition 4 is in fact an isomorphism.

Corollary 2. *Theorems 2 and 3 hold for $\lambda = 0$.*

3. Regularity of the multiplication law

Denote by S_n the $\mathbb{C}^I$-bimodule $(\mathrm{Sym}^n(V^*) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G$, by S the graded algebra $(\mathrm{Sym}(V^*) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G$, by T_n the $\mathbb{C}^I$-bimodule $(V^{*\otimes n} \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G$, and by T the graded algebra $(T(V^*) \otimes \mathrm{End}_{\mathbb{C}}(V_\Sigma))^G$. In this section we will show that all algebras of the family $\Pi^\lambda(Q)$ can be identified with an algebra that is S as a vector space and the multiplication law in it polynomially depends on λ. For every $k = 0, 1, 2, \ldots$, we construct an operator

$$\pi_k^\lambda : T_k \longrightarrow \bigoplus_{i=0}^{k} S_i$$

such that

1. $\pi_k^\lambda(x) = x$ for $x \in S_k$;
2. $\pi_k^\lambda(x) \equiv x \mod \delta\omega - \lambda$ for any $x \in T_k$;
3. $\pi_k^\lambda(x_1 \omega x_2) = \pi_{k-2}^\lambda(x_1 \delta^{-1} \lambda x_2)$ for any $x_1 \in T_i$ and $x_2 \in T_j$ with $i + j = k - 2$;
4. $\pi_k^\lambda(x)$ polynomially depends on λ.

Then the family of operators π_k^λ define an operator π^λ acting from T to S. It is clear that π^λ is a projection with the image S, the second property of π_k^λ guarantees that $\pi^\lambda(x)$ is equivalent to x in the algebra $\Pi^\lambda(Q)$, whereas the third property implies that elements equivalent in $\Pi^\lambda(Q)$ are mapped into identical elements. Combining this gives an isomorphism between $\Pi^\lambda(Q)$ and S as filtered vector spaces, and the multiplication in $\Pi^\lambda(Q)$ is carried over to S to give

$$x \times y = \pi^\lambda(x \otimes y),$$

which polynomially depends on λ. It remains to show that the family of operators with properties (1) – (4) exists.

Clearly, for $k = 0$ and $k = 1$ we can take the identity operators. Then we prove existence of π_k^λ by induction. Fix some $\lambda \in \mathbb{C}^I$ and an integer $k \geq 2$. For $i = 1, \ldots, k-1$, define the operators

$$\tau_i : T_k \oplus \bigoplus_{j=0}^{k-2} S_j \longrightarrow T_k \oplus \bigoplus_{j=0}^{k-2} S_j$$

by setting

$\tau_i(x) = 0$ if $x \in \bigoplus_{j=0}^{k-2} S_j$,

$\tau_i(x) = 0$ if $x \in T_k$ and x is symmetric with respect to the i-th and $(i+1)$-th arguments,

$\tau_i(f\omega g) = f\omega g - \pi_\lambda^{k-2}(f\delta^{-1}\lambda g)$.

This defines τ_i for $x \in T_k$ such that x is antisymmetric with respect to the i-th and $(i+1)$-th arguments. Put $\rho_i = 1 - 2\tau_i$. We prove the following fact.

Proposition 5. *The family of operators (ρ_i) satisfy the following conditions:*

1. $\rho_i^2 = 1$,
2. $\rho_i\rho_j = \rho_j\rho_i$ *for* $|i-j| > 1$,
3. $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$,

so (ρ_i) induce a representation of the group of permutations of k elements.

