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A note on Hall S-permutably embedded subgroups of finite groups

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Abstract. Let G be a finite group. Recall that a subgroup A of G is said to *permute* with a subgroup B if $AB = BA$. A subgroup A of G is said to be *S-quasinormal* or *S-permutable* in G if A permutes with all Sylow subgroups of G. Recall also that H^{sG} is the *S-permutable closure* of H in G, that is, the intersection of all such S-permutable subgroups of G which contains H. We say that H *is Hall S-permutably embedded in* G if H is a Hall subgroup of the S-permutable closure H^{sG} of H in G.

We proof that the following conditions are equivalent: (1) *Every subgroup of G is Hall S-permutably embedded in G;* (2) *The nilpotent residual $G^{\mathfrak{N}}$ of G is a Hall cyclic of square-free order subgroup of G;* (3) *$G = D \rtimes M$ is a split extension of a cyclic subgroup D of square-free order by a nilpotent group M, where M and D are both Hall subgroups of G.*

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. The symbol $G^{\mathfrak{N}}$ denotes the *nilpotent residual* of G, that is, the intersection of all normal subgroups N of G with nilpotent quotient G/N.

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Recall that a subgroup A of G is said to *permute* with a subgroup B if $AB = BA$. A subgroup A of G is said to be *S-quasinormal* or *S-permutable* in G if A permutes with all Sylow subgroups of G.

The S-permutable subgroups possess a series of interesting properties. For instance, the S-permutable subgroups of G form a sublattice of the lattice of all subnormal subgroups of G (Kegel [1]). This important property of S-permutable subgroups allows to introduce the concept of the S-permutable closure a subgroup. The intersection of all such S-permutable subgroups of G which contains a subgroup H of G is called the *S-permutable closure of H in G* and denoted by H^{sG} (see Guo and Skiba [2]).

Recall also that a subgroup H of G is said to be a *Hall normally embedded subgroup* of G [3] if H is a Hall subgroup of the normal closure H^G of H in G. By analogy with it, we say that a subgroup H of G is called a *Hall S-permutably embedded subgroup* of G if H is a Hall subgroup of the S-permutable closure H^{sG} of H in G.

In the paper [4], Shirong Li and Jianjun Liu described groups G such that very subgroup of G is Hall normally embedded in G. Our main goal here is to prove the following generalization of this result.

Theorem 1. *The following conditions are equivalent:*

(1) Every subgroup of G is Hall S-permutably embedded in G.

(2) The nilpotent residual $G^{\mathfrak{N}}$ of G is a Hall cyclic of square-free order subgroup of G.

(3) $G = D \rtimes M$ is a split extension of a cyclic subgroup D of square-free order by a nilpotent group M, where M and D are both Hall subgroups of G.

Corollary 1. *(Shirong Li and Jianjun Liu [4, Theorem 3.4]) Every subgroup of G is Hall normally embedded in G if and only if $G = D \rtimes M$ is a split extension of a cyclic subgroup D of square-free order by a Dedekind group M, where M and D are both Hall subgroups of G.*

Proofs of Theorem 1 and Corollary 1

We will need a few facts about S-permutable subgroups.

Lemma 1. *(See Kegel [1] or [5, Theorem 1.2.14]) Let $H \leq K \leq G$. Then*

(1) If H is S-permutable in G, then H is S-permutable in K.

(2) Suppose that H is normal in G. Then K/H is S-permutable in G/H if and only if K is S-permutable in G.

(3) If H is S-permutable in G, then H is subnormal in G.

Lemma 2. *(See Kegel [1] or [5, Theorem 1.2.19]) The set of all S-permutable subgroups is a sublattice of the subnormal subgroup lattice.*

We write $H^{\cdot\cdot G}$ to denote the *subnormal closure* of H in G, that is, the intersection of all the subnormal subgroups of G which contain H (sf. [6, A, 14.13]).

A subgroup H of G is called a *Hall subnormally embedded subgroup* of G [4, Definition 1.4] if H is a Hall subgroup of the subnormal closure $H^{\cdot\cdot G}$ of H in G. We need also some properties of Hall subnormally embedded subgroups (see in [4, Theorem 3.3]).

Lemma 3. *If every subgroup of G is Hall subnormally embedded in G, then the following statements hold:*

(1) $G = D \rtimes M$, where $D = G^{\mathfrak{N}}$ is the nilpotent residual of G.

(2) D and M are Hall subgroups of G.

(3) M acts irreducibly on each Sylow subgroup of D.

Lemma 4. *(1) If H is a Hall S-permutably embedded subgroup of G, then H is a Hall subnormally embedded subgroup of G.*

(2) If H is a Hall S-permutably embedded subgroup of G, then H is a Hall normally embedded subgroup of G.

Proof. (1) Since every S-permutable subgroup of G is a subnormal subgroup of G by Lemma 1(3), $H^{\cdot\cdot G} \leq H^{sG}$. Moreover, H is a Hall subgroup of H^{sG} by hypothesis, so H is a Hall subgroup of $H^{\cdot\cdot G}$.

