

## On hereditary reducibility of 2-monomial matrices over commutative rings

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**ABSTRACT.** A 2-monomial matrix over a commutative ring  $R$  is by definition any matrix of the form  $M(t, k, n) = \Phi \begin{pmatrix} I_k & 0 \\ 0 & tI_{n-k} \end{pmatrix}$ ,  $0 < k < n$ , where  $t$  is a non-invertible element of  $R$ ,  $\Phi$  the companion matrix to  $\lambda^n - 1$  and  $I_k$  the identity  $k \times k$ -matrix. In this paper we introduce the notion of hereditary reducibility (for these matrices) and indicate one general condition of the introduced reducibility.

### Introduction

This paper is devoted to one class of monomial matrices over commutative rings which first arose in studying indecomposable representations of finite  $p$ -groups over local rings ([1]). They were studied more extensively (in a more generally) in [2]–[6].

Let  $R$  be a commutative ring with Jacobson radical  $J(R) \neq 0$  and  $t$  a non-zero element from  $J(R)$ . An  $n \times n$  matrix over  $R$  is called *2-monomial concerning  $t$* , if it is permutation similar to a matrix of the

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following form:

$$M(t, k, n) := \Phi_n \begin{pmatrix} I_k & 0 \\ 0 & tI_{n-k} \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & t \\ 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & t & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & t & 0 \end{pmatrix},$$

where  $0 < k < n$ ,  $\Phi_n$  is the companion matrix to the polynomial  $x^n - 1$  and  $I_s$  is the identity  $s \times s$  matrix. Such a matrix  $M = M(t, k, n)$  is said to be *hereditary reducible* if it similar to a matrix

$$M' = \begin{pmatrix} M(t, k', n') & * \\ 0 & N \end{pmatrix}, \quad n' \neq n,$$

and *hereditary irreducible* if otherwise.

The aim of this paper is to prove the following result.

**Theorem 1.** *A 2-monomial matrix  $M(t, k, n)$  is hereditary reducible if  $k$  and  $n$  are not coprime.*

In the next section, we indicate a more detailed interpretation of the idea of this statement.

## 1. Generalization of Theorem 1: formulation and proof

In this section we prove a more general theorem (from which Theorem 1 follows). Instead of  $R$  we consider the ring  $\mathbb{Z}[\lambda]$  ((of integer polynomials)). Let  $(n, k)$  denote the greatest common divisor of the natural numbers  $n$  and  $k$ .

**Theorem 2.** *Let  $n > k$  be positive integers, such that  $(n, k) > 1$ . Then for any positive divisors  $d > 1$  of the number  $(n, k)$ , the matrix  $M(\lambda, k, n) \in M(n, \mathbb{Z}[\lambda])$  similar to a matrix of the following form*

$$\begin{pmatrix} M(\lambda, k', n') & B \\ 0 & A \end{pmatrix} \in M(n, \mathbb{Z}[\lambda]),$$

where  $k' = \frac{k}{d}$  and  $n' = \frac{n}{d}$ .

Through this section  $0 < k < n$ ,  $0 < k' < n'$ ,  $0 < n' < n$ . Before we prove this result, we need to provide four other important results needed for the proof.

**Proposition 1.** *Let  $n'|n$ . Then there exists an  $n \times n'$ -matrix*

$$S = \begin{pmatrix} \lambda^{s_1} & 0 & \dots & 0 \\ 0 & \lambda^{s_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_{n'}} \\ \lambda^{s_{n'+1}} & 0 & \dots & 0 \\ 0 & \lambda^{s_{n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_{2n'}} \\ \dots & \dots & \dots & \dots \\ \lambda^{s_{n-n'+1}} & 0 & \dots & 0 \\ 0 & \lambda^{s_{n-n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_n} \end{pmatrix},$$

where  $s_i \geq 0$ ,  $i = 1, \dots, n$ , such that  $M(\lambda, k, n)S = SM(\lambda, k', n')$  if and only if  $\frac{n}{n'} = \frac{k}{k'}$ .

