Cyclic left and torsion-theoretic spectra of modules and their relations

Marta Maloid-Glebova

Communicated by V. V. Kirichenko

Abstract. In this paper, strongly-prime submodules of a cyclic module are considered and their main properties are given. On this basis, a concept of a cyclic spectrum of a module is introduced. This spectrum is a generalization of the Rosenberg spectrum of a noncommutative ring. In addition, some natural properties of this spectrum are investigated, in particular, its functoriality is proved.

Introduction

In this paper, we consider strongly-prime ideals and modules. The concept of strongly-prime ideal was introduced by Beachy in [1]. Also in that paper the author introduced and investigated the concept of a strongly-prime module. Independently, the concept of strongly-prime module and submodule were introduced and investigated by Dauns in his paper [3]. Also, the strongly-prime modules were investigated by Algirdas Kaučikas in [2], where the author studied strongly-prime submodules of cyclic modules, but he did not study the concept of the Rosenberg spectrum for modules. The concept of pre-order on ideals was introduced by Rosenberg, and this concept is a basic one in the definition of cyclic spectrum, whose functoriality is investigated in this paper. Also we consider the notion and some properties of torsion-theoretic spectra of rings and modules. The notion and main properties of torsion-theoretic spectra were introduced by Golan in [5]. The main result of this paper is the

Key words and phrases: strongly-prime ideal, strongly-prime module, cyclic spectrum, torsion-theoretic spectrum, localizations.
proof of the fact that there exists mapping from the cyclic spectrum to
the torsion-theoretic spectrum of module is continuous and surjective.

1. Strongly-prime ideals and modules

Let $R$ be an associative ring with $1 \neq 0$. To have a reference, recall
some necessary concepts from the ring theory that are related to the
concept of spectrum of a noncommutative ring.

A left ideal $p$ of a ring $R$ is called prime, if for every $x, y \in R$, $xRy \subseteq p$
implies $x \in p$ or $y \in p$. Clearly, any left prime ideal is two-sided if and
only if it is prime in the classical way. Set of all two-sided prime ideals is
denoted by Spec($R$) and is called a (prime) spectrum of a ring $R$.

Recall the definition of a pre-order $\leq$ on the set of left ideals of ring
$R$ in the following way: $a \leq b$ for left $R$-ideals $a$ and $b$ if and only if there
exists a finite subset $V$ of ring $R$ such that $(a : V) \subseteq b$. A left prime
ideal $p$ of a ring $R$ is called a left Rosenberg point if $(p : x) \leq p$ for any
$x \in R \setminus p$, [8]. The set of all left Rosenberg points of a ring $R$ is called a
left Rosenberg spectrum of $R$ and is denoted by spec($R$).

The space spec($R$) may be defined in another way: this is the set of
all strongly prime left ideals. Recall that left ideal $p$ of the ring $R$ is called
strongly-prime, if for every $x \in R \setminus p$ there exists a finite set $V$ of ring $R$
such that $(p : V) = \{r \in R: rVx \subseteq p\} \subseteq p$. Clearly, every strongly-
prime left ideal of a ring $R$ is a prime left ideal and every maximal
left ideal is strongly-prime. It is known that if $R$ is noetherian, then
Spec($R$) $\subseteq$ spec($R$).

Now let us recover the information about corresponding analogues of
the above concepts for left modules over a ring $R$.

The concept of strongly-prime module can be given in two ways.

A nonzero left module $M$ over a ring $R$ is called strongly-prime, if for any nonzero $x, y \in M$ there exists a finite subset \{${a_1, a_2, \ldots, a_n}$\} $\subseteq R$
such that Ann$_R$\{${a_1x, a_2x, \ldots, a_nx}$\} $\subseteq$ Ann$_R$\{${y}$\}, \(ra_1x = ra_2x = \cdots = ra_nx = 0\), $r \in R$ implies $ry = 0$.

