Total global neighbourhood domination

S. V. Siva Rama Raju and I. H. Nagaraja Rao

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Abstract. A subset $D$ of the vertex set of a connected graph $G$ is called a total global neighbourhood dominating set (tgnd-set) of $G$ if and only if $D$ is a total dominating set of $G$ as well as $G^N$, where $G^N$ is the neighbourhood graph of $G$. The total global neighbourhood domination number (tgnd-number) is the minimum cardinality of a total global neighbourhood dominating set of $G$ and is denoted by $\gamma_{tgnd}(G)$. In this paper sharp bounds for $\gamma_{tgnd}$ are obtained. Exact values of this number for paths and cycles are presented as well. The characterization result for a subset of the vertex set of $G$ to be a total global neighbourhood dominating set for $G$ is given and also characterized the graphs of order $n \geq 3$ having tgnd-numbers $2, n-1, n$.

Introduction and preliminaries

Domination is an active topic in graph theory and has numerous applications to distributed computing, the web graph and adhoc networks. Haynes et al. gave a comprehensive introduction to the theoretical and applied facets of domination in graphs.

A subset $D$ of the vertex set $V$ is called a dominating set [8] of the graph $G$ if and only if each vertex not in $D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ is the minimum cardinality of the dominating set of $G$.

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Many variants of the domination number have been studied. For instance a dominating set $S$ of graph $G$ is called a total dominating set \[3\] if and only if every vertex in $V$ is adjacent to a distinct vertex in $D$. The total domination number of $G$, denoted by $\gamma_t(G)$ is the smallest cardinality of the total dominating set of $G$. A set $D$ is called a connected dominating set of $G$ if and only if $D$ is a dominating set of $G$ and $\langle D \rangle$ is connected. The connected domination number \[4\] of $G$, denoted by $\gamma_c(G)$ is the smallest cardinality of the connected dominating set of $G$. A dominating set $D$ of connected graph $G$ is called a connected dominating set of $G$ if the induced subgraph $\langle D \rangle$ is connected. The connected domination number of $G$, denoted by $\gamma_c(G)$ is the least cardinality of the connected dominating set of $G$ \[7\].

If $G$ is a connected graph, then the Neighbourhood Graph \[7\] of $G$, denoted by $N(G)$ (or) $G^N$, is the graph having the same vertex set as that of $G$ and edge set being $\{uv/u, v \in V(G), \text{there is } w \in V(G) \text{ such that } uw, wv \in E(G)\}$ \[2\].

In \[5\], a new type of graphs, called semi complete graphs, are introduced as follows. A connected graph $G$ is said to be semi complete if any two vertices in $G$ have a common neighbour.

A subset $D$ of the vertex set $V$ is called a global neighbourhood dominating set \[6\] of the graph $G$ if and only if $D$ is a dominating set of $G$, as well as $G^N$. The global neighbourhood domination number, $\gamma_{gn}(G)$ is the minimum cardinality of the global neighbourhood dominating set of $G$.

In the present paper, we introduce a new graph parameter, the total global neighbourhood domination number, for a connected graph $G$. We call $D \subseteq V$ a total global neighbourhood dominating set(tgnd-set) of $G$ if and only if $D$ is a total dominating set for both $G, G^N$. The total global neighbourhood domination number is the minimum cardinality of a total global neighbourhood dominating set of $G$ and is denoted by $\gamma_{tgnd}(G)$. By a $\gamma_{tgnd}$-set of $G$, we mean a total global neighbourhood dominating set for $G$ of minimum cardinality.

All graphs considered in this paper are simple, finite, undirected and connected. For all graph theoretic terminology not defined here, the reader is referred to \[1\] and \[8\].

In this paper sharp bounds for $\gamma_{tgnd}$ are given. A characterization result for a proper subset of the vertex set of $G$ to be a tgnd-set of $G$ is obtained and also characterized the graphs whose tgnd-numbers are $2, n, n - 1$.