*Proof.* Let  $l_1 = \dots = l_k = 0$ ,  $l_{k+1} = \dots = l_n = 1$ ,  $r_1 = \dots = r_{k'} = 0$  and  $r_{k'+1} = \dots = r_{n'} = 1$  such that

$$M(\lambda, k, n) = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{l_n} \\ \lambda^{l_1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda^{l_k} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \lambda^{l_{k+1}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda^{l_{n-1}} & 0 \end{pmatrix}$$

and

$$M(\lambda, k', n') = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{r_{n'}} \\ \lambda^{r_1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda^{r_{k'}} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \lambda^{r_{k'+1}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda^{r_{n'-1}} & 0 \end{pmatrix}.$$

We denote by  $(i, j)$  the scalar equality

$$(M(\lambda, k, n)S)_{ij} = (SM(\lambda, k', n'))_{ij}.$$

Obviously, in each of matrices  $M(\lambda, k, n)S$  and  $SM(\lambda, k', n')$  there are exactly  $n$  non-zero element, which are in the  $i, j$  positions ( $i$ -th row,  $j$ -th column), where  $i \equiv j + 1 \pmod{n'}$ . Let  $\delta_{n'}(i) = (i - 1) \bmod n' + 1$  or, equivalently,  $\delta_{n'}(i) \equiv i \pmod{n'}$ ,  $1 \leq \delta_{n'}(i) \leq n'$ . Thus  $M(\lambda, k, n)S = SM(\lambda, k', n')$  if and only if scalar equalities

$$\begin{cases} (i + 1, \delta_{n'}(i)) : & \lambda^i \lambda^{s_i} = \lambda^{s_{i+1}} \lambda^{r_{\delta_{n'}(i)}} \quad (i = 1, \dots, n - 1), \\ (1, n') : & \lambda^{l_n} \lambda^{s_n} = \lambda^{s_1} \lambda^{r_{n'}}; \end{cases}$$

hold. Obviously, these equalities are equivalent to the equalities

$$\begin{cases} (i + 1, \delta_{n'}(i)) : & l_i + s_i = s_{i+1} + r_{\delta_{n'}(i)} \quad (i = 1, \dots, n - 1), \\ (1, n') : & l_n + s_n = s_1 + r_{n'}. \end{cases} \quad (1)$$

Assume that for some  $s_i \geq 0$ ,  $i = 1, \dots, n$   $M(\lambda, k, n)S = SM(\lambda, k', n')$ . Then (1) holds. Summing the equations (1), we obtain

$$\sum_{i=1}^{n-1} l_i + \sum_{i=1}^{n-1} s_i + l_n + s_n = \sum_{i=1}^{n-1} s_{i+1} + \sum_{i=1}^{n-1} r_{\delta_{n'}(i)} + s_1 + r_{n'}.$$

But since  $\delta_{n'}(n) = n'$  we have that

$$\sum_{i=1}^n l_i + \sum_{i=1}^n s_i = \sum_{i=1}^n s_i + \sum_{i=1}^n r_{\delta_{n'}(i)},$$

or  $\sum_{i=1}^n l_i = \sum_{i=1}^n r_{\delta_{n'}(i)}$ . This is equivalent to  $\sum_{i=1}^n l_i = \frac{n}{n'} \sum_{i=1}^{n'} r_i$  or  $k = \frac{n}{n'} k'$  and  $\frac{n}{n'} = \frac{k}{k'}$ .

Now, assume that  $\frac{n}{n'} = \frac{k}{k'}$  and we want to prove that for some  $s_i \geq 0$ ,  $i = 1, \dots, n$   $M(\lambda, k, n)S = SM(\lambda, k', n')$ . It remains to prove that the equations in (1) hold for non negative integers  $s_i$ . We will prove it for arbitrary integers  $s_i$  since that addition of any number to  $s_i$  will also be a solution. Let  $s_1 = 0$ ,  $s_{i+1} = l_i + s_i - r_{\delta_{n'}(i)}$  ( $i = 1, \dots, n - 1$ ). It follows immediately that all, except last equation in (1) holds and  $s_n = \sum_{i=1}^{n-1} l_i - \sum_{i=1}^{n-1} r_{\delta_{n'}(i)}$ . If we replace  $s_n$  by  $\sum_{i=1}^{n-1} l_i - \sum_{i=1}^{n-1} r_{\delta_{n'}(i)}$  and  $s_1$  by 0 in the last equation in (1), we obtain the following equation:

$$l_n + \sum_{i=1}^{n-1} l_i - \sum_{i=1}^{n-1} r_{\delta_{n'}(i)} = r_{n'} \text{ or } l_n + \sum_{i=1}^{n-1} l_i = \sum_{i=1}^{n-1} r_{\delta_{n'}(i)} + r_{n'}.$$

This equation is equivalent to  $\sum_{i=1}^n l_i = \sum_{i=1}^n r_{\delta_{n'}(i)}$  and  $\sum_{i=1}^n l_i = \frac{n}{n'} \sum_{i=1}^{n'} r_i$  which in turn is equivalent to  $k = \frac{n}{n'} k'$  or  $\frac{n}{n'} = \frac{k}{k'}$ . The proof is complete.  $\square$

Using a similar argument applied in the previous proof, we can state the following result:

**Proposition 2.** *Let  $n'|n$ ,  $l_i \geq 0$  ( $i = 1, \dots, n$ )  $\sum_{i=1}^n l_i = k$ ,*

$$M = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{l_n} \\ \lambda^{l_1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda^{l_k} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \lambda^{l_{k+1}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda^{l_{n-1}} & 0 \end{pmatrix}.$$

*Then there exists an  $n \times n'$ -matrix*

$$S = \begin{pmatrix} \lambda^{s_1} & 0 & \dots & 0 \\ 0 & \lambda^{s_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_{n'}} \\ \lambda^{s_{n'+1}} & 0 & \dots & 0 \\ 0 & \lambda^{s_{n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_{2n'}} \\ \dots & \dots & \dots & \dots \\ \lambda^{s_{n-n'+1}} & 0 & \dots & 0 \\ 0 & \lambda^{s_{n-n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_n} \end{pmatrix},$$

*where  $s_i \geq 0$ ,  $i = 1, \dots, n$ , such that  $MS = SM(\lambda, k', n')$  if and only if  $\frac{n}{n'} = \frac{k}{k'}$ .*

Next, we provide a result regarding the the similarity of  $M(\lambda, k, n)$  and a certain matrix.

**Proposition 3.** *Let  $n'|n$ ,  $\frac{n}{n'} = \frac{k}{k'}$ . Then  $k' < k$  and  $M(\lambda, k, n)$  is similar*

(over  $\mathbb{Z}[\lambda]$ ) to a matrix of the form

$$M = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{l_n} \\ \lambda^{l_1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda^{l_{k'}} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \lambda^{l_{k'+1}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda^{l_{n-1}} & 0 \end{pmatrix},$$

where  $l_1 = \dots = l_{k'} = 0$ ,  $l_{k'+1} = \dots = l_{k'+n-k} = 1$ ,  $l_{k'+n-k+1} = \dots = l_n = 0$ .

*Proof.* Clearly  $k' < k$  as  $\frac{k}{k'} = \frac{n}{n'} > 1$ . Now, rearrange the rows and columns of the matrix  $M(\lambda, k, n)$  in the order  $k-k'+1, k-k'+2, \dots, n, 1, 2, \dots, k-k'$  and denote the new matrix by  $M$ :

$$M = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{l_n} \\ \lambda^{l_1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda^{l_{k'}} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \lambda^{l_{k'+1}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda^{l_{n-1}} & 0 \end{pmatrix}$$

where  $l_1 = \dots = l_{k'} = 0$ ,  $l_{k'+1} = \dots = l_{k'+n-k} = 1$ ,  $l_{k'+n-k+1} = \dots = l_n = 0$ .  $\square$

The next result connects the previous two results.

