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Locally soluble groups with the restrictions on the generalized norms

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Abstract. The author studies groups with given restrictions on norms of decomposable and Abelian non-cyclic subgroups. The properties of non-periodic locally soluble groups, in which such norms are non-identity and have the identity intersection, are described.

Introduction

In group theory findings related to the study of the impact of properties of the different systems of the subgroups on the group are in focus. This direction includes findings when the restrictions are imposed on the different Σ-norms.

Let Σ be the system of all subgroups of G with a certain theoretical group property. The maximal subgroup of G which normalizes every subgroup of Σ is called Σ-norm of a group G. It is clear that the Σ-norm of a group G coincides with the intersection of the normalizers of all subgroups included in Σ, and contains the center of a group.

In the case when Σ-norm of a group coincides with the group, all subgroups of Σ are normal in the group (assuming that the system Σ is

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non-empty). For the first time non-Abelian groups with such property were considered in the second part of the XIX century by R. Dedekind, who characterized groups, all subgroups of which are normal (nowadays these groups are called Dedekind groups). However, the systematic study of groups with different systems of normal subgroups were continued only in the second part of the XX century, that slowed down the study of Σ-norms. So, the question on the study of the properties of groups, in which the Σ-norm is a proper subgroup, arises naturally.

For the first time such problem was formulated by R. Baer in 30s of the previous century for the system Σ of all subgroups of a group [1]. Such Σ-norm was called the norm $N(G)$ of a group G and denoted as the intersection of normalizers of all subgroups of a group G. Later the findings of R. Baer on the norm of a group were extended on the different systems of subgroups Σ and on the different restrictions, which the Σ-norms satisfies (see e.g. [2]–[7]). It is clear that the norm $N(G)$ is contained in the other Σ-norms, which, in turn, can be regarded as its generalizations.

In the paper, we consider the relations between the norm of decomposable subgroups and the norm of Abelian non-cyclic subgroups of a group. The *norm* N_G^d *of decomposable subgroups* of a group G is the intersection of the normalizers of all decomposable subgroups of a group or group itself, if the system of such subgroups is empty [7]. Recall that a subgroup of a group G is called *decomposable* if it can be representable in the form of the direct product of two non-trivial factors [8].

It is clear, that in the case when $N_G^d = G$, all decomposable subgroups are normal in a group G or the system of such subgroups is empty. Non-Abelian groups with such property were studied in [8] and called *di-groups.*

Obviously, the presence of decomposable subgroups in a group is directly related to the existence of decomposable Abelian subgroups, which in most cases are non-cyclic. So, the norm N_G^d of decomposable subgroups of group G is closely related to the norm N_G^A of Abelian non-cyclic subgroups.

The intersection of normalizers of all non-cyclic Abelian subgroups of a group G (provided that the system of these subgroups is non-empty) is called the *norm of non-cyclic Abelian subgroups of a group* G and denoted by N_G^A (see e.g. [6, 9]). If the norm N_G^A contains at least one Abelian non-cyclic subgroup, then each such a subgroup is normal in N_G^A. Non-Abelian groups with this property were studied by F. Lyman in [10] and called $\overline{HA}$*–group.* So, the norm of Abelian non-cyclic subgroups is Dedekind or non-Hamiltonian $\overline{HA}$-group.

The relations between these norms has been investigated in [7, 11, 12] for quite broad classes of groups. In [7] it was proved that in locally finite groups, that contain an Abelian non-cyclic subgroup, one of the ratios holds: $N_G^A \subseteq N_G^d, N_G^A \supseteq N_G^d$.

In particular, it was found that a periodic locally nilpotent group has the non-Dedekind norm of decomposable subgroups if and only if it is a locally finite p-group and $N_G^d = N_G^A$. The same relations between norms can be traced in an arbitrarily locally finite group with the non-Dedekind locally nilpotent norm N_G^d of decomposable subgroups (see [11]).

In [12] the study of the relations between the norm of decomposable subgroups and the norm of Abelian non-cyclic subgroups was continued in the class of non-periodic locally soluble groups. It was proved the following. When either at least one of the norms N_G^A or N_G^d is non-Dedekind or the subgroup N_G^d is infinite, then one of the following inclusions takes place: $N_G^A \subseteq N_G^d$ or $N_G^d \subseteq N_G^A$.

The purpose of the article is to study the properties of locally soluble groups in which the norm of decomposable subgroups and the norm of Abelian non-cyclic subgroups are nonidentity and have the identity intersection $N_G^d \cap N_G^A = E$.

1. Preliminary Results

The next statements are actively used in the further research.