In [1], the authors introduced such a concept of strongly-prime submodule. A nonzero left module $M$ over a ring $R$ is called strongly-prime, if for any nonzero $x \in M$ there exists a finite subset \{${a_1, a_2, \ldots, a_n}$\} $\subseteq R$
such that Ann$_R$\{${a_1x, a_2x, \ldots, a_nx}$\} = 0. If in this concept we put $M = R$,
we obtain the concept of a strongly-prime ring. Such strongly-prime rings
were studied in [4].

A submodule $P$ of some module $M$ is called strongly-prime, if the
quotient module $M/P$ is a strongly-prime $R$-module. The set of all
strongly-prime submodules of module $M$ is called the left prime spectrum
of $M$ and is denoted by $\text{spec}(M)$. In particular, a left ideal $\mathfrak{p} \subseteq R$ is called
strongly-prime if the quotient module $R/\mathfrak{p}$ is a strongly-prime $R$-module.
In terms of elements, left ideal $\mathfrak{p} \subseteq R$ is strongly-prime if for every $u \notin \mathfrak{p}$
there exists such elements \{${a_1}, \ldots, a_n$\} $\subseteq R$ and a natural number $n = n(u)$
such that $ra_1u, \ldots, ra_nu \in \mathfrak{p}$, $r \in R$ implies $r \in \mathfrak{p}$.

2. Preorder on the set of modules and cyclic left spectrum
of module

It is easy to see that if $R$ is a left noetherian ring and $\mathfrak{p} \in \text{Spec}(R)$,
then $R/\mathfrak{p}$ is a left noetherian prime ring. This implies that it is sufficient
to prove that in a left noetherian prime ring $R$ zero ideal belongs to $\text{spec}(R)$. But taking into account the assumption that $R$ is a prime Goldie
ring, for any $0 \neq x \in R$ any two-sided ideal $RxR$ is essential, thus there
exists a regular element $a = \sum_{i=1}^{n} r_i x s_i \in RxR$ (Using Goldie theorem).

Let $V = \{r_1, \ldots, r_n\}$ and $y \in (0 : V x)$, then $ya = \sum yr_ixs_i = 0$. Since $a$
is regular, it follows that $y = 0$, hence $0 \in \text{spec}(R)$ indeed.

Clearly, it is necessary to demonstrate how to calculate prime left
ideals in an easy example. For this purpose we use the following example.

**Example 1.** Consider the matrix ring $R = M_2(k)$ over a (commutative)
field $k$.

\[ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \]
\[ \begin{bmatrix} a' & b' \\ 0 & 0 \end{bmatrix} \]

Moreover, $[a, b]_k = [a', b']_k$ if and only if there exists such $c \in k$ that
$a = ca'$ and $b = cb'$. Then $\text{spec}(R) = \{[a, b]_k \mid a, b \in k\}$ may be identified
with the projective line $P^1_k$ (with "generic point" $(0) = [0, 0]_k$). (See [6])

As in [8] we introduce a preorder $\leq$ on the set of all left ideals by
putting $K \leq L$ for a pair of left $R$-ideals $L$ and $K$ if and only if there
exists a finite subset $V$ of the ring $R$ such that $(K : V) \subseteq L$.

Let us try to establish a preorder on the modules. Let $R$ be a regular
module over itself with generator 1. Then $M = R \cdot 1$ is a cyclic module.

**Theorem 1.** Every cyclic module is isomorphic to the quotient module
of a regular module by the annihilator of a generator $R \cdot m = R/\text{Ann}(m)$,
where $\text{Ann}(m)$ is the left annihilator of a generator $m$. 
Consider some submodules of a cyclic module $M$ which is presented as $Rm = R/\text{Ann}(m)$ for the generator $m$. Let $L$, $K$ be some submodules. We can represent $L = \mathfrak{A}/\text{Ann}(m)$ and $K = \mathfrak{B}/\text{Ann}(m)$ for some left ideals $\mathfrak{A}$ and $\mathfrak{B}$ of a ring $R$. Then we define $L = \mathfrak{A}/\text{Ann}(m) \leq K = \mathfrak{B}/\text{Ann}(m)$ if and only if $\mathfrak{A} \leq \mathfrak{B}$ as the Rosenberg ideals. All properties are carried out. Thus the spectrum of a cyclic module is the set of all ideals that are in the spectrum of ring $R$.