Proof. Property (1) is easy. Consider the property (2). Assume $j > i$. It is enough to check the property for argument of the form

$$x = f_1\omega f_2\omega f_3 \ \text{ for } f_1 \in T_{i-1},\ f_2 \in T_{j-i-2},\ f_3 \in T_{k-j-1}.$$

Then

$$\begin{aligned}\rho_i\rho_j x - \rho_j\rho_i x &= \pi_\lambda^{k-2}(f_1\omega f_2\delta^{-1}\lambda f_3 - f_1\omega f_2\delta^{-1}\lambda f_3)\\ &= \pi_\lambda^{k-4}(f_1\delta^{-1}\lambda f_2\delta^{-1}\lambda f_3) - \pi_\lambda^{k-4}(f_1\delta^{-1}\lambda f_2\delta^{-1}\lambda f_3) = 0\end{aligned}$$

by the induction hypothesis. Consider the property (3). Denote by ρ_i', $i = 1, 2, \ldots, k-1$, the operator in T_k that acts on $x \in T_k$ by interchanging the i-th and $(i+1)$-th arguments. Then, clearly the operators ρ_i' satisfy conditions (1) – (3). Choose some $i \neq k-1$. Since there is no element of

T_k which is antisymmetric with respect to the arguments i, $i+1$, $i+2$, the following operator vanishes on T_k:

$$1 - \rho'_i - \rho'_{i+1} - \rho'_i\rho'_{i+1}\rho'_i + \rho'_i\rho'_{i+1} + \rho'_{i+1}\rho'_i = 0.$$

If we substitute $\rho'_i = 1 - 2\tau'_i$ we obtain that

$$\tau'_i\tau'_{i+1}\tau'_i = \frac{1}{4}\tau'_i \text{ and } \tau'_{i+1}\tau'_i\tau'_{i+1} = \frac{1}{4}\tau'_{i+1}$$

If we prove that

$$\tau_i\tau_j\tau_i = \frac{1}{4}\tau_i \text{ for } |i-j| = 1,$$

then property (3) would follow. But if $|i-j| = 1$, then

$$\tau_i\tau_j\tau_i = \tau_i^2\tau_j\tau_i = \tau_i\tau'_i\tau'_j\tau'_i = \frac{1}{4}\tau_i\tau'_i = \frac{1}{4}\tau_i^2 = \frac{1}{4}\tau_i,$$

here we consider τ'_m, $m = 1, 2, \ldots, k-1$, as an operator on $T_k \oplus \bigoplus_{i=0}^{k-2} S_i$ which acts as zero on the component $\bigoplus_{i=0}^{k-2} S_i$ and use the equality $\tau_{m_1}\tau_{m_2} = \tau_{m_1}\tau'_{m_2}$ which is valid for $m_1, m_2 = 1, 2, \ldots, k-1$. $\square$

Consider the representation of the group of permutations of k elements, $\mathcal{S}_k$, given by the operators ρ_i. Denote by $\bar{\varepsilon}$ the image of the element

$$\varepsilon = \frac{1}{k!}\sum_{\sigma\in\mathcal{S}_k}\sigma$$

of the group algebra $\mathbb{C}\mathcal{S}_k$. Then we can expand every σ as a product of the operators ρ_i, substitute $\rho_i = 1 - 2\tau_i$, and represent

$$\bar{\varepsilon} = 1 + \sum_{i,j=1}^{k-1}\tau_i x_{ij}\tau_j, \tag{2}$$

where x_{ij} are some operators. Then put $\pi_\lambda^k x = \bar{\varepsilon}x$ for $x \in T_k$. Let us check the required properties for π_λ^k. The property (1) follows from (2) and the fact that all τ_i vanish on elements of S_k. Since all images of τ_i belong to the ideal generated by $\delta\omega - \lambda$, the property (2) follows. The property (3) is true, since $\bar{\varepsilon} = \bar{\varepsilon}\rho_{i+1}$ implies that $\bar{\varepsilon} = \bar{\varepsilon}(1-\tau_{i+1})$ and

$$\bar{\varepsilon}(x_1\omega x_2) = \bar{\varepsilon}(1-\tau_{i+1})(x_1\omega x_2) = \bar{\varepsilon}\pi_\lambda^{k-2}(x_1\delta^{-1}\lambda x_2) = \pi_\lambda^{k-2}(x_1\delta^{-1}\lambda x_2).$$