Lemma 1. *([7]) If a group G contains a nonidentity N_G^d-admissible subgroup H such that $N_G^d \bigcap H = E$, where N_G^d is the norm of decomposable subgroups, then N_G^d is Dedekind.*

Lemma 2. *([6]) If a group G contains an Abelian non-cyclic subgroup H such that $N_G^A \bigcap H = E$, where N_G^A is the norm of Abelian non-cyclic subgroups, then the norm N_G^A is Dedekind (Abelian, if a group is non-periodic).*

Lemma 3. *([6]) Let G be a non-periodic group, N_G^A be the norm of Abelian non-cyclic subgroups and a group G contain a nonidentity N_G^A-admissible subgroup H such that $N_G^A \bigcap H = E$. If the norm N_G^A is non-periodic, then all infinite cyclic subgroups are normal in it.*

The following statement reduces the study of groups, in which the norms N_G^A of Abelian non-cyclic subgroups and the norm N_G^d of decomposable subgroups are nonidentity and $N_G^d \cap N_G^A = E$, to the study of non-periodic groups.

Theorem 1. *If a locally soluble group G contains an Abelian non-cyclic subgroup, the norm N_G^A of Abelian non-cyclic subgroups, the norm N_G^d*

of decomposable subgroups are nonidentity and $N_G^d \cap N_G^A = E$, then G is a non-periodic group.

Proof. Suppose that G is a periodic group. Then it is locally finite and either $N_G^A \supseteq N_G^d$ or $N_G^A \subseteq N_G^d$ by Theorem 1.4 [7], which contradicts the condition of the Theorem. Thus, G is a non-periodic locally soluble group and the Theorem is proved. □

Further we will consider only non-periodic groups in which the norms N_G^d and N_G^A are nonidentity and their intersection is identity. The existence of non-periodic groups with given restrictions on the norm of Abelian noncyclic and the norm of decomposable subgroups is confirmed by the following examples.

Example 1. ([12], example 3.5). Let $G = (\langle a\rangle \leftthreetimes B) \leftthreetimes \langle c\rangle$, where $|a| = p$ (p is prime, $p \neq 2$), B be a group, isomorphic to an additive group of p-adic numbers, $B = B_1\langle x\rangle$, $x^2 \in B_1, x^{-1}ax = a^{-1}, [B_1, \langle a\rangle] = E, |c| = 2, [c, a] = 1, c^{-1}bc = b^{-1}$ for any element $b \in B$.

In this group the norm of decomposable subgroups $N_G^d = \langle a\rangle$ is a cyclic subgroup of prime order. At the same time, the norm of non-cyclic Abelian subgroups is non-Dedekind, $N_G^A = B_1 \leftthreetimes \langle c\rangle$ and $N_G^d \cap N_G^A = E$.

2. The main results

The aim of this section is to study the properties of non-periodic locally soluble groups and the structure of the norms N_G^A and N_G^d, provided that $N_G^d \cap N_G^A = E$. The first of the following theorems characterizes the groups with the non-Dedekind norm N_G^A, respectively, the second theorem describes groups in which the norm N_G^A is the Dedekind.

Theorem 2. *If a non-periodic locally soluble group G has an Abelian non-cyclic subgroup, the norm N_G^A of Abelian non-cyclic subgroups is non-Dedekind, the norm N_G^d of decomposable subgroups is nonidentity and $N_G^d \cap N_G^A = E$, then the following conditions take place:*

1) *$Z(G) = N(G) = E$, where $N(G)$ is the norm of G;*
2) *the norm of decomposable subgroups $N_G^d = \langle c\rangle$ is a cyclic group of a prime odd order p;*
3) *the norm N_G^A of Abelian non-cyclic subgroups is a group of the type $N_G^A = A \leftthreetimes \langle b\rangle$, where A is a group isomorphic to an additive group of p-adic numbers (p is prime, $(p, 2) = 1$), $|b| = 2$ and $b^{-1}ab = a^{-1}$ for any element $a \in A$;*
4) *any infinite cyclic subgroup has a nonidentity intersection with the norm N_G^A;*

5) *a group G does not contain free Abelian subgroups of rank 2;*
6) *a group G does not contain finite non-cyclic Abelian subgroups;*
7) *a group G does not contain periodic non-cyclic locally cyclic subgroups;*
8) *the factor-group G/N_G^A is periodic.*

Proof. Let group G satisfy the conditions of the theorem. Then the first statement of the theorem follows from the inclusions:

$$Z(G) \subseteq N(G) \subseteq N_G^d \cap N_G^A = E.$$

Let's show that the norm N_G^d of decomposable subgroups is a cyclic group. Indeed, since $N_G^d \cap N_G^A = E$, the subgroup N_G^d is Dedekind by Lemma 1. If N_G^d contains non-cyclic Abelian subgroups, then the norm N_G^A of Abelian non-cyclic subgroups is also Dedekind by Lemma 2, which contradicts the condition. Thus, N_G^d does not contain non-cyclic Abelian subgroups.

