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RESEARCH ARTICLE

Bounds for graphs of given girth and generalized polygons

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Abstract. In this paper we present a bound for bipartite graphs with average bidegrees η and ξ satisfying the inequality $\eta \geq \xi^{\alpha}$, $\alpha \geq 1$. This bound turns out to be the sharpest existing bound. Sizes of known families of finite generalized polygons are exactly on that bound. Finally, we present lower bounds for the numbers of points and lines of biregular graphs (tactical configurations) in terms of their bidegrees. We prove that finite generalized polygons have smallest possible order among tactical configuration of given bidegrees and girth. We also present an upper bound on the size of graphs of girth $g \geq 2t+1$. This bound has the same magnitude as that of Erdös bound, which estimates the size of graphs without cycles C_{2t}.

1. Introduction

Let Γ be a simple graph (undirected, no multiple edges, no loops) and let F be a family of graphs none of which is isomorphic to a subgraph of Γ. In this case we say that Γ is F-free.

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Let P be a certain graph theoretical property. By $ex_P(v, F)$ we denote the greatest number of edges of F-free graph on v-vertices, which satisfies property P.

If property P is trivial, that is, valid for all simple graphs we shall omit index P and write $ex(v, F)$.

Extremal graph theory contains several important results on $ex(v, F)$, where F is a finite collection of cycles of different length (see [2], [24]).

The following unpublished result of Paul Erdös is often refereed to as The Even Circuit Theorem (see [2], [24]).

Let C_n denote the cycle of length n. Then

$$\mathrm{ex}(v, C_{2k}) \leq Cv^{1+1/k},$$

where C is positive independent constant. For a proof of this result and its generalization: see [25], [26].

In [24] the upper bound

$$\mathrm{ex}(v, C_3, C_4, \ldots, C_{2k+1}) \leq (1/2)^{1+1/k} v^{1+1/k} + O(v)$$

was established for all integers $k \geq 1$.

In this paper we will prove that

$$\mathrm{ex}(v.C_3, C_4, \ldots, C_{2k}) \leq 1/2v^{1+1/k} + O(v),$$

and obtain the following bound

$$\mathrm{ex}_{P(m)}(C_3, C_4, \ldots C_{2t}) \leq 1/2^{1+1/(m+1)t} v^{1+1/(m+1)t} + O(v) \text{ for even } k,$$

and

$$\mathrm{ex}_{P(m)}(C_3, C_4, \ldots, C_{2t}) \leq v^{1+1/m(t+2)+t-1} + O(v) \text{ for all odd } k,$$

where $m > 1$ is a real number and $P(m)$ is a property: graph is bipartite with average bidegrees η and ξ satisfying inequality $\eta \geq \xi^m$.

Studies of $\mathrm{ex}_{P(m)}(v, C_3, \ldots, C_t)$ is motivated by some problems in Operation Research, Theory of Communication Networks and Cryptography. Among graphs satisfying $P(m)$ are tactical configurations, that is, biregular bipartite graphs.

In section 1 we shall establish some lower bounds for the numbers of points and lines of tactical configurations. In section 2 we shall consider tactical configurations of minimal order. This is a natural generalization of the well known cages (see [3] and further references). Section 3 is devoted to upper bounds for size of tactical configurations. In section 4 we shall develop an important technique for computing walks on bipartite graphs with given average bidegrees. This method allow us to generalize results of section 3 for more general case of graphs with the property $P(m)$. We shall establish $\mathrm{ex}(v, C_3, \ldots, C_{2k})$ and repeat the result of Erdös and Simonovits on $\mathrm{ex}(v, C_3, C_4, \ldots, C_{2k+1})$ in the last section.

2. Some inequalities for tactical configurations

A *tactical configuration* introduced by E. H. Moore [15] almost century ago is a rank two incidence structure $\Delta = \Delta(l, p, a, b)$ consisting of l lines and p points in which each line is incident to a points and each point is incident to b lines. We denote the incidence graph of Δ by $\Gamma = \Gamma(\Delta)$, though when no confusion arise we may abuse terminology and refer to Γ as a tactical configuration. We call bipartite graphs the incidence graphs of the incidence structures. If the structure is a tactical configuration, then the incidence graphs are called biregular with bidegrees a, b.

We shall assume that the graph $\Gamma(\Delta)$ has order $v = l + p$ (number of vertices), and size $e = la = pb$ (number of edges). We also mean, as usual, that the girth g of a graph is the length of its minimal cycle.

The following lemma gives a lower bound on the number of points in a tactical configuration of girth $\geq 2k$. It gives also a lower bound for the number of lines.