**Proposition 4.** *Let  $n'|n$ ,  $k' < k$ .*

$$M = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{l_n} \\ \lambda^{l_1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda^{l_{k'}} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \lambda^{l_{k'+1}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda^{l_{n-1}} & 0 \end{pmatrix},$$

where  $l_1 = \dots = l_{k'} = 0$ ,  $l_{k'+1} = \dots = l_{k'+n-k} = 1$ ,  $l_{k'+n-k+1} = \dots = l_n = 0$ . Then there exists an  $n \times n'$ -matrix

$$S = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & \lambda^{s_{n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_{2n'}} \\ \dots & \dots & \dots & \dots \\ \lambda^{s_{n-n'+1}} & 0 & \dots & 0 \\ 0 & \lambda^{s_{n-n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_n} \end{pmatrix},$$

where  $s_i \geq 0$ , ( $i = n' + 1, \dots, n$ ), such that  $MS = SM(\lambda, k', n')$  if and only if  $\frac{n}{n'} = \frac{k}{k'}$ .

*Proof.* Clearly if  $MS = SM(\lambda, k', n')$ , then  $\frac{n}{n'} = \frac{k}{k'}$  by Proposition 2. Assume that  $\frac{n}{n'} = \frac{k}{k'}$ . Let  $r_1 = \dots = r_{k'} = 0$ ,  $r_{k'+1} = \dots = r_{n'} = 1$  such that

$$M(\lambda, k', n') = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \lambda^{r_{n'}} \\ \lambda^{r_1} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \lambda^{r_{k'}} & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \lambda^{r_{k'+1}} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \lambda^{r_{n'-1}} & 0 \end{pmatrix}.$$

We will prove that  $M(\lambda, k, n)S = SM(\lambda, k', n')$  for some  $s_i \geq 0$  ( $i = n' + 2, \dots, n$ ).

It remains to prove that (1) holds, where  $s_1 = \dots = s_{k'} = s_{k'+1} = 0$ . Let

$$s_1 = 0, \quad s_{i+1} = l_i + s_i - r_{\delta_{n'}(i)} \quad (i = 1, \dots, n-1). \quad (2)$$

Using a similar argument to the one used in (1) all equations hold except the last one. Furthermore, if  $\frac{n-k}{n'-k'} = \frac{n}{n'} = \frac{k}{k'} > 1$  it follows  $n' - k' < n - k$ ,

$$l_1 = \dots = l_{k'} = 0 = r_1 = \dots = r_{k'}$$

and

$$l_{k'+1} = \cdots = l_{n'} = 1 = r_{k'+1} = \cdots = r_{n'}.$$

Therefore  $l_i = r_i$  ( $i = 1, \dots, n'$ ) and  $s_{i+1} = s_i$  ( $i = 1, \dots, n'$ ) by (2). We can also see that  $s_1 = \cdots = s_{n'} = s_{n'+1} = 0$ .

It remains to prove that  $s_i \geq 0$  ( $i = n' + 2, \dots, n$ ). Let  $s_{n+1} = s_1$ . It follows from (2) and last equation from (1), that  $s_{n+1} = l_n + s_n - r_{\delta_{n'}(n)}$ , which is equivalent to  $s_1 = s_{n+1} = 0$  and

$$s_{i+1} = \sum_{j=1}^i l_j - \sum_{j=1}^i r_{\delta_{n'}(j)} \quad (i = 1, \dots, n). \quad (3)$$

Let us consider  $s(i) = s_i$ , as function of an integer  $i$  ( $1 \leq i \leq n+1$ ). Then  $s(i) = 0$  if  $1 \leq i \leq n' + 1$ . Thus,  $s(i)$  is a constant for  $1 \leq i \leq n' + 1$ . If  $n' + 1 \leq i < i + 1 \leq k' + n - k + 1$ , then it follows from (3) that  $s(i + 1) - s(i) = l_i - r_{\delta_{n'}(i)} = 1 - r_{\delta_{n'}(i)} \geq 0$ . Therefore  $s(i)$  either increases or remains constant for each step and  $s(i) \geq s(n' + 1) = 0$ . If  $k' + n - k + 1 \leq i < i + 1 \leq n + 1$ , then it follows from (3) that  $s(i + 1) - s(i) = l_i - r_{\delta_{n'}(i)} = 0 - r_{\delta_{n'}(i)} \leq 0$ . Consequently  $s(i)$  either decreases or remains constant for each step and  $s(i) \geq s(n + 1) = 0$ . Therefore,  $s_i = s(i) \geq 0$  ( $i = n' + 2, \dots, n$ ).  $\square$

Finally, we are in a position to prove our main result.