Since N_G^d is a Dedekind group by the proved above, N_G^d is a finite group and its Sylow p-subgroups are cyclic for $p \neq 2$ and the Sylow 2-subgroup is either a cyclic group or the quaternion group. Taking into account the condition $Z(G) = E$, we make a conclusion that the order of the norm N_G^d is not divided to 2, because otherwise N_G^d contains a central involution. Therefore, N_G^d is a cyclic group, $N_G^d = \langle c \rangle$ and $(|c|, 2) = 1$.

Considering that $\left|N_G^d\right| < \infty$, we obtain $\left[G : C_G(N_G^d)\right] < \infty$ and $x^m \in C_G\left(N_G^d\right), m \in \mathbb{N}$ for an arbitrary element $x \in G, |x| = \infty$. Then the subgroup $\langle x^m, c \rangle$ is Abelian non-cyclic and N_G^A-admissible. If $\langle x \rangle \cap N_G^A = E$, then $\langle x^m, c \rangle \cap N_G^A = E$, and the norm N_G^A is Dedekind by Lemma 2, which is impossible. Therefore, N_G^A is a non-periodic group, and any infinite cyclic subgroup $\langle x \rangle$ of a group G has a nonidentity intersection with the norm N_G^A.

Since the subgroup $\langle c \rangle$ is N_G^A-admissible and $\langle c \rangle \cap N_G^A = E$, all infinite cyclic subgroups are normal in N_G^A by Lemma 3. By the description of such groups (see [5]) N_G^A is a group of the type $N_G^A = A\langle b \rangle$, where A is a non-periodic Abelian group, $|b| \in \{2, 4\}, b^2 \in A$ and $b^{-1}ab = a^{-1}$ for any element $a \in A$.

Let's prove that the norm $N_G^d = \langle c \rangle$ of decomposable subgroups is a group of prime odd order p. Suppose that $\langle c \rangle \supseteq \langle c_1 \rangle \times \langle c_2 \rangle$, where $|c_1| = p, |c_2| = q, (p, q) = 1$, p and q are odd prime. Then by the condition

$$[N_G^d, N_G^A] \subseteq N_G^A \cap N_G^d = E,$$

it follows that $|ac_1| = \infty$ for any arbitrary element $a \in A$, $|a| = \infty$. Since the subgroup $\langle ac_1, c_2 \rangle$ is Abelian non-cyclic, it is N_G^A-admissible. Hence, the subgroup $\langle (ac_1)^q \rangle$ is also N_G^A-admissible. It is clear that the element $b \in N_G^A$ cannot be permutable with the element $(ac_1)^q$, because in this case $[(ac_1)^q, b] = [a^q, b] = 1$, which is impossible. So,

$$b^{-1}(ac_1)^q b = (ac_1)^{-q} = a^{-q}c_1^q = a^{-q}c_1^{-q}$$

and $c_1^{2q} = 1$. We have a contradiction. Therefore, $|c| = p^k$, where p is prime, $k \in \mathbb{N}$.

If $k > 1$, then the subgroups $\langle ac \rangle \times \langle c^{p^{k-1}} \rangle$ and $\langle (ac)^p \rangle = \langle a^p c^p \rangle$ are N_G^A-admissible. Therefore, considering that $|ac| = \infty$ and $b^{-1}ab = a^{-1}$, where $b \in N_G^A$, we have

$$b^{-1}a^p c^p b = (a^p c^p)^{-1} = a^{-p}c^{-p} = a^{-p}c^p.$$

Then $c^{-p} = c^p$ and $c^{2p} = 1$, which is impossible. Therefore, $N_G^d = \langle c \rangle$, where $|c| = p, p \neq 2$.

Let's specify the structure of the subgroup $A \subseteq N_G^A$. Assume that A is a mixed group and $T(A)$ is its periodic part. Since $N_G^d \cap N_G^A = E$, the group G contains an indecomposable non-cyclic Abelian subgroup H which is not N_G^d-admissible. Clearly, H cannot be a complete group, because otherwise $H \subseteq C_G\left(N_G^d\right)$, which contradicts its choice. Therefore, H is the incomplete non-cyclic Abelian torsion-free group of rank 1.