Lemma 1. *Let $\Gamma = \Gamma(\Delta)$ with $\Delta(l, p, s + 1, r + 1)$ of girth $\geq 2k$. Then the following inequalities hold*

1. *If $k = 2t + 1$, then*

$$(1 + r)\sum_{i=0}^{t}(rs)^i \leq p \tag{2.1}$$

$$(1 + s)\sum_{i=0}^{t}(rs)^i \leq l \tag{2.2}$$

2. *If $k = 2t$, then*

$$(1 + r)\sum_{i=0}^{t-1}(rs)^i + (rs)^t \leq p \tag{2.3}$$

$$(1 + s)\sum_{i=0}^{t-1}(rs)^i + (rs)^t \leq l \tag{2.4}$$

Proof. The approach we adopt in the proof has its root in the Theory of Branching Process in Applied Probability, see for example Karlin and Taylor [7]. The idea is to consider an arbitrary vertex v and count the number of vertices at a given distance d, $d \leq [g/2]$, where g is the girth.

Let us assume that we start counting from a point $v = p$. The pass of length $d \leq k$ between two chosen vertices is unique, because of absence of cycles of length $2d$. Thus, we may use the idea of branching processes to count the number of vertices at distance d from p.

If l_{2h+1} refers to the number of lines at distance $2h+1$ from p and p_{2h+2} refers to the number of points at distance $2h+2$ from p, then $l_1 = r+1$, $p_2 = (r+1)s$, $l_3 = (r+1)sr$, ... Finally, we get

$$l_{2h+1} = (s+1)r^h s^h \tag{2.5}$$

$$p_{2h+2} = (r+1)r^h s^{h+1} \tag{2.6}$$

where $h = 0, 1, \ldots, t$.

Summing (2.5) from $h = 0$ to t in case of odd t gives (2.2). Summing (2.6) from $h = 0$ to t in case of even t gives (2.3). By interchanging points and lines in (2.5) and (2.6)together with parameters r and s and using the same a argument as above we obtain (2.1) and (2.4). This completes the proof. □

Remark 1. *If $t+1 = s+1 = k$, then the order of the graph is $v = 2p = 2l$ and the inequalities in Lemma 1 are equivalent to the well known Tutte's inequality for arbitrary regular graph.*

$$v \geq 2(1 + (k-1) + \ldots (k-1)^{(g-2/2)})$$

3. Minimal configurations with prescribed girth and their applications

The well known assignment problem in Operations Research is equivalent to finding the tactical configuration of given bidegees $r+1$ and $s+1$ of minimal order. It is an important special case of the Transport Problem (see, for instance Taha [16]). There is a well known efficient algorithm to solve this problem. In many cases this algorithm can be modified to solve efficiently assignments problem with additional restrictions. In our case this translate to the problem of finding a tactical configuration with minimal number of vertices among graphs satisfying a the list of restrictions.

let us consider the case when the precise list of restrictions is the absence of cycles of length $4, 6, \ldots 2k-2$. One can notice that the incidence graph of tactical configuration does not have cycles of odd length and the last requirement is equivalent to inequality $g \geq 2k$. Unfortunately there is no known modification of existing assignments problem algorithms or other methods for the efficient solution of this problem.

Let $v(r,s,g)$ refers to the minimal order of a tactical configuration with bidegrees $s+1$ and $r+1$ and girth $g \geq 2k$, that is, the solution of the variant of assignment problem as above.

The problem of testing whether or not given tactical configuration $\Delta(r,s,l,p)$ of the girth g is a solution of the problem, that is, checking the condition $p+l = v(s,r,g)$ can be a very difficult one. The computation of function $v(r,s,g)$ is a hard problem of Applied Combinatorics.

We may assume, without loss of generality, that $r \geq s$. It is clear that if both inequalities (2.1) and (2.3) above turn out to be equalities, then our test is trivial and Δ is the solution of our variant of assignment problem. In this special situation we will use term "extra special configuration". In such a configuration we have the "best possible solution" of the problem. Of course , in the case of small bidegrees and girth we may easily find examples where tactical configuration Δ is not an extraspecial, but $p+l = v(r+1,s+1,g)$

We will use term "extraspecial" for graphs of extraspecial tactical configurations and regular graphs (not necessarily bipartite) of Tutte's bound order.

It is important that the totality of extraspecial configurations is non empty. Generalized m-gons were defined by J. Tits in 1959 (see [18], [19] and the survey [17]) as a tactical configurations of bidegrees $s+1$ and $t+1$ of girth $2m$ and diameter m. The pair (s,t) is known as order of generalized m-gon.