Note. If $G$ is a simple graph such that $G$ has isolates, then clearly $\gamma_{tgnd}$-set of $G$ does not exist. So, unless otherwise stated, throughout this paper $G$ stands for a connected graph such that $G^N$ has no isolates.
1. Main results

We give the tgnd-numbers of some standard graphs.

Proposition 1. 1) $\gamma_{tgnd}(K_n) = 2; n = 3, 4, \ldots,$
2) $\gamma_{tgnd}(C_3) = 2$
3) $\gamma_{tgnd}(C_4) = 4$
4) $\gamma_{tgnd}(P_n) = 4; n = 4, 5$
5) $\gamma_{tgnd}(P_n (or) C_n) = 4 \left\lceil \frac{n}{6} \right\rceil + j; n = 6m + j; j = 0, 1, 2, 3.
   = 4 \left\lceil \frac{n}{6} \right\rceil + 4; n = 6m + j; j = 4, 5.$
6) $\gamma_{tgnd}(K_{m,n}) = 4; m, n \geq 2.$
7) $\gamma_{tgnd}(S_{m,n}) = 4.$
8) $\gamma_{tgnd}(C_n o K_2) = n$
9) $\gamma_{tgnd}(K_{1,n})$ does not exist.

Now, we give a characterization result for a total dominating set of $G$ to be a total global neighbourhood dominating set of $G$. Also, we give a relation between connected dominating set and total global neighbourhood dominating set.

Theorem 1. For a graph $G$ the following holds.

(i) A total dominating set $D$ of $G$ is a total global neighbourhood dominating set of $G$ if and only if from each vertex in $D$ there is a path of length two to a vertex in $D$. (characterization result)

(ii) Any connected dominating set for $G$ of cardinality atleast four is a total global neighbourhood dominating set for $G$.

Proof. The proof of (i) is trivial.

The proof of (ii) is as follows. Let $D \subseteq V$ (vertex set of $G$) be a connected dominating set of $G$ with $|D| \geq 4$. It is enough to prove that $D$ is a total dominating set of $G^N$. If $D = V$, we are through. Otherwise, let $v$ be any vertex in $V - D$. Suppose $v$ is adjacent to all the vertices of $D$ (in $G$). Since $\langle D \rangle$ is connected there are $u, w$ in $D$ such that $\langle uvw \rangle$ is a triangle in $G$. This implies $uv, vw$ are in $G^N$ ($u, w$ are in $D$). If $v$ is not adjacent to atleast one vertex in $D$, since $D$ is connected there is $w$ in $D$ such that $vw$ is in $G^N$. Hence in either case there is a $w$ in $D$ such that $vw$ is in $G^N$.

Let $v$ be an arbitrary vertex in $D$. Since $D$ is a connected dominating set of $G$ of cardinality atleast four, there is a $v_1$ in $D$ such that $vv_1$ lies on $C_3$ (in $G$) or $d_G(v, v_1) = 2$. In either case $vv_1$ is in $G^N$.

Hence $D$ is a total dominating set of $G^N$. \qed
Remark. For any connected graph $G$ of order $n \geq 4$, we have $\gamma_t(G) \leq \gamma_{tgn}(G) \leq \gamma_c(G)$.

Lemma 1. If $H$ is a spanning subgraph of a connected graph $G$, then $\gamma_{tgn}(G) \leq \gamma_{tgn}(H)$.

Lemma 2. For a graph $G$ with $n \geq 1$ vertices, we have $2 \leq \gamma_{tgn}(G) \leq n$.

proof. The proof follows by the characterization result. □

Now, we characterize the graphs attaining lower bound.

Theorem 2. $\gamma_{tgn}(G) = 2$ if and only if there is an edge $uv$ in $G$ that lies on $C_3$ such that any vertex in $V - \{u, v\}$ is adjacent to at least one of $u, v$.