Since $[\langle a \rangle, H] \subseteq T(A) \cap H = E$ for an arbitrary non-identity element $a \in T(A)$, the subgroup $\langle a \rangle \times H$ is Abelian decomposable and, therefore, N_G^d-admissible. Then by the condition

$$[N_G^d, H] \subseteq N_G^d \cap (\langle a \rangle \times H) = E,$$

the subgroup H is also N_G^d-admissible, which is impossible. Thus, A is an Abelian torsion-free group. Since $b^2 \in A, |b| = 2$ and $N_G^A = A \leftthreetimes \langle b \rangle$, where $b^{-1}ab = a^{-1}$ for an arbitrary element $a \in A$.

Suppose that the group A contains a free abelian subgroup $\langle a_1 \rangle \times \langle a_2 \rangle$, where $|a_1| = |a_2| = \infty$. Since H is a non-periodic Abelian torsion-free group of rank 1 and $N_G^A \cap H \neq E$ by the proved, at least one of the subgroups $\langle a_1 \rangle$ or $\langle a_2 \rangle$ has an identity intersection with H. Let $\langle a_1 \rangle \cap H = E$. Considering that H is a N_G^A-admissible subgroup, let $a_1^{-1}h_1 a_1 = h_2$, where $h_1, h_2 \in H$. Then by the condition $A \cap H \neq E$ we have $h_1^k \in A$ for some positive integer k. Thus, $a_1^{-1}h_1^k a_1 = h_1^k = h_2^k$ $h_1 = h_2$ and $[\langle a_1 \rangle, H] = E$.

It is clear that the subgroup $\langle a_1^m \rangle \times H$ is N_G^d-admissible for an arbitrary positive integer number m. Therefore, $H = \bigcap_{m=1}^{\infty}(\langle a_1^m \rangle \times H)$ is also N_G^d-admissible subgroup, which contradicts the choice of H. So, the subgroup A does not contain free Abelian subgroups of rank 2 and is an Abelian torsion-free group of rank 1.

Let us consider the group

$$G_1 = N_G^A \times N_G^d = A \leftthreetimes \langle b \rangle \times \langle c \rangle,$$

where A is a torsion-free group of rank 1, $|b| = 2, b^{-1}ab = a^{-1}$ for any element $a \in A$ and $|c| = p$, $p \neq 2$. Since N_G^A is a subgroup of the norm $N_{G_1}^A$ of Abelian non-cyclic subgroups of the group G_1 and $c \in Z(G_1)$, we have $G_1 = N_{G_1}^A$ and G_1 is a $\overline{HA}$-group. By the description of such groups (see, e.g. [10]) A is an infinite cyclic group or a group isomorphic to an additive group of p-adic numbers.

Assume that $A = \langle a \rangle$ is an infinite cyclic group. Then $\left(N_G^A\right)' = \langle a^2 \rangle \lhd G$, $C = C_G(\langle a^2 \rangle) \lhd G$ and $[G : C] \leq 2$. Since $b \notin C$, we have $G = C \leftthreetimes \langle b \rangle$, where $|b| = 2$. By the results of [13] the centralizer C contains all elements of infinite order of a group G and all its Abelian non-cyclic subgroups. Moreover, the periodic part $T(C)$ of the subgroup C is normal in G and $C/T(C)$ is an Abelian torsion-free group of rank 1. Therefore, in this case the commutant C' is periodic and $C' \subseteq T(C)$.

As $[N_G^d, N_G^A] = E$, it follows that $N_G^d = \langle c \rangle \subseteq T(C)$. Taking into account the non-Dedekindness of the norm N_G^A and Lemma 2, we conclude that $T(C)$ does not contain Abelian non-cyclic subgroups, so $|T(C)| < \infty$. Let u be a non-identity element of $T(C)$. Since the group $\langle u, a^2 \rangle = \langle u \rangle \times \langle a^2 \rangle$ and its characteristic subgroup $\langle u \rangle$ are N_G^A-admissible,

$$[\langle u \rangle, N_G^A] \subseteq N_G^A \cap \langle u \rangle = E.$$

Hence, $[T(C), N_G^A] = E$ and $[u, b] = 1$ for an arbitrary element $u \in T(C)$.

Suppose that $T(C)$ contains an involution z. Then subgroup $\langle b \rangle \times \langle z \rangle$ is Abelian non-cyclic and, therefore, N_G^A-admissible. Thus,

$$\langle b \rangle = N_G^A \cap (\langle b \rangle \times \langle z \rangle) \lhd N_G^A,$$

which is impossible. So, $2 \notin \pi(T(C)$.