The following result is well known (see [3])

Theorem 1. *A finite generalized n-gon of order (s,t) has $n \in \{3,4,6,8,12\}$ unless $s=t=1$. If $s>1$ and $t>1$, then*

1. $n \neq 12$
2. *If $n=4$, then $s \leq t^2$, $t \leq s^2$;*
3. *If $n=6$, then st is a square and $s \leq t^3$, $t \leq s^3$;*
4. *If $n=8$, then $2st$ is a square and $s \leq t^2$, $t \leq s^2$;*

This is the original Feit-Higman theorem [6] combined with well known inequalities.

The known examples of generalized n-gons of bidegrees ≥ 3 and $m \in \{3,4,6,8\}$ are rank 2 incidence graphs of geometries of finite simple groups of Lie type. The regular incidence graphs are $m=3$ (group $A_2(q)$), $m=4$ (group $B_2(q)$ or $C_2(q)$), $m=6$ (group $G_2(q)$). In all cases $s=r=q$, where q is prime power.

The biregular but not regular generalized n-gons have parameters $s = q^\alpha$ and $t = q^\beta$, where q is some prime power. The list is below:

1. $n = 4$

   $s = q, r = q^2$ and q is arbitrary prime power,

   $s = q^2, r = q^3$ and q is arbitrary prime power

2. $n = 6$

   $s = q^2$, $t = q^3$ and $q = 3^{2k+1}$, $k > 1$

3. $n = 8$

   $s = q$, $t = q^2$ and $q = 2^{2k+1}$.

Besides finite generalized polygons related to simple groups of Lie type, which we consider above, there are important "nonclassical examples": nondezargezian projective plane, nonclassical generalized quadragons and hexagons (see [17] and further references).

Theorem 2. *Finite generalized polygons are extraspecial configurations.*

Proof. The order of regular generalized m-gons of degree $q + 1$ is $1 + q + q^2 + \cdots + q^{m-1}$ and reaches the Tutte's bound for graphs of girth $m - 2$. The finite irregular tactical configurations which are generalized polygons have to be of even diameter $m = 2k$. If their degrees are $r + 1$ and $s + 1$ then the numbers of points p and number of lines l can be computed by the formulas

$$\begin{aligned} p &= 1 + r + rs + r^2 s + r^2 s^2 + ... + r^k s^k + r^{k+1} s^k, \\ l &= 1 + s + sr + s^2 r + s^2 r^2 + ... + s^k r^{k+1} + s^{k+1} r^{k+1}, \end{aligned}$$

where k have to be an element of $\{2, 3, 4, 6\}$. They are at bounds (2.1) and (2.2) for points and lines.

Thus finite generalized m-gone is a perfect cage configuration. □

Application in Operations Research need explicit constructions of tactical configurations of given girth and bi-degrees of "small size", that is, close to bounds (3.1)-(3.4). General constructions of that kind are presented in [21].

3.1. Cages and $v(r, s, g)$ for $r = s$

We shall next examine the function $v(k, k, g)$ and regular extraspecial configurations. A cage (see [3]) is a $k = t + 1$-regular graph of given girth with the minimal number $v(k, g)$ of vertices. As it follows from definitions of functions $v(r, s, g)$, which is the minimal order of tactical configuration with bidegrees $r + 1$ and $s + 1$ of girth (see section 2) and $v(k, g)$

$$v(t, t, g) \geq v(t + 1, g)$$

Remark. We use same name for two functions but number of variables shall allow to distinguish them.

If we are dealing with t-regular extraspecial configuration, then $v(t, t, g)$ is same as $v(t, g)$ which achieves Tutte's bound.

The cage whose number of vertices is equal to this bound and whose girth is odd is called Moore graph. The only Moore's graph of degree 2 are $2n + 1$-gons. An m-gon is just a totality of vertices (points) and edges (lines) of ordinary cycle of length m with the natural incidence. A Moore graph of degree $k \geq 3$ has diameter 2 and $k \in \{3, 7, 51\}$.

We are interested in the case of even girth because tactical configurations are bipartite graphs and have no odd cycles. When the degree is 2, then we have a $2n$-gone which is an example of extraspecial configurations. In fact, the $(2, g)$-cage is the g-circuit, and $v(g, 2) = g$.

Let us list some well known families of cages of even girth.

(i) the $(k, 4)$-cage is the complete bipartite graph $K_{k,k}$ and $v(k, 4) = 2k$.

If $k = q + 1$ for a prime power q, then

(ii) a $(k, 6)$-cage is the incidence graph of a projective plane $PG(2, q)$, and $v(k, g) = 2(q^2 + q + 1)$;

(iii) a $(k, 8)$-cage is the incidence graph of a generalized quadrangle $CQ(q, q)$, and $v(k, g) = 2(q^3 + q^2 + q + 1)$;

(iv) a $(k, 12)$-cage is the incidence graph of a generalized hexagon $GH(q, q)$, and $v(k, q) = 2(q + 1)(q^4 + q^2 + 1)$

The $(3, 8)$-cage is the Tutte - Coxeter graph (v=30) [20].