Proof. Assume that $\gamma_{tgn}(G) = 2$. So there is a pair of vertices $u, v$ in $V$ such that $\{u, v\}$ is a total dominating set for $G, G^N$. This implies $u, v$ are adjacent in $G, G^N$. Hence $uv$ lies on a cycle $C_3 = \langle uvwv \rangle$ in $G$. Since $\{u, v\}$ is a total dominating set for $G$, for $x \in V - \{u, v\}$, $xv$ or $xu$ is an edge in $G$.

The inverse implication is clear. □

Now, we characterize the graphs attaining upper bound.

Theorem 3. $\gamma_{tgn}(G) = n$ if and only if $G = C_4$ or $P_4$.

Proof. Assume that $\gamma_{tgn}(G) = n$. Suppose that diam$(G) \geq 4$. Then $d_G(u, v) \geq 4$ for some $u, v$ in $G$. Clearly $u$ or $v$ is not a cut vertex in $G$. Hence $V - \{u\}$ or $V - \{v\}$ is connected dominating set of cardinality at least four. By Theorem 1(ii), $V - \{u\}$ or $V - \{v\}$ is a tgn-set of $G$ of cardinality $n - 1$, a contrary to our assumption.

Suppose that diam$(G) = 3$. Without loss of generality assume that $d_G(u, v) = 3$ for some $u, v$ in $G$. Let $P = \langle uv_1v_2v \rangle$ be a diametral path in $G$. Form a spanning tree of $G$ say $G'$ by preserving the diametral path. Clearly diam$(G') \geq 3$. If $G' \neq P$, then $V - \{w\}$ ($w$ is a pendant vertex in $G'$) is a tgn-set of $G'$. By Lemma 1, $V - \{w\}$ is a tgn-set of $G$ of cardinality $n - 1$, a contrary to our assumption. If $G' = P$, then $G$ is not cyclic. This implies that $G$ is tree with diameter three. Clearly $G$ cannot have more than two pendant vertices. Hence $G = P_4$.

Suppose that diam$(G) = 2$. By hypothesis, $G$ cannot be acyclic. Also $G$ cannot have pendant vertices. Therefore $G$ is cyclic and each vertex lies on a cycle. Suppose that $g(G) = 3$. If $G = C_3$, $\gamma_{tgn}(G) = 2 < 3$ a contradiction. If $G \neq C_3$, then $V(C_3) \subset V$. If all the vertices in $V - V(C_3)$
are adjacent to $C_3$, then $\gamma_{tgn}(G) = 3 < n$, a contrary to our assumption. If there is at least one vertex $v_4$ in $V - V(C_3)$ not adjacent to $C_3$, then $V - \{v_4\}$ is a tgd-set of $G$ which is again a contradiction. Hence $g(G) \neq 3$.

Suppose that $g(G) = 4$. Let $C_4 = \langle v_1v_2v_3v_4 \rangle$ be a cycle in $G$. If $V = V(C_4)$, then we have two possibilities $G = C_4, G \neq C_4$. If $G \neq C_4$, we have $g(G) = 3$ which is not possible. If $G = C_4$, then $\gamma_{tgn}(G) = 4(= n)$. If $V \neq V(C_4)$ (i.e. $V(C_4) \subset V$), notice that $G$ has no pendant vertices. Since $g(G) = 4$, any vertex in $V - V(C_4)$ can be adjacent to exactly two non adjacent vertices of $C_4$. If all the vertices in $V - V(C_4)$ are adjacent to vertices in $C_4$, then $\gamma_{tgn}(G) < n$, a contrary to our assumption. If there is a vertex $v_5$ in $V - V(C_4)$ not adjacent to $C_4$, then $V - \{v_5\}$ is a tgd-set of $G$, a contrary to our assumption. Hence by our assumption $g(G) = 4$ implies $G = C_4$.