The property (4) is obvious, so we have proved

Proposition 6. *A family of operators π_λ^k satisfying properties (1) – (4) exists.*

An immediate corollary is

Corollary 3. *Every algebra $\Pi^\lambda(Q)$ is isomorphic, as a filtered algebra, to S with multiplication law $\times^\lambda$ which polynomially depends on λ and is such that for any homogeneous x of degree i and homogeneous y of degree j the term of degree $i+j$ in $x \times^\lambda y$ does not depend on λ.*

4. Generic λ

Using Corollary 3 we identify $\Pi^\lambda(Q)$ with S equipped with a multiplication that depends on λ polynomially. Denote this multiplication by $\times^\lambda$. Sometimes, if λ is fixed, we will omit the sign $\times^\lambda$ and simply write xy instead of $x \times^\lambda y$ keeping in mind that the result depends on λ polynomially. In this section we will prove the statements of Theorems 2 and 3 for some algebraically dense subset of the set

$$\eta = \{\lambda \in \mathbb{C}^I : \lambda \cdot \delta = 0\}.$$

Proposition 7. *There exist elements $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ in S and rational functions $\alpha_1, \ldots, \alpha_n$ defined on η such that*

$$\sum_{i=1}^{n} \alpha_i(\lambda) f_i \times^\lambda e_0 \times^\lambda g_i = 1$$

for each λ from some algebraically dense subset of η.

Proof. It easily follows from the definition of the deformed preprojective algebra that

$$\Pi^\lambda(Q)/\Pi^\lambda(Q)e_0\Pi^\lambda(Q) \cong \Pi^{\lambda'}(Q'),$$

where Q' is the Dynkin diagram obtained from Q by deleting the vertex 0 and λ' is the restriction of λ to the vertices of Q'. It was proved in [2] that the deformed preprojective algebra of a Dynkin diagram is always finite dimensional and is zero for all parameters except for a number of hyperplanes. We will use the following facts:

1. the homogeneous subspace $S \times^0 e_0 \times^0 S$ of S has finite codimension,
2. there exists $\lambda_0 \in \eta$ such that $S \times^{\lambda_0} e_0 \times^{\lambda_0} S = S$.

Choose some basis in $S \times^0 e_0 \times^0 S$ of the form $(a_i \times^0 e_0 \times^0 b_i)$ where i ranges over the set of positive integers and all a_i and b_i are homogeneous elements of S. It follows from the first statement that we can add some finite number of homogeneous elements of S, $x_1, x_2, \ldots, x_n$, such that

x_i and $a_i \times^0 e_0 \times^0 b_i$ together form a basis of S. Now, for $\lambda \in \eta$ consider the set

$$B(\lambda) = \{x_i | i = 1, \ldots, n\} \cup \{a_i \times^\lambda e_0 \times^\lambda b_i | i = 1, 2, \ldots\}.$$

It is again a basis of S, because each $a_i \times^\lambda e_0 \times^\lambda b_i$ is equal to the sum of $a_i \times^0 e_0 \times^0 b_i$ and some terms of lower degree. Moreover, every element of S, being expanded with respect to this basis, has all coefficients that are polynomial in λ.

It follows from the statement (2) that there exists some λ_0 such that for $i = 1, \ldots, n$,

$$x_i = \sum_{k=1}^{K_i} f_i^k \times^{\lambda_0} e_0 \times^{\lambda_0} g_i^k,$$

where all f_i^k and g_i^k are elements of S. Consider elements $y_i(\lambda) \in S$ for $i = 1, \ldots, n$ defined by

$$y_i(\lambda) = \sum_{k=1}^{K_i} f_i^k \times^\lambda e_0 \times^\lambda g_i^k.$$