One has $v(7, 6) = 90$ and the unique $(7, 6)$ cage was independently found in [8], [5]. Finally, there are 3 distinct $(3, 10)$- cages, all of them are bipartite [9], and $v(3, 10) = v(2, 2, 10) = 70$.

The problem of determining $v(k,g)$ was posed in 1959 by F. Kartesi who noticed that $v(3,5) = 10$ was realized by the Petersen graph. Sachs showed that $v(k,g)$ is finite and Erdös and Sashs gave the upper bound. This bound was improved in [10] for the best known general bound see [14]. For the case of bipartite graphs similar problem had been considered in [12]. A lower bound is given by Tutte's formula.

Applications in Operations Research, Cryptography, Networking also need constructions of regular graphs of a given gorth with the lowest known order. There are some interesting examples of cubic graphs of that kind (see [22] and further references).

4. Bounds for the size of tactical configurations

The minimization problem for the order of a graph with prescribed bidegrees r, s and girth g is equivalent to the maximization of the size (number of edges) of a graph with parameters r, s and g. The maximal number of edges of the graph of order v without cycles C_{2k} is estimated by Erdös Even Circuit Theorem.

Let $ex(v,n)$ be, as usual, the greatest number of edges (size) in a graph on v vertices, which contains no cycles C_3, C_4, …, C_n.

As it was mentioned in the introduction, from Erdös' Even Circuit Theorem and its modifications (see [2]) it follows that

$$ex(v,2k) \leq Cv^{1+1/k} \tag{4.1}$$

where C is a positive constant.

In the case of tactical configuration with the restriction on bidegrees it is possible to get a stronger bounds than the one given by the Even Cycle Theorem.

Let us consider some corollaries of the Lemma 1. Without loss of generality we will assume $r = a^m, s = a$, where $m \geq 1$

In case of $k = 2t$, we may omit all terms of the left hand side of (1.3) and (1.4) except highest terms, $a^{mt}a < p$, and $a^t a^{mt} < l$.

Adding last inequalities

we get $a^{(m+1)t} < v/2$, or $a < (v/2)^{1/((m+1)t)}$. We also have $l(a+1) = e$ or $la = e - l$. Thus $e - l < l(v/2)^{1/((m+1)t)}$.

Put v instead of l to get $e < v(v/2)^{(1/((m+1)t)} + v$, which leads to the next theorem

Theorem 3.

$$e \leq (1/2)^{(1/(m+1)t)} v^{(1+1/(m+1)t)} + v \tag{4.2}$$

Remark 2. *If $m = 1$ the magnitude of right hand side is same as that of Erdös Even Circuit Theorem, but the constant is better. The constant has monotonic dependence on m, and is always < 1. If $m > 1$, then (2.4) is stronger than Erdos inequality in magnitude. Of course (2.4) is applicable only to bipartite biregular graphs.*

Let us consider the case $k = 2t + 1$. If we discard some of the summands on the left hand side of (2.1) we get $r^t s^t + r^{t+1} s^t < p$. Set as before $r = a^m$, and $s = a$ to get $a^{mt+1}(a^m + 1) < p$. Also, $l(p + 1) = p(a^m + 1) = e$ gives $a^{mt+t}(l/p)(a + 1)l < p$ or $a^{mt+t}(a + 1)l < p^2 = l^2(a + 1)^2/(a^m + 1)^2$.
Simplifying last inequality we obtain

$$a^{mt+t}(a^m + 1)^2/(a + 1) < l.$$

Note that the function $f(a) = (a^m + 1)^2/(a + 1)$ is increasing.

Thus $f(a-1)a^{mt+t} < l$ or $a^{mt+t-1}[(a-1)^2+1] < l$. The last inequality then leads to $(a - 1)^{mt+2m+t-1} < l$ or $a - 1 < e^{(mt+2m+t-1)^{-1}}$. But we know that $l(a + 1) = e$. So $l(a - 1) = e - 2l$, and multiplication of two sides of the last inequality by l produces

$$e < l^{l+(m(t+2)+t-1)^{-1}} + 2l.$$

Order $v = p + l$ is $\geq l$, thus substitution of v instead of l gives us a slightly weaker inequality.

Theorem 4.

$$e \leq v^{1+1/(m(t+2)+t-1)} + 2v \tag{4.3}$$

Remark 3. *If $m = 1$, then the above bound has the same magnitude as that of Erdös bound in Even Circuit Theorem, but the constant is better than in (3.1). In fact we can improve the constant by substitution $l = v/2$ into inequality 3 to get.*

$$e \leq (1/2)^{1+1/(2t+1)} v^{1+1/(2t+1)} + v \tag{4.4}$$

If $m > 1$, then magnitude of (3.3) is better than that of Erdös bound.