Let us prove that $T(C) = N_G^d$. Suppose for a contradiction that there exists an element $u \in T(C) \backslash \langle c \rangle$. Since $T(C)$ does not contain non-cyclic Abelian subgroups, then $|u| \neq p$ and the element c is contained in each cyclic p-subgroup of the composite order. Therefore, if $|u| = p^k > p, k \in \mathbb{N}$, then $\langle c \rangle \subseteq \langle u \rangle$ and $[u, c] = 1$. Now let $(|u|, p) = 1$. Thus,

$$[u, c] \in (\langle c \rangle \cap (\langle u \rangle \times \langle a^2 \rangle)) = E$$

and again $[u, c] = 1$. Since the subgroup $\langle ua^2 \rangle \times \langle c \rangle$ is Abelian non-cyclic and therefore N_G^A-admissible, we conclude that the subgroup $\langle ua^2 \rangle^p = \langle u^p a^{2p} \rangle$ is also N_G^A-admissible. Then by the condition $[b, a] \neq 1$ we have that $[b, u^p a^{2p}] \neq 1$ and

$$b^{-1} u^p a^{2p} b = (u^p a^{2p})^{-1} = u^{-p} a^{-2p}.$$

On the other hand, $b^{-1}u^pa^{2p}b = u^pa^{-2p}$, because $[u,b]=1$. Therefore, $u^{-p}=u^p$ and $u^{2p}=1$, which is impossible. So, $T(C)=N_G^d=\langle c\rangle$, where $|c|=p, p\neq 2$.

Let $C_1 = C_C(N_G^d)$ be the centralizer of the subgroup $N_G^d=\langle c\rangle$ in C. Since $C'\subseteq\langle c\rangle$, the group C_1 is Abelian with the complementary subgroup $\langle c\rangle$, e.g., $C_1=\langle c\rangle\times Y$, where Y is an Abelian torsion-free group of rank 1. By the proved above C contains all elements of infinite order of a group. Therefore, $H\subseteq C$, where H is a non-cyclic Abelian torsion-free subgroup that is not N_G^d-admissible, and $C_1\neq C$.

By the cyclicity of the factor group C/C_1 and the previous considerations, we obtaine that the subgroup Y is non-cyclic. Suppose that it contains an infinite sequence of subgroups

$$\langle y_1\rangle\subset\langle y_2\rangle\subset ...\subset\langle y_n\rangle\subset ...,$$

where $y_n=y_{n+1}^{k_{n+1}}, (k_{n+1},p)=1$ for all $n\in\mathbb{N}$. Then the isolator I (see [14]) of the subgroup $\langle cy_1\rangle$ is non-cyclic and hence, I is a N_G^A-admissible subgroup. Therefore,

$$b^{-1}(cy_1)b\in(I\cap(\langle c\rangle\times\langle cy_1\rangle))=\langle cy_1\rangle.$$

Since $\langle y_1\rangle\cap\langle a\rangle\neq E$ and $b^{-1}y_1b=y_1^{-1}$, then $b^{-1}cy_1b=(cy_1)^{-1}=c^{-1}y_1^{-1}=cy_1^{-1}$ and $c^2=1$, which is impossible. So, Y does not contain such chains and hence is a group isomorphic to an additive group of p-adic numbers.

Let's prove that the subgroup $\langle c\rangle$ is complemented in C. By the proved above we have $1\neq[C:C_1]=k$, where $k|(p-1)$. Since we can uniquely find the root of k degree for each element of the subgroup $\langle c\rangle$ and $\langle c\rangle$ is complemented in its centralizer, it is also complemented in C (Theorem 1, [15]), e.g. $C=\langle c\rangle\leftthreetimes D$, where D is an incomplete Abelian group of rank 1.

It is obvious, that the group $G=(\langle c\rangle\leftthreetimes D)\leftthreetimes\langle b\rangle$ does not contain periodic Abelian non-cyclic subgroups, all mixed Abelian subgroups belong to the group $\langle c\rangle\leftthreetimes D$, contain $\langle c\rangle$ and are normal in G. Moreover, all tortion-free Abelian non-cyclic subgroups are contained either in the subgroup D or in subgroups $g^{-1}Dg, g\in G$, conjugated to this subgroup. Then the normalizer of each Abelian non-cyclic subgroup of rank 1 contains a subgroup $Y\lhd G$ and, as a consequence, the norm N_G^A contains this subgroup, which contradicts the assumption of its structure. Therefore, A cannot be an infinite cyclic group. So, $N_G^A=A\leftthreetimes\langle b\rangle$, where A is a group isomorphic to an additive group of p-adic numbers, $(p,2)=1, |b|=2$ and $b^{-1}ab=a^{-1}$ for any element $a\in A$.