(2) See the proof of (1). □

Lemma 5. *(See Deskins [7] or [5, Theorem 1.2.14]) If the subgroup H of G is S-permutable in G, then H/H_G is nilpotent.*

Lemma 6. *(See [8, Lemma 2.4]) Let H be a Hall S-permutably embedded subgroup of G. Then the following statements hold:*

(1) If $H \leq K \leq G$, then H is Hall S-permutably embedded in K.

(2) If $N \lhd G$, then HN/N is Hall S-permutably embedded in G/N.

Lemma 7. *Let $G = D \rtimes M$, where D is a Hall cyclic of square-free order subgroup of G and M is a nilpotent (respectively Dedekind) subgroup of G. Then every subgroup of G is Hall S-permutably embedded (respectively Hall normally embedded) in G.*

Proof. Since D is of square-free order, D is supersoluble by [9, IV, Theorem 2.8]. Moreover, since M is nilpotent, G is a soluble group. Let H be a subgroup of G. Then H is soluble. Since D is a Hall π-subgroup of G, for some π, $|H \cap D|$ is a π-number. Also,

$$|H : H \cap D| = |DH/D|$$

is a π'-number, so $D_1 = H \cap D$ is a normal Hall π-subgroup of H. Then, by the Hall theorem, D_1 has a complement M_1 in H. On the other hand, since D is soluble and D_1 is a Hall subgroup of D, D_1 has a complement D_2 in D.

Since $M \simeq G/D$ is nilpotent (respectively Dedekind), all subgroups of G/D is S-permutable (respectively normal) in G/D. Then DH/D is S-permutable (respectively normal) in G/D. Hence by Lemma 1(2), DH is S-permutable (respectively normal) in G. Therefore $H \leq H^{sG} \leq DH$ (respectively $H \leq H^G \leq DH$) and by Lemma 2 and the definition H^{sG}, H^{sG} is S-permutable (respectively normal) in G.

Now we will show that H is a Hall subgroup of H^{sG} (respectively of H^G). Since

$$|DH : H| = \frac{|D_1 D_2 H|}{|H|} = \frac{|D_2 H|}{|H|} = \frac{|D_2||H|}{|D_2 \cap H||H|} = |D_2|,$$

$(|H|, |DH : H|) = 1$. Thus H is a Hall subgroup of DH, therefore H is a Hall subgroup of H^{sG} (respectively of H^G). Hence H is Hall S-permutable embedded (respectively Hall normally embedded) in G. □

Lemma 8. *(See [5, Theorem 1.2.16]) Let H be a p-subgroup of G, where p is a prime. Then H is S-permutable in G if and only if*

$$O^p(G) \leq N_G(H).$$

Now we are in position to proof the main result.

Proof. Let $D = G^{\mathfrak{N}}$. (1)$\Rightarrow$ (2)

Assume that this is false and let G be a counterexample of minimal order.

(a) *If N is a minimal normal subgroup of G, then the hypothesis holds for every quotient G/N and so Condition (2) is true for G/N.*

Let H/N be any subgroup of G/N. Then H is Hall S-permutably embedded in G by hypothesis. Hence H/N is Hall S-permutably embedded in G/N by Lemma 6(2). Therefore the hypothesis holds for G/N. Hence, since

$$|G/N| < |G|,$$

the choice of G implies that Condition (2) is true for G/N.

(b) *G is is soluble.*

Assume that this is false. Claim (a) implies that G/N is soluble for every minimal normal subgroup N of G, so N is the unique minimal normal subgroup of G and N is non-abelian. Hence $N \nleq \Phi(G)$. Let X be a maximal subgroup of G such that $N \nleq X$. Then $G = NX$.

Let p be a prime dividing the order of G. Then there exist a Sylow p-subgroup N_p of N and a Sylow p-subgroup X_p of X such that $P = N_pX_p$ is a Sylow p-subgroup of G. We have that either $P = X_p$ and then X contains a Sylow p-subgroup of G or there exists a maximal subgroup K of P such that X_p is contained in K. Suppose the second possibility is true. By hypothesis, K is Hall S-permutably embedded in G. Hence we can find a subgroup B of G such that B is S-permutable in G and K is a Sylow p-subgroup of B. Suppose that $B_G = 1$. Then B is a nilpotent group by Lemma 5. Moreover, B is subnormal in G by Lemma 1(3). This implies that B is contained in $F(G)$. But $F(G) = 1$ because N is non-abelian. Therefore $X_p = 1$ and $P = N_p$ is a Sylow p-subgroup of G. We see that in this case P is cyclic of order p. In order to prove it, let A be a maximal subgroup of $P = N_p$. Then A is Hall S-permutably embedded in G. Hence A is a Sylow p-subgroup of some S-permutable subgroup W of G. Since $F(G) = 1$ and W is subnormal in G, in the case $A \neq 1$ we have $W_G \neq 1$. Then $N \leq W_G$ and so $A \cap N = A$ is a Sylow p-subgroup of N. In particular $|A| = |P|$, a contradiction. Consequently, $W_G = 1$ and so $A = 1$. This is to say that P is a cyclic group of order p.