Consider an $n \times n$ matrix $Z(\lambda) = (z_{ij}(\lambda))$, where $z_{ij}(\lambda)$ is the value of the coefficient at x_i in the expansion of $y_j(\lambda)$ with respect to the basis $B(\lambda)$. We have the following expansion of $y_j(\lambda)$ with respect to the basis $B(\lambda)$:

$$\sum_{k=1}^{K_j} f_j^k \times^\lambda e_0 \times^\lambda g_j^k = \sum_{i=1}^{n} z_{ij}(\lambda) x_i + \sum_{k=1}^{L_j} c_{jk}(\lambda) a_k \times^\lambda e_0 \times^\lambda b_k$$

for some polynomial functions of λ $c_{jk}(\lambda)$. Rewrite this as

$$\sum_{i=1}^{n} z_{ij}(\lambda) x_i = \sum_{k=1}^{K_j} f_j^k \times^\lambda e_0 \times^\lambda g_j^k - \sum_{k=1}^{L_j} c_{jk}(\lambda) a_k \times^\lambda e_0 \times^\lambda b_k$$

and consider it as a system of linear equations with indeterminates $x_1, \ldots, x_n$. Clearly it can be solved for λ if $\det Z(\lambda) \neq 0$ and the solution will depend on λ rationally. If we expand 1 with respect to the basis $B(\lambda)$ and then use this solution we obtain the required expansion. The set of $\lambda \in \eta$ for which $\det Z(\lambda) \neq 0$ is open. It is nonempty since $Z(\lambda_0)$ is the identity matrix, hence this set is dense. This completes the proof. $\square$

Denote by η' the subset of η for which we the proposition above holds.

Proposition 8. *For every $\lambda \in \eta'$ and every $x \in \mathcal{O}^\lambda(Q) = e_0\Pi^\lambda(Q)e_0$ there exists $z(x)$ in the center of $\Pi^\lambda(Q)$ such that $e_0z(x)e_0 = x$.*

Proof. Put
$$z(x) = \sum_{i=1}^{n} \alpha_i(\lambda) f_i x g_i.$$

Then
$$e_0z(x)e_0 = \sum_{i=1}^{n} \alpha_i(\lambda)e_0 f_i x g_i e_0 = \sum_{i=1}^{n} \alpha_i(\lambda) x f_i e_0 g_i e_0 = x,$$

since $\mathcal{O}^\lambda(Q)$ is commutative. Again, using commutativity of $\mathcal{O}^\lambda(Q)$ for any $y \in S$ we have

$$yz(x) = \sum_{i=1}^{n} \alpha_i(\lambda) y f_i x g_i = \sum_{i,j=1}^{n} \alpha_i(\lambda)\alpha_j(\lambda) f_j e_0 g_j y f_i x g_i$$
$$= \sum_{i,j=1}^{n} \alpha_i(\lambda)\alpha_j(\lambda) f_j x g_j y f_i e_0 g_i = \sum_{j=1}^{n} \alpha_j(\lambda) f_j x g_j y = z(x)y.$$

□

Proposition 9. *For every $\lambda \in \eta'$ and every $q \in I$, the algebra $e_q\Pi^\lambda(Q)e_q$ has a standard identity of degree $2\delta_q$.*

Proof. For $x \in S$, construct an $n \times n$ matrix $M(x)$ over $\mathcal{O}^\lambda(Q)$ with the elements
$$m_{ij}(x) = \alpha_i(\lambda) e_0 g_i x f_j e_0.$$
Then for $x, y \in S$ the matrix $M(x)M(y)$ has elements

$$\sum_{k=1}^{n} m_{ik}(x) m_{kj}(y) = \sum_{k=1}^{n} \alpha_i(\lambda) e_0 g_i x f_k e_0 \alpha_k(\lambda) e_0 g_k y f_j e_0$$
$$= \alpha_i(\lambda) e_0 g_i x y f_j e_0 = m_{ij}(xy),$$