By the proved above, every infinite cyclic subgroup has a nonidentity intersection with the norm N_G^A. On the other hand, the norm N_G^A does not contain free Abelian subgroups of rank 2. So, the group G also does not contain such subgroups. A similar statement holds for non-cyclic Abelian subgroups of finite order. Indeed, if a group G contains finite Abelian non-cyclic subgroups, then their intersection with N_G^A, is a finite subgroup normal in N_G^A, which is impossible, or is an identity subgroup, which contradicts Lemma 2.

Suppose that G contains a periodic non-cyclic locally cyclic subgroup P. If P contains an infinite subgroup which has the identity intersection with the norm N_G^A, then the norm N_G^A is Dedekind by Lemma 2, which is impossible. So, P is a quasicyclic subgroup and $P \cap N_G^A \neq E$. But in this case $(P \cap N_G^A) \lhd N_G^A$, which contradicts the structure of the norm N_G^A. Therefore, the group G does not contain periodic non-cyclic locally cyclic subgroups.

Finally, since the intersection $\langle x \rangle \cap N_G^A$ is nonidentity for an arbitrary element $x \in G, |x| = \infty$, the factor-group G/N_G^A is periodic. The Theorem is proved. □

Theorem 3. *If a non-periodic locally soluble group G has an Abelian non-cyclic subgroup, the norm N_G^A of Abelian non-cyclic subgroups is Dedekind, $N_G^A \neq E$, $N_G^d \neq E$ and $N_G^d \cap N_G^A = E$, then:*

1) $Z(G) = N(G) = E$;
2) *the norm N_G^A of Abelian non-cyclic subgroups is an Abelian torsion-free group of rank 1;*
3) *the norm N_G^d of decomposable subgroups is a cyclic group, $N_G^d = \langle c \rangle, (|c|, 2) = 1$.*

Proof. The first statement is proved in the same way as in Theorem 2. By the condition $N_G^d \cap N_G^A = E$ and Lemma 1 we have that the norm N_G^d is Dedekind. Moreover, the group G contains a non-primary, not N_G^A-admissible cyclic subgroup $\langle g \rangle$ and an indecomposable Abelian non-cyclic subgroup H, which is not N_G^d-admissible.

Suppose that the norm N_G^d is non-periodic and an element $c \in N_G^d$ such that $|c| = \infty$ exists. Since the subgroup $\langle g \rangle$ is N_G^d-admissible, the subgroup $\langle g, c^k \rangle = \langle g \rangle \times \langle c^k \rangle$ is Abelian non-cyclic for some positive integer k, and therefore N_G^A-admissible. So, the subgroup $\langle g \rangle$ is also N_G^A-admissible, which contradicts its choice. Therefore, the norm N_G^d is a periodic Dedekind group.

Assume that N_G^d does not satisfy the minimal condition for Abelian subgroups. Then the intersection $C_G(g) \cap N_G^d$ contains non-cyclic Abelian subgroups A_1 and A_2 such that $(A_1 \cup A_2) \cap \langle g \rangle = E$. Since the subgroups

$A_1 \times \langle g\rangle$ and $A_2 \times \langle g\rangle$ are non-cyclic Abelian, they are N_G^A-admissible. So, the group

$$\langle g\rangle = (A_1 \times \langle g\rangle) \cap (A_2 \times \langle g\rangle)$$

is also N_G^A-admissible, which is impossible. This contradiction shows that N_G^d is a group with the minimal condition for Abelian subgroups. Moreover, since the subgroup N_G^d is Dedekind it follows from Corollary 4.2 [16] that N_G^d is a finite extension of the direct product of a finite number of quasicyclic subgroups.

Let denote the subgroup generated by elements of the prime order of the norm N_G^d by $\omega(N_G^d)$. By the proved above $\left|\omega(N_G^d)\right| < \infty$, so $[G : C_G(\omega(N_G^d))] < \infty$. If an indecomposable non-cyclic Abelian subgroup H, which is not N_G^d-admissible, is complete, then $H \subseteq C_G(\omega(N_G^d))$ and the group $B = H \cdot \omega(N_G^d)$ is Abelian. If B is decomposable, then it is N_G^d-admissible. But in this case the subgroup

$$B^{\left|\omega(N_G^d)\right|} = H^{\left|\omega(N_G^d)\right|} = H$$

is also N_G^d-admissible, which contradicts its choice. Thus, B is a non-decomposable Abelian group and as a consequence, H is a quasicyclic p-group. So, $\omega(N_G^d) \subseteq H$ and $\left|\omega(N_G^d)\right| = p$. Since N_G^d is Dedekind and contains an only one subgroup of prime order by the proved above, it is either a cyclic or a quasicyclic p-group. In both cases we conclude that $H \subseteq C_G(N_G^d)$. Therefore, the subgroup H is N_G^d-admissible, which is impossible. Hence, H is an incomplete non-cyclic Abelian torsion-free group of rank 1.