Suppose that $g(G) = 5$. Then we have two possibilities, $G = C_5$, $G \neq C_5$. If $G = C_5$, then $\gamma_{tgn}(G) = 4$, a contrary to our assumption. If $G \neq C_5$, then $V = V(C_5)$ or $V \neq V(C_5)$. If $V = V(C_5)$, then $g(G) < 5$ a contradiction to our supposition. If $V \neq V(C_5)$ (i.e. $V(C_5) \subset V$). Since $g(G) = 5$, each vertex in $V - V(C_5)$ is adjacent to at most one vertex in $C_5$. If all the vertices in $V - V(C_5)$ are adjacent to $C_5$, then $V(C_5)$ is a tgd-set of $G$, a contrary to our assumption. Suppose that there is a vertex $v_7$ in $V - V(C_5)$ adjacent to $C_5(= \langle v_1v_2v_3v_4v_5v_1 \rangle)$. Since $diam(G) = 2$, $v_7$ is at a distance two from each vertex of $C_5$. Then $V - \{v_6v_7\}$ ($\langle v_1v_6v_7 \rangle$ is a path) is a tgd-set of $G$, a contrary to our assumption. So $g(G) \neq 5$. Clearly $diam(G) \neq 1$. Hence we have $G = C_4$ or $G = P_4$. The inverse implication is clear.

**Theorem 4.** If $diam(G) \neq 2, 3$. Then, $\gamma_{tgn}(G) = n - 1$ if and only if $G = P_5$ or $C_3$.

**Proof.** Assume that $\gamma_{tgn}(G) = n - 1$. Suppose that $diam(G) \geq 5$. Forming a spanning tree $G'$ of $G$, we get $\gamma_{tgn}(G) \leq \gamma_{tgn}(G') \leq n - m < n - 1$, a contrary to our assumption (here $m$ is the number of pendant vertices in $G'$). Therefore $diam(G) \leq 4$.

Suppose that $diam(G) = 4$. If $G = P_5$, then $\gamma_{tgn}(G) = 4 = 5 - 1$. If $G \neq P_5$, forming a spanning tree $G'$ of $G$, we have $\gamma_{tgn}(G) < n - 1$, a contrary to our assumption. Hence $G = P_5$. Suppose that $diam(G) = 1$. This implies $G = K_n(n \geq 3)$. By Theorem 2, $\gamma_{tgn}(G) = 2 < n - 1$ whenever $n \geq 4$, a contrary to our assumption. So $G = C_3$. The inverse implication is clear.

\[\square\]
Theorem 5. Suppose $n \geq 5$ and $\text{diam}(G) = 2$. Then, $\gamma_{tgn}(G) = n - 1$ if and only if $G = C_5$ or $G$ is isomorphic to $H$ given by

\[ \begin{array}{c}
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\end{array} \]

Proof. Assume that $\gamma_{tgn}(G) = n - 1$. By hypothesis $g(G) \leq 5$. Suppose that $g(G) = 5$. If $G = C_5$, then $\gamma_{tgn}(G) = n - 1 (n = 5)$. If $G \neq C_5$, then $V = V(C_5)$ or $V \neq V(C_5)$. If $V = V(C_5)$, then $g(G) < 5$, a contradiction. If $V \neq V(C_5)$ i.e., $V(C_5) \subset V$. Clearly $G$ has no pendant vertices. By hypothesis any vertex in $V - V(C_5)$ is adjacent to atleast two non adjacent vertices of $C_5$ or at a distance two from each vertex of $C_5$. From the former case or later case we get $g(G) = 4$, a contradiction. So $V \neq V(C_5)$. Hence $g(G) = 5$ implies $G = C_5$.