Remark 4. *Theorems 3 and 4 give slightly better bounds than the upper bounds given in [21] (better constants but the same magnitude). This, we shall generalize for graphs with average bidegrees in next section.*

5. Bipartite graphs with given average bidegrees

Here, we assume that we have a random tactical configuration $\Delta = \Delta(l, p, a(\omega), b(\omega))$ consisting of l lines and p points laid out as a Branching Process. We shall assume, without loss of generality, that level zero consists of some line x_0, say. This line is incident to m points with probability $p(m)$, where $E(M) = \eta + 1$, where M denotes the random variable representing the outcomes m and $\eta \geq 1$ and is known. Here, E denotes the usual expectation operator.

Now, let ${X_n}^l$,(${X_n}^p$) be the number of lines (points, respectively) at level n. We shall assume from level 1 onwards that each line is incident to $a(\omega)$ points with probability $p(a(\omega))$, where $a(\omega)$ takes the integer values $0, ..., p$, with $E(a) = \eta$. Similarly, we have each point is incident to $b(\omega)$ lines with probability $p(b(\omega))$, with $E(b) = \xi$,where $\xi \geq 1$ and known.

If the girth of our graph is $> 2t$, then there is at most one pass between any two vertices at a distance $\leq t$. Points of level k are precisely at distance $2k+1$ from the initial line. The line of level k are at distance $2k$. Thus, computation of$X_k l$, $2k \leq k$ and X_k^p can be done by branching process.

We have

$$
\begin{aligned}
X_0^p &= M, \\
X_n^p &= \sum_{i=1}^{X_n^l} Z_i \\
X_n^l &= \sum_{j=1}^{X_{n-1}^l} Y_j
\end{aligned}
$$

where $Z_i's$ are i.i.d random variables, with mean η and variance σ_Z^2, corresponding to $a(\omega)$. The variables Y_j are i.i.d random variables corresponding to $b(\omega)$, with mean ξ and variance σ_Y^2. We shall be interested in finding a closed form for the means and the variances of the random variables X_n^p, X_n^l defined above.

The next two lemmas provide an answer to our query.

Lemma 2. $X_0^l = 1$,

(i) $E[X_n^p] = (\eta+1)(\eta\xi)^n, \;\; n = 1, ...$

Proof. The proof is standard: see Karlin and Taylor [7] for similar ideas. Note that

$$E[X_n^p] = E[E[X_n^p | X_n^l]].$$

Now, consider $E[X_n^p | X_n^l = x] = E[\sum_{i=1}^{x} Z_i] = xE[Z] = \eta x$,
because of the independence of Z_i. Hence

$$E[X_n^p] = \eta E[X_n^l]. \tag{5.1}$$

Now, we compute $E[{X_n}^l]$. Again $E[X_n^l] = E[E[X_n^l | X_{n-1}^p]]$. But,

$$E[X_n^l | X_{n-1}^p = x] = E[\sum_{j=1}^{x} Y_j] = \xi x,$$

by the independence, whence

$$E[X_n^l] = \xi E[X_{n-1}^p]. \tag{5.2}$$

by independence. Hence, combining (5.1) and (5.2) we get

$$E[X_n^p] = \eta\xi E[X_{n-1}^p] = (\eta\xi)^n E[X_0^p]$$

But $E[X_0^p] = \eta + 1$. Hence,

$$E[X_n^p] = (\eta + 1)(\eta\xi)^n. \tag{5.3}$$

To show part (ii) note that $E[X_n^l] = \xi E[X_{n-1}^p] = (\eta+1)\xi(\eta\xi)^{n-1}$, by (5.3), as required. □

The next lemma gives a bound on the variances of the random variables X_n^p and X_n^l.

Lemma 3.

(i) $Var(X_n^p) \leq \tilde{V}\left\{(\eta+1)(\xi+\eta^2)(\xi\eta)^{n-1}\frac{(\eta\xi)^{n-1}-1}{(\eta\xi)-1} + (\eta\xi)^{2(n-1)}\right\}$

(ii) $Var(X_n^l) \leq \tilde{V}\left\{\xi(\eta+1)(\xi+\eta^2)(\xi\eta)^{n-2}\frac{(\eta\xi)^{n-1}-1}{(\eta\xi)-1} + (\eta\xi)^{2(n-1)}\right\}$,

where $\tilde{V} = \max\{Var(X), Var(Z), Var(X_0^p), Var(X_0^l)\}$.