so
$$M(xy) = M(x)M(y).$$

Denote by p the matrix $M(1)$. Clearly p is an idempotent and M defines a homomorphism from $\Pi^\lambda(Q)$ to $p\,\mathrm{Mat}(n, \mathcal{O}^\lambda(Q))p$, where $\mathrm{Mat}(n, \mathcal{O}^\lambda(Q))$ denotes the algebra of $n \times n$ matrices over $\mathcal{O}^\lambda(Q)$. Construct the inverse map $N : \mathrm{Mat}(n, \mathcal{O}^\lambda(Q)) \longrightarrow S$. Let $A = (a_{ij})$ and set

$$N(A) = \sum_{i,j=1}^{n} \alpha_j(\lambda) f_i a_{ij} g_j.$$

Then we can check that

$$N(M(x)) = \sum_{i,j=1}^{n} \alpha_j(\lambda) f_i \alpha_i(\lambda) e_0 g_i x f_j e_0 g_j = x$$

and

$$m_{ij}(N(A)) = \sum_{k,l=1}^{n} \alpha_i(\lambda) e_0 g_i \alpha_l(\lambda) f_k a_{kl} g_l f_j e_0,$$

which implies that

$$M(N(A)) = pAp.$$

This proves that M is an isomorphism. The algebra $\mathcal{O}^\lambda(Q)$ is a domain (see [2]). Hence it can be embedded into its field of fractions, F. So the algebra $p\,\mathrm{Mat}(n, \mathcal{O}^\lambda(Q))p$ can be embedded into $p\,\mathrm{Mat}(n, F)p$ that is isomorphic to $\mathrm{Mat}(r, F)$ where r is the rank of p in $\mathrm{Mat}(n, F)$. Denote by p_q the matrix $M(e_q)$ for $q \in I$. In a similar way, $e_q \Pi^\lambda(Q) e_q$ can be embedded into $\mathrm{Mat}(r_q, F)$ where r_q is the rank of p_q in $\mathrm{Mat}(n, F)$. On the other hand, $r_q = \mathrm{tr}\, p_q$ which is a rational function of λ. Since r_q can accept only a finite number of values on the dense set η', namely $1, 2, \ldots, n$, it is constant. In $\Pi^\lambda(Q)$,

$$\sum_{a \in \mathrm{Arrows}(Q)} [a, a^*] = \sum_{q \in I} \lambda_q e_q.$$

Hence

$$\sum_{q \in I} \lambda_q r_q = \mathrm{tr} \sum_{q \in I} \lambda_q p_q = 0.$$

Since this equality holds for all λ from η', which is dense in η, there is a constant $c \in \mathbb{C}$ such that $r_q = c\delta_q$ for $q \in I$. For $q = 0$,

$$p_0 = M(e_0) = (\alpha_i(\lambda) e_0 g_i e_0 f_j e_0),$$

so p_0 has rank 1. This implies that $c = 1$ and $r_q = \delta_q$. We have proved that the algebra $e_q \Pi^\lambda e_q$ for $\lambda \in \eta'$, $q \in I$, is isomorphic to some subalgebra of the algebra of $\delta_q \times \delta_q$ matrices over the field F, so a standard identity of degree $2\delta_q$ is satisfied by the Amitsur-Levitzki theorem. $\square$

5. Extending to the whole hyperplane

To finish the proof of Theorems 2 and 3, we need to take several steps.

Proposition 10. *For any $\lambda \in \mathbb{C}^I$ such that $\lambda \cdot \delta = 0$ and any $i \in I$, the algebra $e_i \Pi^\lambda(Q) e_i$ satisfies a standard identity of degree $2\delta_i$.*

Proof. For $x_1, \dots x_{2\delta_i} \in e_i S e_i$, the sum

$$\sum_{\sigma \in \mathcal{S}_{2\delta_i}} \operatorname{sign}(\sigma) x_{\sigma(1)} \times^\lambda \dots \times^\lambda x_{\sigma(2\delta_i)}$$

is zero on an algebraically dense subset of $\lambda \in \mathbb{C}^I$, $\lambda \cdot \delta = 0$. Since it is polynomial in λ, it is zero for all $\lambda \in \mathbb{C}^I$, $\lambda \cdot \delta = 0$. □