Suppose that $g(G) = 4$. Then $G = C_4$ or $G \neq C_4$. If $G = C_4$, $\gamma_{tgn}(G) = 4 = n > n - 1$ a contrary to our assumption. If $G \neq C_4$, then we have $V = V(C_4)$ or $V \neq V(C_4)$. If $V = V(C_4)$, then $g(G) = 3$ a contradiction to our supposition. If $V \neq V(C_4)$ i.e., $V(C_4) \subset V$. By hypothesis and our supposition $G$ has no pendant vertices and any vertex $v$ in $V - V(C_4)$ is adjacent to exactly two non adjacent vertices of $C_4$ or $v$ is at a distance two from each vertex of $C_4$ or $v$ is at a distance two from a vertex of $C_4$ and adjacent to a vertex of $C_4$, non adjacent to the former. Except in the first case we can form a spanning tree $G'$ of $G$ with $\text{diam}(G') \geq 5$. So $\gamma_{tgn}(G) \leq \gamma_{tgn}(G') \leq n - m < n - 1$, a contrary to our assumption(here $m$ is the number of pendant vertices in $G'$). If $G$ has more than one vertex of first kind, then $\gamma_{tgn}(G) < n - 1$ a contrary to our assumption. If $G$ has exactly one vertex of first kind, then $\gamma_{tgn}(G) = n - 1$ and $G$ is isomorphic to $H$.

Suppose that $g(G) = 3$. Clearly $G$ has a cycle $C_3 (= \langle v_1v_2v_3v_1 \rangle)$. If $G \neq C_3$, then $V(C_3) \subset V$. Clearly $G$ cannot have more than one pendant vertex. Suppose $G$ has exactly one pendant vertex, say $v$. Since $\text{diam}(G) = 2$, there is a vertex $w$ on $C_3$ such that $vw$ is in $G$. Without loss of generality assume that $w = v_1$. Clearly $\{v_1, v_2\}$ or $\{v_1, v_3\}$ or $\{v_1, v\}$ is a $tgn$-set for $G$ a contrary to our assumption. This implies $G$ has no pendant vertices. So any vertex in $V - V(C_3)$ is adjacent to $C_3$ or at a distance two from atleast one vertex of $C_3$. In either case $\gamma_{tgn}(G) < n - 1$. The inverse implication is clear. \qed
Theorem 6. Suppose that $n \geq 5$ and $\text{diam}(G) = 3$. Then $\gamma_{\text{tgn}}(G) = n - 1$ if and only if $G$ is isomorphic to $H$ given by

\begin{figure}[h]
\centering
\includegraphics[width=0.1\textwidth]{triangle.png}
\end{figure}

Proof. Assume that $\gamma_{\text{tgn}}(G) = n - 1$. Clearly $g(G)$ is not greater than 6.

Suppose that $g(G) = 5$. By hypothesis $G \neq C_5$. This implies $V(C_5) \subset V$. Clearly $G$ cannot have more than two pendant vertices. Suppose $G$ has exactly one pendant vertex, say $v$. By hypothesis $v$ is adjacent to a vertex of $C_5(\langle v_1v_2v_3v_4v_5v_1 \rangle)$, say $v_1$. Then clearly $V - \{v_2, v_3\}$ is a tgnd-set of $G$, a contrary to our assumption. Suppose $G$ has exactly two pendant vertices.

Since $\text{diam}(G) = 3$ they are adjacent to a vertex on $C_5$ or adjacent to end vertices of an edge in $C_5$. In either case $\gamma_{\text{tgn}}(G) = n - 2 < n - 1$, a contrary to our assumption. So $G$ cannot have pendant vertices. Since $\text{diam}(G) = 3$ and $g(G) = 5$, $G$ cannot have more than two cycles. Hence $g(G) \neq 5$.

Suppose $g(G) = 3$. Clearly $G \neq C_3$ (since $n \geq 5$). Also $G$ cannot have pendant vertices. If $|V(G)| = 5$ or all the vertices in $V - V(C_3)$ are adjacent to $C_3$ or there is a vertex at a distance two from $C_3$, we get a contrary to our assumption. Hence $g(G) \neq 3$.