Assume that $B_G \neq 1$. Then N is contained in B. Now $K \cap N$ is a Sylow p-subgroup of N and so

$$|K \cap N| = |N_p|$$

and

$$K = X_p(K \cap N_p).$$

On the other hand, $K \cap N$ is a normal subgroup of K and then $X_p(K \cap N)$ is a subgroup of K containing $X_p(K \cap N_p)$. This implies that

$$K = X_p(K \cap N) = X_p(K \cap N_p).$$

Moreover, $P \cap N = N_p$ and so

$$X_p \cap N = X_p \cap N_p.$$

Then

$$|K| = (|X_p||K \cap N|)/|X_p \cap N| = |X_p||N_p|/(|X_p \cap N_p|) = |P|,$$

a contradiction.

Therefore we have proved that if p divides the order of G, then it follows that either X contains a Sylow p-subgroup of G or N contains a Sylow p-subgroup of G. In the second case, this Sylow p-subgroup should be cyclic of order p.

Denote $\pi = \pi(N)$. Since G/N is supersoluble by Claim (a), it follows that

$$G/N = XN/N \cong X/(N \cap X)$$

is supersoluble. In particular, $X/(N\cap X)$ is soluble. Let H be a subgroup of G such that $H/(N\cap X)$ is a Hall π-subgroup of $X/N\cap X$. Suppose that NH is a proper subgroup of G. Then by Lemma 6(1) that the hypothesis of the theorem holds in NH. By the minimal choice of G, it follows that NH is supersoluble, a contradiction. Hence we have $G = NH$ and G is a π-group. Suppose that for each prime $p \in \pi$, the Sylow p-subgroups of N are Sylow p-subgroups of G. Then $G = N$ and, by the above argument, every Sylow subgroup of G is cyclic. By [9, IV, 2.9], G is soluble, a contradiction.

(c) *G is supersoluble.*

Assume that this is false. Then, since the class of all supersoluble groups is a saturated formation, Claim (a) implies that G has a unique minimal normal subgroup, say N, and $N \not\leq \Phi(G)$. Moreover, since by Claim (b) G is soluble,

$$N = O_p(G) = C_G(N)$$

is a non-cyclic abelian p-group for some prime p. Let P be a Sylow subgroup of G containing N. Let A be a maximal subgroup of P not containing N. Since A is Hall S-permutably embedded in G, we can find an S-permutable subgroup of G, B say, such that A is a Sylow p-subgroup of B. From the fact that N is not contained in A, we have $B_G = 1$. By Lemma 5, we know that B is nilpotent. So B is contained in $F(G) = N$ because B is subnormal in G. Therefore if A is a maximal subgroup of P, we have either A is contained in N or N is contained in A. Since N is not contained in $\Phi(P)$, it follows that there exists a maximal subgroup A of P such that $A \leq N$. Moreover A is S-permutable in G because $A = B$. Since A is normal in P and normalized by $O^p(G)$ by Lemma 8, we have that A is a normal subgroup of G and so either $A = 1$ or $A = N$ because N is a minimal normal subgroup of G. In the first case P is cyclic and in the second one N is the unique maximal subgroup of P. In both cases P is cyclic. So N is cyclic and G is supersoluble, contradiction.

Since G is supersoluble, D is nilpotent. Moreover, by Lemmas 3(1),(2) and 4(1), D is a Hall subgroup of G.

Since D is nilpotent, each Sylow subgroup of D is normal in D and therefore each Sylow subgroup of D is characteristic in D. Hence each Sylow subgroup of D is normal in G. Let $V \neq 1$ be a Sylow subgroup of D and let R be a minimal normal subgroup of G contained in V. Then

$1 < R \leq V$. Since M acts irreducible on each Sylow subgroup of D by Lemmas 3(3) and 4(1), $R = V$. Therefore, since G is supersoluble, $|R| = |V|$ is a prime. Hence D is a cyclic group of square-free order.

(2)⇒ (3) Since D is a Hall subgroup of G, D has a complement M in G by the Schur-Zassenhuas theorem. Finally, since

$$M \simeq G/D = G/G^{\mathfrak{N}},$$

M is a Hall nilpotent subgroup of G.

(3)⇒ (1) This directly follows from Lemma 7.

The theorem is proved. □

Finally, we proof Corollary 1.

Proof. Necessity. In view of Lemma 4(2), Theorem 1 and [5, 1.4], it is enough to show that G is a T-group. Let H be a subnormal subgroup of G. Then H is subnormal in H^G by [6, 14.8]. Then, since H is a Hall subgroup of H^G by hypothesis, H is characteristic in H^G. Hence H is a normal subgroup of G, so G is a T-group.

Sufficiency. This directly follows from Lemma 7.

The corollary is proved. □

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