Proof of Theorem 2. Recall that  $0 < k < n$ ,  $\frac{k}{k'} = \frac{n}{n'} = d > 1$  and  $0 < n' < n$ ,  $0 < k' < k$ . By Proposition 3,  $M(\lambda, k, n)$  is similar (over  $\mathbb{Z}[\lambda]$ ) to a matrix of the form

$$M = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda^{l_n} \\ \lambda^{l_1} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda^{l_{k'}} & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \lambda^{l_{k'+1}} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \lambda^{l_{n-1}} & 0 \end{pmatrix}$$

where  $l_1 = \cdots = l_{k'} = 0$ ,  $l_{k'+1} = \cdots = l_{k'+n-k} = 1$ ,  $l_{k'+n-k+1} = \cdots = l_n = 0$ .



By Proposition 4 there exists an  $n \times n'$ -matrix

$$S = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & \lambda^{s_{n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_{2n'}} \\ \dots & \dots & \dots & \dots \\ \lambda^{s_{n-n'+1}} & 0 & \dots & 0 \\ 0 & \lambda^{s_{n-n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_n} \end{pmatrix} = \begin{pmatrix} I_{n'} \\ S' \end{pmatrix},$$

where

$$S' = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \lambda^{s_{n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_{2n'}} \\ \dots & \dots & \dots & \dots \\ \lambda^{s_{n-n'+1}} & 0 & \dots & 0 \\ 0 & \lambda^{s_{n-n'+2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^{s_n} \end{pmatrix},$$

$I_{n'}$  is the identity  $n' \times n'$  matrix,  $s_i \geq 0$  ( $(i = n' + 1, \dots, n)$ ) such that  $MS = SM(\lambda, k', n')$ . Now,

$$\begin{aligned} C^{-1}MC &= \begin{pmatrix} I_{n'} & 0 \\ S' & I_{n-n'} \end{pmatrix}^{-1} M \begin{pmatrix} I_{n'} & 0 \\ S' & I_{n-n'} \end{pmatrix} = \\ &= \begin{pmatrix} I_{n'} & 0 \\ -S' & I_{n-n'} \end{pmatrix} M \begin{pmatrix} I_{n'} & 0 \\ S' & I_{n-n'} \end{pmatrix}. \end{aligned}$$

If we omit the last  $n - n'$  columns of the last matrix, we obtain

$$\begin{aligned} \begin{pmatrix} I_{n'} & 0 \\ -S' & I_{n-n'} \end{pmatrix} M \begin{pmatrix} I_{n'} \\ S' \end{pmatrix} &= \begin{pmatrix} I_{n'} & 0 \\ -S' & I_{n-n'} \end{pmatrix} \begin{pmatrix} I_{n'} \\ S' \end{pmatrix} M(\lambda, k', n') = \\ &= \begin{pmatrix} I_{n'} \\ 0 \end{pmatrix} M(\lambda, k', n') = \begin{pmatrix} M(\lambda, k', n') \\ 0 \end{pmatrix}. \end{aligned}$$

In conclusion, we note that matrix  $M$  and the matrix  $M(\lambda, k, n)$  are similar (over  $\mathbb{Z}[\lambda]$ ) to a matrix of the form

$$\begin{pmatrix} M(\lambda, k', n') & B \\ 0 & A \end{pmatrix} \in M(n, \mathbb{Z}[\lambda]),$$

as claimed.

Note that Theorem 1 follows from the last theorem and the existence of the homomorphism of rings  $f : \mathbb{Z}[\lambda] \rightarrow R$  where  $f(1) = 1$  and  $f(\lambda) = t$ .

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