Proof. We shall only prove (i). The proof of (ii) is similar. Note that

$$Var(X_n^p) = E[(X_n^p)^2] - (E[X_n^p])^2. \tag{5.4}$$

Let us compute $E[(X_n^p)^2]$. We have $E[(X_n^p)^2] = E[E[(X_n^p)^2 | X_n^l]]$, and

$$\begin{aligned} E[(X_n^p)^2 | X_n^l &= x] = E\left[(\sum_{i=1}^{x} Z_i)^2\right] = Var\left[\sum_{i=1}^{x} Z_i\right] + \left(E\left[\sum_{i=1}^{x} Z_i\right]\right)^2 \\ &= xVar(Z) + (x\eta)^2, \end{aligned}$$

by independence. Hence

$$E[(X_n^p)^2] = Var(Z)E[X_n^l] + \eta^2 E[(X_n^l)^2]. \tag{5.5}$$

Now, we are required to compute $E[(X_n^l)^2]$. The same argument used above gives

$$E[(X_n^l)^2] = Var(Y)E[X_{n-1}^p] + \xi^2 E[(X_{n-1}^p)^2]. \tag{5.6}$$

Combining (5.4), (5.5), and (5.6) gives

$$\begin{aligned} Var(X_n^p) &= Var(Z)E[X_n^l]+ \\ &\quad +\eta^2\left\{Var(Y)E[X_{n-1}^p] + \xi^2 E[(X_{n-1}^p)^2]\right\} - (E[X_n^p])^2. \end{aligned}$$

This can be shown to be equal to:

$$Var(Z)E[X_n^l] + \eta^2 Var(Y)E[X_{n-1}^p] + (\eta\xi)^2 Var(X_{n-1}^p).$$

Using the previous lemma the above is less than or equal to:

$$\max\{Var(Y), Var(Z)\}(\eta+1)(\eta\xi)^{n-1}(\xi+\eta^2) + (\eta\xi)^2 Var(X_{n-1}^p).$$

Now, we use induction to get

$$Var(X_n^p) \le \tilde{V}\left\{(\eta+1)(\xi+\eta^2)(\xi\eta)^{n-1}\frac{(\eta\xi)^{n-1}-1}{(\eta\xi)-1} + (\eta\xi)^{2(n-1)}\right\},$$

where $\tilde{V} = \max\{Var(X), Var(Z), Var(X_0^p), Var(X_0^l)\}$, as required. □

The next lemma (which is a direct consequence of lemmas 3 and 4 and Chebeychev inequality: see [7]) gives confidence intervals for both X_n^p and X_n^l.

(i) The confidence interval for X_n^p is

$$(\eta+1)(\eta\xi)^n \pm k s_p,$$

for some nonnegative $k > 0$ and

$$s_p = (\tilde{V}\left\{(\eta+1)(\xi+\eta^2)(\xi\eta)^{n-1}\frac{(\eta\xi)^{n-1}-1}{(\eta\xi)-1} + (\eta\xi)^{2(n-1)}\right\})^{1/2}. \tag{5.7}$$

(ii) The confidence interval for X_n^l is

$$(\eta+)\xi(\eta\xi)^{n-1} \pm k' s_l,$$

for some nonnegative $k > 0$, and

$$s_l = (\tilde{V}\left\{\xi(\eta+1)(\xi+\eta^2)(\xi\eta)^{n-2}\frac{(\eta\xi)^{n-1}-1}{(\eta\xi)-1} + (\eta\xi)^{2(n-1)}\right\})^{1/2}. \quad (5.8)$$

Remark 5. *Note that (4.7) shows that the order of X_n^p is at most $O((\eta+1)(\eta\xi)^n)$, while the order of X_n^l is at most $O((\eta+1)\xi(\eta\xi)^{n-1})$.*

6. On the size of general bipartite graphs

In this section we will consider upper bounds for the size of bipartite graphs G_i of increasing order $v = v_i$ without cycles of girth $g > 2k$ satisfying inequality $\eta_i \geq \xi_i^m$, where $m \geq 1$ is some positive real number.

and superlinear size without cycles C_{2k}.

We have a free choice which partition set is the point set. So we may assume that $\eta_i \geq \xi_i$ (the average degree for points is greater than or equal to average degree of lines). Thus our result for $m = 1$ estimates size of general bipartite graphs of a given girth.

Theorem 5. *Let G_i, $i = 1, \ldots$ be a family of bipartite graphs without even cycles $C_4, \ldots, C_{2k}$ such that average degrees η_i and ξ_i of lines and points satisfy the inequality: $\eta_i \geq {\xi_i}^m$. Then, we have:*

(i) $e \leq (1/2)^{(1/(m+1)t)} pv^{(1/(m+1)t)} + O(v)$

(ii) $e \leq p^{1+1/(m(t+2)+t-1)} + O(v)$

hold for cases $k = 2t$ and $k = 2t + 1$ respectively.