Let's prove that the norm N_G^A of Abelian non-cyclic subgroups is a torsion-free Abelian group. Indeed, otherwise, there exists a nonidentity element $x \in N_G^A, |x| < \infty$. Then, taking into account that the norm N_G^A is Dedekind and the subgroup H is N_G^A-admissible, we have

$$[\langle x\rangle, H] \subseteq T(N_G^A) \cap H = E,$$

where $T(N_G^A)$ is the periodic part of the subgroup N_G^A. Therefore, the subgroup $\langle x\rangle \times H$ is decomposable Abelian and, as a consequence, N_G^d-admissible. But in this case

$$[N_G^d, H] \subseteq N_G^d \cap (\langle x\rangle \times H) = E.$$

Hence, H is N_G^d-admissible subgroup, which contradicts its choice. So, N_G^A is an Abelian torsion-free group.

If $N_G^A \cap H = E$, then $[N_G^A, H] = E$ and for any element $a \in N_G^A, |a| = \infty$ the subgroup $\langle a\rangle \times H$ is Abelian decomposable and, hence, N_G^d-admissible. But then $[N_G^d, H] \subseteq N_G^d \cap (\langle a\rangle \times H) = E$, which is impossible, because in this case H will be N_G^d-admissible subgroup. Therefore,

$N_G^A \cap H \neq E$, and for an arbitrary element $h \in H$ there exists a non-zero integer k such that $h^k \in N_G^A$.

Suppose that the norm N_G^A contains a free Abelian subgroup $\langle a_1\rangle \times \langle a_2\rangle$, where $|a_1| = |a_2| = \infty$. Then by the proved at least one of the subgroups $\langle a_1\rangle$ or $\langle a_2\rangle$ has the identity intersection with H. Let $\langle a_1\rangle \cap H = E$. Since H is a N_G^A-admissible subgroup, then $a_1^{-1}h_1a_1 = h_2$, where $h_1, h_2 \in H$. Moreover, by the condition $N_G^A \cap H \neq E$ we have $h_1^k \in N_G^A$ for some integer $k \neq 0$. Hence, $a_1^{-1}h_1^k a_1 = h_1^k = h_2^k$, and $h_1 = h_2$. Therefore, $[\langle a_1\rangle, H] = E$ and the subgroup $\langle a_1^m\rangle \times H$ is N_G^d-admissible for an arbitrary natural m. Thus, the subgroup $H = \bigcap_{m=1}^{\infty}(\langle a_1^m\rangle \times H)$ is also N_G^d-admissible, which contradicts its choice. So, the norm N_G^A does not contain free abelian subgroups of rank 2 and is an Abelian torsion-free group of rank 1.

Let $\langle g\rangle$ be a non-primary subgroup, which is not N_G^A-admissible. It is clear that at least one of its Sylow p-subgroups is also not N_G^A-admissible. Let it be a subgroup $\langle g\rangle_p$, where p is prime. Since the factor-group $G/C_G(N_G^A)$ is isomorphic to a subgroup of automorphisms of an Abelian torsion-free group of rank 1 with the periodic part of order 2 ([17], p. 294], we conclude that $\langle g\rangle_p = \langle g\rangle_2 = \langle \bar{g}\rangle$ is a 2-group.

Let's prove that all Sylow p-subgroups of N_G^d are cyclic. Suppose that N_G^d contains an elementary Abelian subgroup N of order $p^2, p \neq 2$. Since

$$[N, \langle \bar{g}\rangle] \subseteq (N_G^d)_p \cap \langle \bar{g}\rangle = E,$$

where $\left(N_G^d\right)_p$ is a Sylow p-subgroup of the norm N_G^d, the subgroup $N \times \langle \bar{g}\rangle$ is an Abelian non-cyclic and $\langle \bar{g}\rangle$ is N_G^A-admissible as its characteristic subgroup, which is impossible. Therefore, any Sylow p-subgroup of the norm N_G^d for $p \neq 2$ contains a unique subgroup of prime order, so, it is a cyclic or a quasicyclic p-group.

Suppose that the norm N_G^d contains quasicyclic p–subgroup P for some prime $p \neq 2$. Then $P \times \langle \bar{g}\rangle$ is Abelian non-cyclic, and hence, N_G^A-admissible group. Thus, the subgroup $\langle \bar{g}\rangle$ is N_G^A-admissible, which contradicts its choice. So, any Sylow p-subgroup of the norm N_G^d is cyclic for $p \neq 2$.