It is well known that any module is the sum of its cyclic submodules. Then the cyclic spectrum of an arbitrary module $M$ is defined as the union of all spectra of its cyclic submodules. The cyclic spectrum of module $M$ is denoted by $\text{Cspec}(M)$. Then we can define $L \leq K \iff \text{Cspec}(L) \subseteq \text{Cspec}(K)$ for all submodules of the module $M$ and obtain a preorder on the family of such submodules.

**Example 2.** Let $M = \{(a, b) | a, b \in k\}$ be module of columns with height 2 over ring $R = M_2(k)$, where $k$ is commutative field.

This module is cyclic with generator $e = (\frac{1}{0})$, that is, $M = R \times (\frac{1}{0})$. Then $\text{Ann}((\frac{1}{0})) = \{(0, b, 0, d) | b, d \in k\}$, thus $M/\text{Ann}((\frac{1}{0})) \cong \{(a, 0, 0, c) | a, c \in k\}$. The maximal submodule is $\{(0, 0, 0, 0)\}$, hence cyclic spectrum consists of one point.

**Lemma 1.** Let $L$ and $K$ be left cyclic $R$-modules. Then $L \leq K$ if and only if there exists a cyclic left $R$-module $X$, a monomorphism $X \hookrightarrow L^n$ and an epimorphism $X \twoheadrightarrow K$. In other words, there exists a diagram $(L)^n \hookrightarrow X \twoheadrightarrow K$.

**Proof.** Recall the definition of preorder for submodules of a cyclic module. Let $L$, $K$ be some submodules. We can represent $L = \mathfrak{A}/\text{Ann}(m)$ and $K = \mathfrak{B}/\text{Ann}(m)$ for some left ideals $\mathfrak{A}$ and $\mathfrak{B}$ of the ring $R$. Then we define $L = \mathfrak{A}/\text{Ann}(m) \leq K = \mathfrak{B}/\text{Ann}(m)$ iff $\mathfrak{A} \leq \mathfrak{B}$ as Rosenberg ideals. Thus consider two cyclic modules $L$ and $K$. They are fully represented by their ideals $\mathfrak{A}$ and $\mathfrak{B}$. Than if $\mathfrak{A} \leq \mathfrak{B}$ by the definition, than there exists a finite subset $V \subseteq R$, such that $(\mathfrak{A} : V) \leq \mathfrak{B}$. Put $V = \{v_1, \ldots, v_n\}$ and let $X = R\vec{v}$ be a cyclic module, where $\vec{v} = \{v_1, \ldots, v_n\} \in (L)^n$. Than we have $(0 : \vec{v}) = \cap_{i=1}^n (\mathfrak{A} : v_i) = (\mathfrak{A} : V) \subseteq \mathfrak{B}$, which implies that there exists a surjection $X \twoheadrightarrow K$.

On the other hand, assume that there exists a diagram $(L)^n \hookrightarrow^\alpha X \twoheadrightarrow^\beta K$. Thus we can find such element $x \in X$, that $\beta(x) = \vec{1}$. Put
\[ \alpha(x) = (\vec{v}_1, \ldots, \vec{v}_n) \in (L)^n, \text{ where } (\vec{v}_1, \ldots, \vec{v}_n) \in \mathfrak{A} \text{ for some } v_i \in R. \]

Put \( V = \{v_1, \ldots, v_n\} \) and then we have
\[
(\mathfrak{A} : V) = \cap_{i=1}^n (\mathfrak{A} : v_i) = (0 : \vec{v}) = (0 : x) \subseteq \mathfrak{B},
\]
so \( \mathfrak{A} \leq \mathfrak{B} \) and \( L \leq K \).

Usually from the preorder \( \leq \) we obtain an equivalence relation \( \sim \) as follows: \( K \sim L \) iff \( K \leq L \) and \( L \leq K \). The equivalence class of the submodule \( L \) will be denoted by \([L]\).