Proposition 11. *For every $\lambda \in \eta$ and every $x \in \mathcal{O}^\lambda(Q)$ there exists a unique $z(x)$ in the center of $\Pi^\lambda(Q)$ such that $e_0 z(x) e_0 = x$.*

Proof. First note that if such $z(x)$ exists, then it is unique. Suppose the contrary. Then there exists a in the center of $\Pi^\lambda(Q)$ such that $e_0 a = 0$. Suppose $e_i a \neq 0$. Then, since $\Pi^\lambda(Q)$ is prime (see [2]), there exists $y \in \Pi^\lambda(Q)$ such that $e_0 y e_i a \neq 0$. Rewrite it as $e_0 a y e_i$ and get a contradiction.

Then note that the degree of $z(x)$ is not greater than that of x. Let $z(x)'$ be the term with a maximal degree of $z(x)$ and suppose that the degree of $z(x)'$ is greater than that of x. Clearly $z(x)'$ belongs to the center of $\Pi^0(Q)$, but $e_0 z(x)' e_0 = 0$ which, contradicts the previous remark.

The algebra $\Pi^\lambda(Q)$ is finitely generated and for any x, since the degree of $z(x)$ is bounded, the problem of finding such $z(x)$ for any fixed x is equivalent to solving some finite system of linear equations. Coefficients of the system depend on λ polynomially. Suppose that the system has m equations and n indeterminates. Consider the set W of λ for which the system has a unique solution. The system has a unique solution if and only if there exist equations $i_1, i_2, \dots, i_n$ in the system such that the subsystem $i_1, i_2, \dots, i_n$ is nondegenerate (the set U of λ for which this is true is open) and a solution of the equations $i_1, i_2, \dots, i_n$ satisfies other equations (the set of λ for which this is true is closed in U). Thus we obtain a sequence of open sets $U_1, U_2, \dots, U_N$ and a sequence of sets $V_1, V_2, \dots, V_N$, each V_i being closed in the corresponding U_i. It follows that W is covered by $U_1, U_2, \dots, U_N$ and the intersection of W with each U_i is closed. So W is a closed set in the union of $U_1, U_2, \dots U_n$, hence it is an intersection of some open set and some closed set.

Applying Proposition 8 and the first remark in this proof we obtain that W is an open set. Using Proposition 4 and the first remark we obtain that W contains some neighborhood of zero. So for any $x \in e_0 S e_0$ and any λ there exists some constant $c \in \mathbb{C}$ such that there is $z'(x) \in S$ that belongs to the center of $\Pi^{c\lambda}(Q)$ and $e_0 z'(x) e_0 = x$. Let x be a homogeneous element of degree k. Define an operator ϕ on T as the multiplication by $c^{\frac{n}{2}}$ on each T_n. Then ϕ is an automorphism of the algebra T and it maps $\delta\omega - c\lambda$ to $c\delta\omega - c\lambda$. It follows that $\phi(z'(x))$ belongs to the center of $\Pi^\lambda(Q)$ and $e_0\phi(z'(x))e_0 = c^{\frac{k}{2}}x$, so $z(x) = \phi(z'(x))c^{-\frac{k}{2}}$ belongs to the center of $\Pi^\lambda(Q)$ and $e_0z(x)e_0 = x$. □