Proof. Let G_i, $i = 1, \ldots$ be a family of graphs satisfying the restrictions on the bidegrees and the girths as above. It follows from [12], that the size of the bipartite graphs of a given girth, with the restrictions on the bidegrees as stated above, is superlinear function Cv^α, $\alpha > 1$ of order v. Thus, we may assume that function ξ_i is unbounded. Else, we may bound the number of edges of the graphs by a linear expression in v. In fact, we shall conduct all computations up to $O(v)$. We will also keep the notations of the previous section: η_i and ξ_i will be the average degrees for the lines and the points respectively. Without loss of generality we may assume $\eta_i \geq \xi_i$. $i = 1, \ldots$.

Let us consider case $k = 2t$. In this case, there is not more then one pass of length $\leq 2t$ between two given elements at distance $2t$. Hence, we may apply result (4.8) to get, if ξ is "sufficiently large"

$$X_t^l = (\eta+1)\xi(\eta\xi)^{t-1} - C_1\eta^{(t-1/2)}\xi^{n-3/2} \leq l.$$

The expression one left hand side gives us the number of lines at distance $2t$ from chosen line, where C_1 is some constant. If we swap points and lines together with their average degrees we get"

$$(\xi+1)\eta(\eta\xi)^{t-1} - (\xi\eta)^{t-1} \leq p.$$

Addition of last two inequalities gives us

$$2(\eta\xi)^t + [(\eta+\xi)(\eta\phi)^{t-1} - C_1(\eta\xi)^{t-1}(\eta/\xi)^{1/2} - (\eta\xi)^{t-1}] \leq (p+l) = v$$

when ξ is sufficiently large, expression in parenthesis is positive and we are getting

$$2(\eta\xi)^t < v.$$

For $\xi = a$ and $\eta \geq a^m$ we may write

$$a < (v/2)^{1/(m+1)^t}.$$

Analogously to similar case for biregular graphs we are getting $pa \leq p(v/2)^{1/(m+1)^t}$. The last inequality together with $p(a+1) = e$ gives us the following bound for the size e

$$e < p(v/2)^{1/(m+1)t} + O(v).$$

Let us consider the case $k = 2t+1$. It follows from (4.8) that X_t^l is at least $(\eta+1)\eta^t\xi^t - C\eta^t\xi^{t-3/2}$, where C is some positive constant. Thus instead of the inequality ${X^l}_t + {X^l}_{t-1} \leq l$ we can write:

$$\eta^{t+1}\xi^t + \eta^t\xi^t + (\eta^t\xi^{t-1} + \eta^{t-1}\xi^{t-1} - C\eta^t\xi^{t-3/2}).$$

For sufficiently large ξ, the expression in brackets in the previous formula will be negative and we get $\eta^{t+1}\xi^t + \eta^t\xi^t < l$. Setting as before $\eta \geq a^m$, $\xi = a$. Thus $a^{mt+1}(a^m+1) < l$. From $p\xi = l\eta$, we get $e = p(a+1) > l(a^m+1)$ and $(a^m+1) \leq p(a+1)/la^{mt+t}(p/l)(a+1)l < l$ or

$$^{mt+t}(a+1)l < l^2 = p^2(a+1)^2/(a^m+1)^2 a.$$

Simplifying the last inequality we obtain

$$a^{mt+t}(a^m+1)^2/(a+1) < p.$$

We can notice that the function $f(a) = (a^m+1)^2/(a+1)$ is increasing. Thus $f(a-1)a^{mt+t} < l$ or $a^{mt+t-1}[(a-1)^2+p] < l$. From the last inequality we get $(a-1)^{mt+2m+t-1} < l$ or $a-1 < l^{(mt+2m+t-1)^{-1}}$. We know that $p(a+1) = e$. So, $p(a-1) = e - 2l$ and multiplication of the two sides of the last inequality by l gives $e < p^{p(m(t+2)+t-1)^{-1}} + O(p)$ of lines and the number of pints of G_i satisfy the inequality $\eta_i \geq {\xi_i}^m$ for certain real number $m > 1$. Then, $e \leq (1/2)^{1+1/k} v^{1+1/k} + O(v)$ in the case of k even and $e \leq v^{1+1/k} + O(v)$ if k is odd. □

Remark 6. *The bounds in the theorem 5 are sharp up to constant when we deal with families of generalized $k+1$-gons. In particular for $m > 1$, we have the following list: $(m = 2, k = 3)$, $(m = 3/2, k = 3$, $(m = 3/2, k = 5)$, $(m = 2, k = 7)$.*

7. On the size of general graphs of high girth

In this section, we shall be concerned with the size of graphs of large girth as function of the order.