**Lemma 2** (11). If \( \mathfrak{P} \) is a strongly-prime module, then for any element \( x \in M \) the following properties are equivalent:

1. \( x \notin \mathfrak{P} \);
2. \( (\mathfrak{P} : x) \leq \mathfrak{P} \);
3. \( (\mathfrak{P} : x) \in [\mathfrak{P}] \).

**Lemma 3.** Let \( M \) be cyclic module. If \( \mathfrak{P} \in \text{Cspec}(M) \) and if \( L \) and \( K \) are submodules such that \( L \cap K \leq \mathfrak{P} \), then either \( L \leq \mathfrak{P} \) or \( K \leq \mathfrak{P} \).

**Proof.** Let \( L \notin \mathfrak{P} \) and \( K \notin \mathfrak{P} \) and let \( L \cap K \leq \mathfrak{P} \). Thus, by the definition, there exist ideals \( \mathfrak{A}, \mathfrak{B} \) and \( p \) of the ring \( R \), such that \( L = \mathfrak{A}/\text{Ann}(m) \), \( K = \mathfrak{B}/\text{Ann}(m) \) and \( P = p/\text{Ann}(m) \). Then there exists a finite subset \( V \) of the ring \( R \), such that \( (\mathfrak{A} \cap \mathfrak{B} : V) \subseteq p \). Since \( \mathfrak{A} \notin p \), this implies that \( (\mathfrak{A} : F) \notin p \) for some finite subset \( F \) of the ring \( R \). Thus, if we take \( F = V \), we obtain the fact, that \( (\mathfrak{A} : V) \notin p \). Now, if \( x \in (\mathfrak{A} : V) - p \), then there exists a finite set \( W \subseteq R \) with the property that \( (p : Wx) \subseteq p \). Since \( K \notin p \), we have \( b \notin p \), get fact that \( (\mathfrak{B} : F) \notin p \) for any finite set \( F \subseteq R \).

In particular, this holds for \( F = WxV \), thus we can find an element \( y \in (\mathfrak{B} : WxV) - p \). Finally, \( x \in (\mathfrak{A} : V) \) implies that \( yWxV \subseteq \mathfrak{B} \), and \( y \) belongs to the set \( (\mathfrak{B} : WxV) \). Certainly, \( yWxV \subseteq \mathfrak{B} \), then \( yWxV \subseteq \mathfrak{A} \cap \mathfrak{B} \) and \( yWx \subseteq (\mathfrak{A} \cap \mathfrak{B} : V) \subseteq p \). Thus, \( y \in (p : Wx) \subseteq p \), that contradicts to the fact, that \( y \notin p \).

Similarly

**Lemma 4.** If \( \mathfrak{P} \in \text{Cspec}(R) \) and if \( L \) and \( K \) are submodules such that \( LK \leq \mathfrak{P} \), then either \( L \leq \mathfrak{P} \) or \( K \leq \mathfrak{P} \).

Recall the operation of multiplication of the submodules of cyclic module \( R/c \). Any submodule of cyclic module can be viewed as the quotient-module of some left ideal by some other left ideal. Let we have two such submodules \( L \cong a/c \) and \( K \cong b/c \). Then \( L \cdot K = a/c \cdot b/c = ab/c \).
Lemma 5. Let $\mathcal{P}$ and $\mathcal{Q}$ be strongly-prime submodules of the cyclic module $M$. Then the following holds:

1. If $\mathcal{P} \sim \mathcal{Q}$, then $\mathcal{P} \cap \mathcal{Q}$ is a strongly-prime module and $\mathcal{P} \sim \mathcal{P} \cap \mathcal{Q}$;
2. If $\mathcal{P} \cap \mathcal{Q}$ is a strongly-prime module, then either $\mathcal{P} \subseteq \mathcal{Q}$ or $\mathcal{P} \supseteq \mathcal{Q}$ or $\mathcal{P} \sim \mathcal{Q}$.