Proof of Theorem 2. For any $\lambda \in \mathbb{C}^I$, $\lambda\cdot\delta = 0$, take a map ϕ_λ from $\mathcal{O}^\lambda(Q)$ to the center of $\Pi^\lambda(Q)$ such that $e_0\phi_\lambda(x)e_0 = x$ for all $x \in \mathcal{O}^\lambda(Q)$. By Proposition 11, ϕ_λ is uniquely defined by this property, so it is linear. If $x, y \in \mathcal{O}^\lambda(Q)$, then $\phi_\lambda(x)\phi_\lambda(y)$ belongs to the center of $\Pi^\lambda(Q)$ and $e_0\phi_\lambda(x)\phi_\lambda(y)e_0 = xy$, so again by Proposition 11, $\phi_\lambda(xy) = \phi_\lambda(x)\phi_\lambda(y)$. Clearly, $\phi_\lambda(e_0) = 1$. So ϕ_λ is a homomorphism. The homomorphism ϕ_λ is an inclusion, since for any $x \in \mathcal{O}^\lambda(Q)$, $x = e_0\phi_\lambda(x)e_0$.

For any $i \in I$, put $\phi^i_\lambda(x) = e_i\phi_\lambda(x)$ for $x \in \mathcal{O}^\lambda(Q)$. Then it is easy to check that ϕ^i_λ is a homomorphism from the algebra $\mathcal{O}^\lambda(Q)$ to the center of $e_i\Pi^\lambda(Q)e_i$. It is an inclusion, since $\Pi^\lambda(Q)$ is prime (see [2]), so if $x \neq 0$ belongs to the center of $\Pi^\lambda(Q)$, then there exists $y \in \Pi^\lambda(Q)$ such that $e_iyx \neq 0$ and, hence, $e_ix \neq 0$.

To prove that ϕ^i_λ is surjective, suppose that x belongs to the center of $e_i\Pi^\lambda(Q)e_i$, x does not belong to the image of ϕ^i_λ, and has the smallest possible degree. Let x' be the term of highest degree of x (we again identify $\Pi^\lambda(Q)$ with S). Then x' belongs to the center of $e_i\Pi^0(Q)e_i$ and thus there is a homogeneous $y \in \mathcal{O}^\lambda(Q)$ such that $x' = \phi^i_0(y)$ (it follows at once from Corollary 2 that ϕ^i_0 is surjective). Consider $z = \phi_\lambda(y)$ and z', the term of the highest degree in z. Then z' is in the center of $\Pi^0(Q)$ and $e_0z'e_0$ is zero or equal to y. The first case is impossible due to Proposition 11. Thus $z' = \phi_0(y)$ and the term of the maximal degree of $\phi^i_\lambda(y) = e_ize_i$ equals x'. It follows that $x - \phi^i_\lambda(y)$ has degree lower than x and does not belong to the image of ϕ^i_λ, thus obtaining a contradiction. □

Proof of Theorem 3. The statement of Theorem 3 follows from Proposition 10 and the fact that if k is such that

$$\sum_{\sigma\in\mathcal{S}_k} \operatorname{sign}(\sigma)x_{\sigma(1)} \times^\lambda \dots \times^\lambda x_{\sigma(k)} = 0$$

for any $x_1, x_2, \dots, x_k \in e_iSe_i$, then denoting by x'_i the term of the maximal degree of x_i we get

$$\sum_{\sigma\in\mathcal{S}_k} \operatorname{sign}(\sigma)x'_{\sigma(1)} \times^0 \dots \times^0 x'_{\sigma(k)} = 0,$$

so from Corollary 2 we get that $k \geq 2\delta_i$. □

6. The proof of theorem 1

Consider a quiver C_n with n vertices, $I = \{1, 2, \ldots, n\}$, which form a chain,

$$n \xleftarrow[a_{n-1}]{} n-1 \xleftarrow[a_{n-2}]{} n-2 \longleftarrow \cdots \xleftarrow[a_1]{} 1$$

Suppose we have a sequence of complex numbers $\lambda = (\lambda_i)$, $i = 1, \ldots, n-1$. Consider the algebra

$$R_n^\lambda = e_n \left(\mathbb{C}\bar{C}_n / (\sum_{i=1}^{n-2} [a_i, a_i^*] - a_{n-1}^* a_{n-1} - \sum_{i=1}^{n-1} \lambda_i e_i) \right) e_n.$$