Suppose $g(G) = 4$. Since $\text{diam}(G) = 3$, $G \neq C_4$. Clearly $G$ cannot have more than two pendant vertices. If $|V(G)| = 5$, since $\text{diam}(G) = 3$ the vertex in $V - V(C_4)$ is a pendant vertex. This implies $\gamma_{\text{tgn}}(G) = 4 = 5 - 1 = n - 1$. Suppose $|V(G)| \geq 6$. If all the vertices in $V - V(C_4)$ are adjacent to $C_4$ (each vertex in $V - V(C_4)$ can be adjacent to exactly two non adjacent vertices in $C_4$ (since $g(G) = 4$)), then $\gamma_{\text{tgn}}(G) = 4 \leq n - 2 < n - 1$ a contrary to our assumption. If not, there is atleast one vertex in $V - V(C_4)$ at a distance two from $C_4$ (say $v$). Then $V - \{v, v_5\}$ is a tgnd-set of $G$ ($C_4 = \langle v_1v_2v_3v_4 \rangle$, $v_1v_5$ is an edge in $G$) which is again a contradiction. So $|V(G)|$ is not greater than or equal to 6. Hence $G \cong H$.

\begin{figure}[h]
\centering
\includegraphics[width=0.1\textwidth]{square.png}
\end{figure}

Theorem 7. Suppose $g(G) = 3$ and $\text{diam}(G) = 2$. Then $\gamma_{\text{tgn}}(G) = n - 2$ if and only if $G = K_4$ or $G \cong K_4 - \{e\}$ or $G$ is isomorphic to $H$ given by

\begin{figure}[h]
\centering
\includegraphics[width=0.1\textwidth]{square.png}
\end{figure}
Case 1: \( v_4 \neq v_5 \neq v_6 \). Then \( V - \{ v, v_4, v_5 \} \) is a tgn-d-set of \( G \).

Case 2: two of them are equal. Without loss of generality assume that \( v_4 = v_5 \). Clearly \( G \) cannot have pendant vertices. Then \( V - \{ v, v_4, v_1 \} \) is a tgn-d-set of \( G \).

Case 3: \( v_4 = v_5 = v_6 \). Clearly \( G \) cannot have pendant vertices. Then \( V - \{ v_1, v_2, v_3 \} \) is a tgn-d-set of \( G \).

In each of the three cases, we get a contradiction with our assumption. So our supposition is false. Hence all the vertices in \( V - V(C_3) \) are adjacent to \( C_3 \). Clearly \( C_3 \) has exactly one neighbour in \( V - V(C_3) \), say \( v \). If \( v \) is adjacent to exactly one vertex of \( C_3 \), then \( \gamma_{tgn}(G) = 2 = 4 - 2 \) and \( G \cong H \).

If \( v \) is adjacent to exactly two vertices of \( C_3 \), then \( \gamma_{tgn}(G) = 2 = 4 - 2 \) and \( G \cong K_4 - \{ e \} \). If \( v \) is adjacent to all vertices of \( C_3 \), then \( \gamma_{tgn}(G) = 2 = 4 - 2 \) and \( G = K_4 \).

The inverse implication is clear. \( \square \)

**Theorem 8.** If \( \delta(G) \geq 3 \) and \( g(G) > 4 \), then

\[
2e - n(n - 3) \leq \gamma_{tgn}(G) \leq n - \Delta(G) + 1.
\]

**Proof.** Suppose that \( D \) is a \( \gamma_{tgn} \)-set of \( G \). Since \( g(G) > 4 \), for each vertex in \( V \) there is a vertex in \( D \) which is not adjacent to the former. This implies \( e \leq nC_2 - [n - \gamma_{tgn}] - \frac{\gamma_{tgn}}{2} \). Hence \( 2e - n(n - 3) \leq \gamma_{tgn}(G) \).

Suppose \( d_G(v) = \Delta(G) \) for some \( v \) in \( V \). Let \( N_G(v) = \{ v_1, v_2, \ldots, v_{\Delta(G)} \} \). Now consider the set \( D = [V - N_G(v)] \cup \{ v_i : \Delta(G) - 1 \} \). Without loss of generality assume that \( D = [V - N_G(v)] \cup \{ v_{\Delta(G)} \} \). Let \( u_1 \in V \).

Case 1: \( u_1 \in V - D \). This implies \( u_1 \in \{ v_1, v_2, \ldots, v_{\Delta(G) - 1} \} \). Without loss of generality assume that \( u_1 = v_1 \). Clearly \( u_1v \) is in \( G \).