Let us consider the Sylow 2-subgroup $\left(N_G^d\right)_2$ of the norm N_G^d. If $\left(N_G^d\right)_2 \cap \langle \bar{g}\rangle = E$, then for an arbitrary element $c \in \left(N_G^d\right)_2$ the subgroup $\langle c, \bar{g}\rangle = \langle c\rangle \times \langle \bar{g}\rangle$ is Abelian non-cyclic, and therefore, is N_G^A-admissible. Then $[\langle \bar{g}\rangle, N_G^A] \subseteq (\langle c\rangle \times \langle \bar{g}\rangle) \cap N_G^A = E$, which is impossible. Thus, $\left(N_G^d\right)_2 \cap \langle \bar{g}\rangle \neq E$.

Let's denote the lower layer of the Sylow 2-subgroup $\left(N_G^d\right)_2$ by M and consider the group $G_2 = \langle \bar{g}\rangle M$. Let M be a non-cyclic group. Then by the condition $\langle \bar{g}\rangle \lhd G_2$ we have $[\langle \bar{g}\rangle, M] \subseteq M \cap \langle \bar{g}\rangle = \langle c_1\rangle$, where

$c_1 \in M, |c_1| = 2$. As $\langle \bar{g} \rangle = C_{G_2}(\langle \bar{g} \rangle)$, it follows $M = \langle c_1 \rangle \times \langle c_2 \rangle$ and $[\langle \bar{g} \rangle, \langle c_2 \rangle] = \langle c_1 \rangle$. Then by $M \lhd G$, $[G : C_G(M)] = 2$ and $\bar{g} \notin C_G(M)$, we conclude that $G = C_G(M) \langle \bar{g} \rangle$. However, in this case $c_1 \in Z(G)$, which is impossible. Therefore, the lower layer of $\left(N_G^d\right)_2$ contains one involution, which again contradicts the condition $Z(G) = E$. So, $2 \notin \pi(N_G^d)$ and $N_G^d = \langle c \rangle, (|c|, 2) = 1$. The Theorem is proved. □

The following example confirms the existence of groups satisfying the conditions of Theorem 3 and generalizes Example 3.4 of [12]. Let's note that the order of the norm of decomposable subgroups in this case can be a composite number (unlike the groups satisfying the conditions of Theorem 2).

Example 2. Let $G = (\langle a \rangle \leftthreetimes B) \leftthreetimes \langle c \rangle$, where $|a| = m > 1, (m, 2) = 1$, B is a group isomorphic to an additive group of q-adic fractions, q is prime, $(q, 2m) = 1$, $B = B_1\langle x \rangle$, $x^2 \in B_1, x^{-1}ax = a^{-1}$, $[B_1, \langle a \rangle] = E$, $|c| = 2, [c, a] = 1$ and $c^{-1}bc = b^{-1}$ for any element $b \in B$.

In this group, all periodic decomposable subgroups have the order $2d$, $d|m, d > 1$ and are groups of the form $\langle a^{\frac{m}{d}s} c b_1^k \rangle$, where $b_1 \in B_1, (s, d) = 1, k \in \{0, 1\}$. Thus, all nonperiodic decomposable subgroups are mixed and contained in the group $B_1 \times \langle a \rangle$ and, hence, they are normal in G. Since $N_G(\langle a^{\frac{m}{d}s} c b_1^k \rangle) = \langle a c b_1^k \rangle$, we conclude that $N_G^d = \langle a \rangle$.

Let's determine the norm N_G^A of non-cyclic Abelian subgroups of the group G. It is obvious that G does not contain periodic non-cyclic Abelian subgroups but all mixed Abelian subgroups contain $\langle a^{\frac{m}{d}s} \rangle$, and are subgroups of the group $B_1 \times \langle a \rangle$. It is easy to prove that all these subgroups are normal in G. Further, all non-cyclic Abelian subgroups of rank 1 are contained either in the subgroup B or in the subgroups $g^{-1}Bg, g \in G$, conjugate to this subgroup, or in the group $B_1 \times \langle a \rangle$. Let's consider an infinite sequence of subgroups in B_1:

$$\langle y_1 \rangle \subset \langle y_2 \rangle \subset ... \subset \langle y_n \rangle \subset ...,$$

where $y_n = y_{n+1}^{k_{n+1}}, (k_{n+1}, m) = 1$ for all $n \in \mathbb{N}$.

It is easy to prove that the isolator I of the subgroup $\langle a y_1 \rangle$ is non-cyclic because the root of the element a of any power co-prime for m exists. Moreover, $N_G(I) = B_1 \times \langle a \rangle$. Since $N_G(B) = B \leftthreetimes \langle c \rangle$, we conclude that $N_G^A = B_1$ is a torsion-free Abelian group of rank 1 and $N_G^d \cap N_G^A = E$.

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