Proof. Let $\mathcal{P}$ and $\mathcal{Q}$ be strongly-prime submodules of a cyclic module $M$. Thus, for every submodule of a cyclic module there exist ideals $\mathcal{P} = p/\text{Ann}(m)$ and $\mathcal{Q} = q/\text{Ann}(m)$, where $\mathcal{P} \leq \mathcal{Q}$ if and only if $p \leq q$ as Rosenberg ideals. Similarly, we can formulate the definition of equivalence relation. Thus let $p \sim q$ and $x / p \not\in \mathcal{P} \cap \mathcal{Q}$. Let $x \not\in p$, thus there exists a finite subset $V \subseteq R$, such that $(p : V x) \subseteq p$. If $x \not\in q$, then $(q : W x) \subseteq q$ for some finite subset $W$ of the ring $R$. Let $U = V \cup W$, then $(p \cap q : U x) \subseteq p \cap q$. If $x \in q$, then $(q : V x) = R$, hence $(p \cap q : V x) \subseteq p$. Since $p \sim q$ by the assumption, $p \leq q$, and thus $(p : U) \subseteq q$ for some finite subset $U \subseteq R$, and since we may assume that $1 \in U$, we obtain

$$(p \cap q : UV x) = ((p \cap q : V x) : U) \subseteq (p : U) \subseteq q.$$  

Moreover, since $V \subseteq UV$, we also have

$$(p \cap q : UV x) \subseteq (p \cap q : V x) \subseteq p,$$

hence $(p \cap q : UV x) \subseteq p \cap q$, thus $p \cap q$ is a strongly prime ideal. Clearly $p \cap q \leq p$. On the other hand, since $p \leq q$, there exists a finite subset $V \subseteq R$, with $(p : V) \subseteq q$. We may obviously assume that $1 \in V$, thus we have $(p : V) \subseteq p$. Hence $(p : V) \subseteq p \cap q$, so $p \leq p \cap q$ and $p \sim p \cap q$.

Let us now assume that $p \cap q$ is a strongly-prime ideal while $p \not\subseteq q$ and $p \not\supseteq q$. Sinc such a $p \not\subseteq q$ there exists an element $x \in p - q$. Thus $x \not\in p \cap q$ and we may find a finite subset $V \subseteq R$ such that $(p \cap q : V x) \subseteq p \cap q$. Since $(p : V x) = R$, this yields $(q : V x) \subseteq p \cap q \subseteq p$, hence $p \leq q$. By symmetry $p \geq q$, and thus $p \sim q$. \hfill \square

We easy obtain the following corollary:

Corollary 1. Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be a finite family of strongly-prime modules, such that $\mathcal{P}_1 \sim \cdots \sim \mathcal{P}_n$, then $\bigcap_{i=1}^n \mathcal{P}_i$ is a strongly-prime module and $\mathcal{P}_1 \sim \bigcap_{i=1}^n \mathcal{P}_i$.

For any left module $M$, it’s submodule $N$ is called strongly two-sided, if left annihilator of every element of $N$ is two-sided ideal. Clearly, new
submodule is two-sided. Thus the set of such submodules is not empty, because the zero submodule is strongly two-sided submodule. The sum of all strongly two-sided submodules is called the bound of the submodule $N$. In other words, the bound of the module is the largest submodule among those that have two-sided left annihilators for all their elements. In the case when $M = N$ we are talking about the concept of a bound of the module. As follows, the bound of the module $M$ is the largest strongly two-sided submodule of the module $M$. Denote the bound of a submodule $N$ by $b(N)$, the bound of the module $M$ by $b(M)$.

**Lemma 6.** For every strongly-prime left submodule $\mathfrak{p}$ of the module $M$ we have $b(\mathfrak{p}) \in \text{Cspec}(M)$.

**Proof.** Let $x, y \in M$ by elements, such that $xRy \subseteq b(\mathfrak{p})$. Assume that $y \notin b(\mathfrak{p})$. Then there exists such an element $s \in R$ with $ys \notin \mathfrak{p}$. For every $r \in R$, $(xr)R(ys) \subseteq (xRy)s \subseteq b(\mathfrak{p})s \subseteq b(\mathfrak{p}) \subseteq \mathfrak{p}$. Hence $rx \in \mathfrak{p}$. Thus $xR \subseteq b(\mathfrak{p})$, which proves the assertion.