Case 2: \( u_1 \in D \). This implies \( u_1 \notin \{ v_1, v_2, \ldots, v_{\Delta(G) - 1} \} \). If \( u_1 = v \) or \( u_1 = v_{\Delta(G)} \), then \( u_1v_{\Delta(G)} \) or \( u_1v \) is in \( G \). If not since \( \delta(G) \geq 3 \) and \( g(G) > 4 \) there is \( u_2 \notin \{ v, v_1, v_2, \ldots, v_{\Delta(G)} \} \) such that \( u_1u_2 \) is in \( G \). Hence \( D \) is a total dominating set of \( G \).

We now show that \( D \) is a total dominating set of \( G^N \). Let \( u_1 \in V \).

Case 1: \( u_1 \in V - D \). This implies \( u_1 \in \{ v_1, v_2, \ldots, v_{\Delta(G) - 1} \} \). Since \( d_G(u_1, v_{\Delta(G)}) = 2 \), \( i = 1, 2, \ldots, \Delta(G) - 1 \) we have \( v_1v_{\Delta(G)} \) is in \( G^N \). So \( v_1, v_{\Delta(G)} \) is in \( G^N \).
Case 2: $u_1 \in D$. This implies $u_1 \notin \{v_1, v_2, \ldots, v_{\Delta(G)-1}\}$. Suppose $u_1 = v$. Since $\delta(G) \geq 3$ and $g(G) > 4$ we have $vu_2$ is in $G^N$. for some $u_2 \in N(N(v))$ and $u_2 \in D$. If $u_1 = v_{\Delta(G)}$. Suppose $u_1 \notin \{v, v_1, v_2, \ldots, v_{\Delta(G)}\}$. If $u_1 \in N(v_i)$ for some $i = 1, 2, \ldots, \Delta(G)$, then $u_1v$ is an edge in $G^N$. If $u_1 \notin N(v_i)$ for any $i$.

Subcase a: $u_1 \in N(N(v_i))$ for some $i = 1, 2, \ldots, \Delta(G)$. Without loss of generality assume that $u_1 \in N(N(v_1))$. Since $\delta(G) \geq 3$, there are $u_2$ and $u_3$ in $G$, adjacent to $u_1$. Since $g(G) > 4$, $u_2$ and $u_3$ cannot be adjacent to $\{v_1, v_2, \ldots, v_{\Delta(G)}\}$. This implies there is $u_4$ in $D$ such that $u_2u_4$ or $u_3u_4$ is in $G$. Hence $u_1u_4$ is in $G^N$.

Subcase b: $u_1 \in V - [\{N(N(v_i)) : i = 1, 2, \ldots, \Delta(G)\} \cup \{v_1, v_2, \ldots, v_{\Delta(G)}\}]$. By hypothesis there is a $u_2$ in $D - \{v, v_{\Delta(G)}\}$ such that $u_1u_2$ is in $G^N$. $D$ is a total dominating set of $G^N$.

Hence $D$ is a tgnd-set of $G$ whose cardinality is $n - \Delta(G) + 1$. So $\gamma_{tgn}(G) \leq n - \Delta(G) + 1$. This completes the proof.

**Notation.** For $n \geq 4$ and $k = 2, 3$ define a family of graphs $G_k$ as follows. $G \in G_k$ if and only if there is $D \subseteq V$ such that $|D| = k$ satisfying:

(i) $\langle D \rangle$ is connected;

(ii) at least two vertices of $D$ lie on the same $C_3$;

(iii) each vertex in $V - D$ is adjacent to a vertex in $D$.