Theorem 6. *Let $F = \{G_j\}$, $j = 1, \ldots$ be a family of bipartite graphs without of cycles C_i, $3 \leq i \leq n$, $n \geq 3$, that is, a family of graphs of girth $g > n$. Let e and v be the size and the order of graphs from F respectively.*

Then

(i)$e \leq (1/2) v^{1+1/k} + O(v)$ for $n = 2k$

(ii) $e \leq (1/2)^{1+1/k} v^{1+1/k} + O(v)$ for $n = 2k+1$

Proof. Let η be the average degree of graph G_i. Let us consider the case $n = 2t+1$. If the girth $g \geq 2k+2$, then there is at most 1 pass between vertices at distance k, we can choose adjacent vertices v and u and count the number of passes at distance $\leq t$ via branching process. Let $Y_l(u)$ $(Y_l(w))$be the totality of vertices x of length l, $l \leq k+1$ such that the pass between u and v does not contain w (u respectively). It is clear, that $|Y_l(u)| = |Y_l(v)| = y_l$. We have $|Y_l(u) \cap Y_s(v)| = 0$, because common point for $Y_l(u)$ and $Y_s(w)$ corresponds to cycle of length $l+s+1 \leq 2k+1$. The induced subgraph of G_i with the union of all $Y_l(u)$ and $Y_s(v$ is a tree which is a bipartite graph. Thus, we can estimate the number y_l via the technique of section 5. We need just to take in account that in our case $\eta = \xi$ and at the first step of branching process we have η instead of $\eta + 1$. Thus $y_l \geq \eta^{l-1} - C\eta^{l-2}$, $l = 1, \ldots, k+1$. After summation of the above inequalities and multiplication by 2 we get that the number Ind of all vertices for our tree is at least

$$2\eta^k + 2(1 + \eta + \ldots \eta^{t-1} - C1\eta^{t-2}).$$

If parameter i for G_i and related η are "sufficiently large" then The expression in brackets above will be positive. Thus $\eta k < Ind < (v/2)$,

$\eta < (v/2)^{1/k}$ and $(v/2)\eta(v/2)^{1+1/k}$. But $(v/2)(\eta+1) = e$ or $(v/2)\eta = e - v/2$. Finally $e < (1/2)^{1+1/k}v^{1+1/k} + O(v)$ and the statement (i) of the theorem is proven.

Let us consider the case of $n = 2k$. There is at most one pass between two given vertices at the distance l, $l \leq 2k$ otherwise we have a cycle C_{2l-2}. Thus we may choose a vertex v and count number X_l vertices at distance l from v by branching process with $\eta = \xi$. As it follows from results of section 4 s_p (s_l, respectively) is less then $C\eta^{k-2}$, where k is a highest degree of η in the expression for X_p^n (respectively X_l^n) for appropriate n , where C is a certain constant. Thus

$$X_l \geq (\eta+1)\eta^{k-1} - C\eta^{k-2}.$$

So from $X_l + X_{l-1} \leq v$ we can obtain

$$\eta^k + [2\eta^{k-1} + \eta^{k-2} - C1\eta^{k-2}] \leq v.$$

If η is sufficiently large then the expression in parenthesis is positive and we can write simply $\eta^k \leq v$. We can get $\eta \leq v^{1/k}$. Multiplication by v of both sides of last inequality together with $2e = (\eta+1)v$ gives us $e \leq 1/2v^{1+1/k} + O(v)$. □

8. Conclusion

Let us reformulate main results in terms of ex notations.

We presented an upper bound on the $\mathrm{ex}(v, C_3, \ldots, C_{2n})$ and a bound $\mathrm{ex}_{P(m)}(v, C_3, \ldots, C_t)$, where $P(m)$ is a property of the bipartite graph whose average bidegrees η and ξ satisfy the inequality $\eta \geq \xi^m$, $m \geq 1$. We proved that the sizes of the tactical configurations of finite generalized polygons are exactly on that bound. In fact, we proved that finite generalized polygons have minimal possible order among tactical configurations of the same bidegrees and girth.

Upper bounds for $ex(v, C_{2n})$ are known to be sharp up to constant in case of $n \in \{2, 3, 5\}$. The question on the sharpness of this bound for other n is still open. We conjecture that our bound for the $\mathrm{ex}_{P(m)}(v, C_3, \ldots C_{2m})$ is sharp if and only if (m, n) belongs to the following list: $(2,3), (3/2,3), (3/2,5), (2,14)$.

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