**Lemma 7.** If $L \leq K$ are left $R$-modules, then $b(L) \subseteq b(K)$. Conversely, if $R$ is a left noetherian fully-bounded ring, and if $b(L) \subseteq b(K)$, then $L \leq K$.

**Proof.** Since $L \leq K$, there exists a representation $L = \mathfrak{A}/\text{Ann}(m)$ and $K = \mathfrak{B}/\text{Ann}(m)$ for some left ideals $\mathfrak{A}$ and $\mathfrak{B}$ of the ring $R$. Then $\mathfrak{A} \subseteq \mathfrak{B}$. Thus there exist a finite subset $V \subseteq R$, that $(\mathfrak{A} : V) \subseteq \mathfrak{B}$. Then for every elements $r \in b(L)$ and $s \in R$, we have $rs \in \mathfrak{A}$, therefor $r \in (\mathfrak{A} : s)$. Thus $r \in (\mathfrak{A} : V) = \cap_{s \in V}(\mathfrak{A} : s)$. Since the former is contained in $\mathfrak{B}$, we have $b(L) \subseteq K$, hence $b(L) \subseteq b(K)$.

On the other hand, if $R$ is a left noetherian fully-bounded ring, then there exists a finite subset $V = \{v_1, \ldots, v_n\} \subseteq R$ such that $b(L) = \cap_{i=1}^n(\mathfrak{A} : v_i) = (\mathfrak{A} : V)$. Hence $(\mathfrak{A} : V) = b(\mathfrak{A}) \subseteq b(\mathfrak{B}) \subseteq \mathfrak{B}$, and $\mathfrak{A} \subseteq \mathfrak{B}$, therefore $L \leq K$.

**Corollary 2.** Let $L$ and $K$ be left modules such that $L \sim K$, then $b(L) = b(K)$. Moreover if $R$ is a left noetherian fully-bounded ring, then the converse is also true.

3. **Functoriality of cyclic spectrum of module**

The cyclic spectrum construction can be regarded as a contravariant functor from the category of modules to the category of sets,

$$\text{CSpec} : \text{Mod} \to \text{Set}. $$
A contravariant functor $\text{CSpec}$ is a rule assigning to each module $M$ over an associative ring $R$ the set $\text{CSpec}(M)$, the cyclic spectrum, i.e. the set of submodules that are related in that spectrum, and to each module homomorphism $f: M_1 \to M_2$ the map of sets

$$\text{Cspec}(M_1) \to \text{Cspec}(M_2),$$

$$P \mapsto f^{-1}(P).$$

Consider the endomorphism ring $E = \text{End}(M)$, and also consider the center of that ring, denoted by $C = \{c \in E \mid cr = rc, \forall r \in E\}$. Consider the construction of partial algebra over the ring $C$. It is the set $Q$ with a reflexive, symmetric binary relation $\perp \subseteq Q \times Q$ (called commeasurability), partial addition and multiplication operations "+$" and "$\cdot$", that are functions $I \to Q$, a scalar multiplication operation $E \times Q \to Q$, and elements $0, 1 \in C$, such that the following axioms are satisfied:

1. for all $q \in Q$, $a \perp 0$ and $a \perp 1$;
2. the relation $\perp$ is preserved by the partial binary operations: for all $q_1, q_2, q_3 \in Q$, with $q_i \perp q_j$ ($1 \leq i, j \leq 3$) and for all $\lambda \in C$, one has $(q_1 + q_2) \perp q_3$, $(q_1 \cdot q_2) \perp q_3$ and $(\lambda q_1) \perp q_2$;
3. if $q_i \perp q_j$ for $1 \leq i, j \leq 3$, then the values of all polynomials in $q_1, q_2$ and $q_3$ form a commutative algebra.

Commeasurability subalgebra of a partial $C$-algebra $Q$ is a subset $Z \subseteq Q$ consisting of pairwise commeasurable elements that is closed under $C$-scalar multiplication and the partial binary operations of $Q$.