**Theorem 9.** For $n \geq 4$, $\gamma_{tgn}(G) = 3$ if and only if $G \in G_3 - G_2$.

**Proof.** Assume that $\gamma_{tgn}(G) = 3$. Then there is a $\gamma_{tgn}$-set of $G$ such that $|D| = 3$ and $\langle D \rangle$ is connected. By the characterization result for tgnd-set there is a path of length 2 between a pair of adjacent vertices in $D$. This implies at least two vertices of $D$ lie on the same $C_3$. So $G \in G_3$. Since $D$ is a $\gamma_{tgn}$-set, $G \in G_2$. Hence $G \in G_3 - G_2$. The inverse implication is clear.

Before considering the next result, for convenience we introduce the following. For $n \geq 6$, define a family of trees $T_k$ as $T \in T_k$ if and only if there is a $D \subseteq V$ with $|D| = k$ satisfying:

(i) $\langle D \rangle$ is connected in $G$;

(ii) each vertex in $V - D$ is adjacent to a vertex in $D$ (in $G$).

**Theorem 10.** $\gamma_{tgn}(T) = 4$ if and only if $T \in T_4 - T_3$.

**Proof.** Suppose $\gamma_{tgn}(T) = 4$. Then there is a $\gamma_{tgn}$ - set of $T$ (say $D$) such that $D$ satisfies (i) and (ii) of the above mentioned family. This implies
$T \in \mathcal{T}_4$. Clearly by characterization theorem $T \notin \mathcal{T}_3$. Hence $T \in \mathcal{T}_4 - \mathcal{T}_3$. The inverse implication is clear. \hfill \Box

**Theorem 11.** $\gamma_{tgn}(T) = 5$ if and only if $T \notin \mathcal{T}_5 - \mathcal{T}_4$.

**Theorem 12.** If $G$ is a graph satisfying the following two conditions:
(i) each edge of $G$ lies on $C_3$ or $C_5$;
(ii) there is no path of length four between any pair non adjacent vertices in $G$, then

$$\frac{\gamma_t(G) + \gamma_t(G^N)}{2} \leq \gamma_{tgn}(G) \leq \gamma_t(G) + \gamma_t(G^N)$$

**Proof.** By the hypothesis, we have $G = G^{NN}$. Clearly $\gamma_t(G) \leq \gamma_{tgn}(G)$, $\gamma_t(G^N) \leq \gamma_{tgn}(G^N) = \gamma_{tgn}(G)$. Hence $\frac{\gamma_t(G) + \gamma_t(G^N)}{2} \leq \gamma_{tgn}(G)$. Clearly $\gamma_{tgn}(G) \leq \gamma_t(G) + \gamma_t(G^N)$. Thus the result follows. \hfill \Box

**Theorem 13.** Assume that $D$ is a $\gamma_t$-set of $G$. If there is a $v$ in $V - D$ adjacent to all the vertices in $D$, then $\gamma_{tgn}(G) \leq 1 + \gamma_t(G)$.

**Proof.** Clearly $D \cup \{v\}$ is a tgnd-set of $G$. Hence, the theorem follows. \hfill \Box

**Theorem 14.** If $G$ is a semi complete graph, then $D \subseteq V$ is a total dominating set of $G$ if and only if $D$ is a tgnd-set of $G$.

**Proof.** The proof follows from the fact that each edge in a semi complete graph lies on $C_3$. \hfill \Box

**Theorem 15.** If $G$ is a semi complete graph, then a set $D \subseteq V$ with $\delta_G(D) \geq 1$ is a global dominating set of $G$ if and only if $D$ is a tgnd-set of $G$.

**Proof.** The proof follows from the fact that, for a semi complete graph $G$, we have $G^c = G^N$. \hfill \Box

**References**


CONTACT INFORMATION

S. V. Siva Rama Raju

Academic Support Department, Abu Dhabi Polytechnic, Al Ain, United Arab Emirates; Department of Information Technology, Ibra college of Technology, Ibra, Sultanate of Oman
E-Mail(s): venkata.sagiraju@adpoly.ac.ae, shivram2006@yahoo.co.in

I. H. Nagaraja Rao

Laxmikantham Institute of Advanced Studies, G.V.P. College of Engineering, Visakhapatnam, India
E-Mail(s): ihnrao@yahoo.com

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