Proposition 12. *The algebra R_n^λ is isomorphic to the algebra $\mathbb{C}[x]/P(x)$ via an isomorphism sending x to $a_{n-1}a_{n-1}^*$, where $P(x)$ is a polynomial given by*

$$P(x) = x(x + \lambda_{n-1})(x + \lambda_{n-1} + \lambda_{n-2}) \ldots (x + \sum_{i=1}^{n-1} \lambda_i).$$

Proof. If $n = 1$ both algebras are isomorphic to $\mathbb{C}$. We proceed by induction. For $n > 1$, the algebra R_n^λ splits as a vector space,

$$R_n^\lambda = \mathbb{C} \oplus a_{n-1} e_{n-1} \left(\mathbb{C}\bar{Q} / (\sum_{i=1}^{n-1} [a_i, a_i^*] - a_{n-1}^* a_{n-1} - \sum_{i=1}^{n-1} \lambda_i e_i) \right) e_{n-1} a_{n-1}^*.$$

Then,

$$\begin{aligned} &e_{n-1} \left(\mathbb{C}\bar{C}_n / (\sum_{i=1}^{n-1} [a_i, a_i^*] - a_{n-1}^* a_{n-1} - \sum_{i=1}^{n-1} \lambda_i e_i) \right) e_{n-1} \\ &\cong (R_{n-1}^\lambda * \mathbb{C}[a_{n-1}^* a_{n-1}]) / (a_{n-2} a_{n-2}^* - a_{n-1}^* a_{n-1} - \lambda_{n-1} e_{n-1}), \end{aligned}$$

where $*$ denotes the free product of algebras. By the induction hypothesis, the latter is isomorphic to

$$\begin{aligned} &(\mathbb{C}[a_{n-2}a_{n-2}^*]/P^-(a_{n-2}a_{n-2}^*) * \mathbb{C}[a_{n-1}^* a_{n-1}]) \\ &\qquad /(a_{n-2}a_{n-2}^* - a_{n-1}^* a_{n-1} - \lambda_{n-1} e_{n-1}) \end{aligned}$$

for

$$P^-(x) = x(x + \lambda_{n-2})(x + \lambda_{n-2} + \lambda_{n-3}) \ldots (x + \sum_{i=1}^{n-2} \lambda_i),$$

so

$$e_{n-1}\left(\mathbb{C}\bar{C}_n/(\sum_{i=1}^{n-1}[a_i,a_i^*]-a_{n-1}^*a_{n-1}-\sum_{i=1}^{n-1}\lambda_i e_i)\right)e_{n-1}$$
$$\cong \mathbb{C}[a_{n-1}^*a_{n-1}]/P^-(a_{n-1}^*a_{n-1}+\lambda_{n-1}),$$

and therefore,

$$R_n^\lambda \cong \mathbb{C}[a_{n-1}a_{n-1}^*]/(P^-(a_{n-1}a_{n-1}^*+\lambda_{n-1})a_{n-1}a_{n-1}^*)$$

and it can be easily seen that

$$P^-(a_{n-1}a_{n-1}^*+\lambda_{n-1})a_{n-1}a_{n-1}^*=P(a_{n-1}a_{n-1}^*).$$

□

The theorem is now valid because $e_c\Pi^\lambda(Q)e_c$, defined as in the statement of the theorem, is isomorphic to the free product of the algebras $R_{\deg P_i-1}^{\lambda^i}$ factored by the relation

$$\sum_{i=1}^{n}a_{i1}a_{i1}^*=\mu e_c,$$

where

$$\lambda^i=(\alpha_{i\deg P_i-2}-\alpha_{i\deg P_i-1},\dots,\alpha_{i1}-\alpha_{i2},-\alpha_{i1})$$

and, by Proposition 12, each $R_{\deg P_i-1}^{\lambda^i}$ is isomorphic to

$$\mathbb{C}[a_{i1}a_{i1}^*]/P_i(a_{i1}a_{i1}^*).$$

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