Given functors $K: \mathcal{A} \to \mathcal{B}$ and $S: \mathcal{A} \to \mathcal{C}$, we recall that the (right) Kan extension of $S$ along $K$ is a functor $L: \mathcal{B} \to \mathcal{C}$ with a natural transformation $\varepsilon: LK \to S$ such that for any other functor $F: \mathcal{B} \to \mathcal{C}$ with a natural transformation $\eta: FK \to S$ there is a unique natural transformation $\delta: F \to L$, such that $\eta = \varepsilon \circ (\delta K)$.

**Theorem 2.** The functor $\text{Cspec}: \text{Mod}^{\text{op}} \to \text{Set}$, with the identity natural transformation $\text{Cspec} |_{\text{Comm Mod}^{\text{op}}} \to \text{CSpec}$ is the Kan extension of the functor $\text{Cspec}: \text{Comm Mod}^{\text{op}} \to \text{Set}$ along the embedding $\text{Comm Mod}^{\text{op}} \subseteq \text{Mod}^{\text{op}}$.

**Proof.** Let $F: \text{Mod} \to \text{Set}$ be a contravariant functor with a fixed natural transformation $\eta: F|_{\text{Comm Mod}} \to \text{Spec}$. Consider functor $C$-$\text{Spec}: \text{Comm Mod} \to \text{CSpec}$. We need to show that there
is a unique natural transformation $\delta: F \to \text{CSpec}$, that induces $\eta: F|_{\text{Comm Mod}} \to \text{CSpec}$ upon a restriction to $\text{Comm Mod} \subseteq \text{Mod}$. To construct it, fix ring $R$ and module $M$ over it. For every submodule $N \subseteq M$ over ring $R$ the inclusion $N \subseteq M$ given a morphisms of sets $F(M) \to F(N)$, and $\eta$ provides a morphisms $\eta_N: F(N) \to \text{CSpec}(N)$; these compose to give morphisms $F(M) \to \text{CSpec}(N)$. By naturality of the morphisms involved, these maps of $F(M)$ collectively form a cone over the diagram obtained for submodules of module. By the universal property of limit, there exists a unique arrow making corresponding diagram commutative for all $N \subseteq M$.

Defined morphisms $\delta_M$ form the components of a natural transformation $\delta: F \to \text{CSec}$. By construction, $\delta$ induces $\eta$ when restricted to $\text{Comm Mod}$. Uniqueness of $\delta$ is guaranteed by the uniqueness of the indicated arrow used to define $\delta_M$ above.

4. **Localisations**

Recall some definitions. By a *torsion-theoretic spectrum* we mean the space of all prime torsion theories (or prime Gabriel filters of a main ring) in the category of left $R$-modules with Zarisky topology. Recall that *prime torsion theory* $\pi \in R - \text{tors}$ is a torsion theory, for which $\pi = \chi(R/I)$ for some critical ideal $I$ of the ring $R$, where $R - \text{tors}$ is class of all torsion theories of the category $R$-mod and $\chi(R/I)$ is the torsion theory, cogenerated by module $E(R/I)$. If $\tau$ is torsion theory of the category $R$-mod, then left $R$-module $M$ is called *torsion free module* if and only if there exist $R$ from $M$ into some member of $\tau$. Class of all torsion free modules for some $\tau$ is denote by $F_\tau$. Further information about the prime torsion theories can be fund in [5].

**Remark 1.** The class of all torsion theories $R$-tors can be partially ordered by setting $\tau \leq \tau'$ if and only if $\Sigma_\tau \subseteq \Sigma_{\tau'}$, namely, the class of all torsion modules of one torsion theory is contained in the class of all torsion modules of other torsion theory.

Introduce the notion of torsion-theoretic spectrum of a module $M$. Use the concepts of torsion-theoretic spectrum of a ring $R$ introduced above. Introduce the concept of support of module $M$: $\text{supp}(M) = \{\sigma|\sigma(M) \neq 0\}$. *Torsion-theoretic spectrum of module* $M$, $R$$\text{-Sp}(M)$ is defined as $R$$\text{-sp}(R) \cap \text{supp}(M)$.

If $M$ is a left $R$-module, denote by $\xi(M)$ the smallest torsion theory such that $M$ will be a torsion module, by $\chi(M)$ the largest torsion theory,
that $M$ will be a torsion-free module. Clearly, $\mathcal{T}_{\chi(M)}$ consists of $R$-modules $N$ such that $\text{Hom}_R(N, E(M)) = 0$, where $E(M)$ is the injective hull of a module $M$.

**Lemma 8.** If $\sigma$ is a torsion theory and $\mathfrak{P}$ is a left Rosenberg point of a cyclic module $M$, then $M/\mathfrak{P}$ is either a $\sigma$-torsion module or a $\sigma$-torsion free module.

**Proof.** Assume that $M/\mathfrak{P} \not\in \mathcal{F}_{\sigma}$. If $\mathfrak{P}$ is a left Rosenberg point, then there exists ideal $\mathfrak{p}$ of a ring $R$ such that $\mathfrak{P} = \mathfrak{p} / \text{Ann}(m)$. Pick an element $0 = \bar{x} \in \sigma(R/\mathfrak{p})$. Thus, there exists a finite subset $V$ of the ring $R$ with $(\mathfrak{p} : Vx) \subseteq \mathfrak{p}$. Obviously, $V\bar{x} \subseteq \sigma(R/\mathfrak{p})$, hence, for every element $v \in V$ there exists left ideal $L_v \in \mathcal{L}(\sigma)$ such that $L_v v x \subseteq \mathfrak{p}$. Let $L = \cap_{v \in V} L_v$, then $L \in \mathcal{L}(\sigma)$ and $LV x \subseteq \mathfrak{p}$. Hence $L \subseteq (\mathfrak{p} : Vx) \subseteq \mathfrak{p}$ and $\mathfrak{p} \in \mathcal{L}(\sigma)$, and therefore $M/\mathfrak{P}$ is $\sigma$-torsion module.

**Proposition 1.** If $M$ is a fully bounded left noetherian module and $\mathfrak{P} \in \text{Cspec}(M)$, then the torsion theory $\tau_{\mathfrak{P}} = \chi(M/\mathfrak{P})$ cogenerated by module $M/\mathfrak{P}$ is prime.

**Proof.** Obviously, $\mathfrak{P} \not\in \mathcal{L}(\tau_{\mathfrak{P}})$, therefore $M/\mathfrak{P}$ is a $\tau_{\mathfrak{P}}$-torsion free module. Thus, since $\chi(M/\mathfrak{P})$ is the largest torsion theory for which $M/\mathfrak{P}$ is torsion free module. We have $\chi(M/\mathfrak{P}) \leq \tau_{\mathfrak{P}}$. Conversely, assume that $\mathcal{L}(\chi(M/\mathfrak{P})) \not\subseteq \mathcal{L}(\tau_{\mathfrak{P}})$. Take $L \in \mathcal{L}(\chi(M/\mathfrak{P})) - \mathcal{L}(\tau_{\mathfrak{P}})$, then $L \leq \mathfrak{P}$. Thus, by the definition, $\mathfrak{A} \leq \mathfrak{p}$ for some ideals $\mathfrak{A}$ and $\mathfrak{p}$ of the ring $R$. Thus there exists a finite subset $U \subseteq R$ such that $\cap_{u \in U} (\mathfrak{A} : u) = (\mathfrak{A} : U) \subseteq \mathfrak{p}$. Hence $\mathfrak{p} \in \mathcal{L}(\chi(M/\mathfrak{P}))$, contradicting the definition of $\chi(M/\mathfrak{P})$.

The previous statements imply the following result.

**Theorem 3.** The mapping $\Phi : \text{Cspec}(M) \to \text{M-sp}$, where $\Phi(\mathfrak{P}) = \chi(M/\mathfrak{P})$ is continuous and surjective.

**References**


CONTACT INFORMATION

M. Maloid-Glebova  Ivan Franko National University of L’viv
E-Mail(s): martamaloid@gmail.com

Received by the editors: 05.10.2015
and in final form 22.12.2015.