On the genus of the annihilator graph of a commutative ring*

T. Tamizh Chelvam and K. Selvakumar

Communicated by V. V. Kirichenko

Abstract. Let $R$ be a commutative ring and $Z(R)^*$ be its set of non-zero zero-divisors. The annihilator graph of a commutative ring $R$ is the simple undirected graph $AG(R)$ with vertices $Z(R)^*$, and two distinct vertices $x$ and $y$ are adjacent if and only if $\text{ann}(xy) \neq \text{ann}(x) \cup \text{ann}(y)$. The notion of annihilator graph has been introduced and studied by A. Badawi [7]. In this paper, we determine isomorphism classes of finite commutative rings with identity whose $AG(R)$ has genus less or equal to one.

1. Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past twenty years, leading to many fascinating results and questions. I. Beck[8] began the study of associating a graph called the zero-divisor graph $\Gamma_0(R)$ to a commutative ring $R$ and was mainly interested in the coloring of the zero-divisor graph. For a commutative ring $R$, the zero-divisor graph is the simple graph with $R$ as the vertex set and two distinct elements $x$ and $y$ are adjacent if and only if $xy = 0$ [8]. D. F. Anderson and P. S. Livingston[3] modified and

*This work was supported by the UGC-BSR One-time grant and UGC Major Research Project (F. No. 42-8/2013(SR)) of University Grants Commission, Government of India through first and second authors respectively.

2010 MSC: 05C99, 05C15, 13A99.

Key words and phrases: commutative ring, annihilator graph, genus, planar, local rings.
studied the zero-divisor graph $\Gamma(R)$ as the graph with vertex set as the nonzero zero-divisors $\mathcal{Z}(R)^*$ of $R$ and two distinct elements $x, y \in \mathcal{Z}^*(R)$ are adjacent if and only if $xy = 0$. Thereafter, several graphs have been associated with commutative rings. These graphs exhibit the interplay between the algebraic properties of $R$ and graph theoretical properties of the associated graph. For $a \in R$, let $\text{ann}(a) = \{d \in R : da = 0\}$ be the annihilator of $a \in R$. In 2014, A. Badawi [7] introduced the annihilator graph $\text{AG}(R)$ as the simple graph with vertex set $\mathcal{Z}(R)^*$, and two distinct vertices $x$ and $y$ are adjacent if and only if $\text{ann}(xy) \neq \text{ann}(x) \cup \text{ann}(y)$. One can see that the zero-divisor graph $\Gamma(R)$ is a subgraph of the annihilator graph $\text{AG}(R)$.

The main objective of topological graph theory is to embed a graph into a surface. Let $S_k$ denote the sphere with $k$ handles, where $k$ is a nonnegative integer, that is, $S_k$ is an oriented surface of genus $k$. The genus of a graph $G$, denoted $g(G)$, is the minimal integer $n$ such that the graph can be embedded in $S_n$. Intuitively, $G$ is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. A graph $G$ with genus 0 is called a planar graph, whereas a graph $G$ with genus 1 is called a toroidal graph. Further note that if $H$ is a subgraph of a graph $G$, then $g(H) \leq g(G)$. For details on the notion of embedding a graph in a surface, see [26].

Many research articles have appeared on the genus of zero divisor graphs of commutative rings. In particular, there are many papers [1, 2, 9, 15, 24], where the planarity of zero-divisor graphs has been discussed. The question addressed in these papers is this: For which finite commutative rings $R$ is $\Gamma(R)$ planar? A partial answer was given in [1], but the question remained open for local rings of order 32. In [15], and independently in both [9] and [24], it was shown that no local ring of order 32 has the planar zero divisor graph. Furthermore, Smith [15] gave a complete list of finite planar rings; this list included the 2 infinite families $\mathbb{Z}_2 \times F$ and $\mathbb{Z}_3 \times F$, where $F$ is any finite field, and the 42 other isomorphism classes of finite planar rings. H.J. Wang determined rings of the forms $\mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_r^{\alpha_r}}$ and $\mathbb{Z}_n[x]/(x^m)$ that have genus at most one [24, Theorems 3.5 and 3.11]. Further H.J. Wang and N.O. Smith obtained all commutative rings whose zero divisor graph has genus at most one [23, Theorem 3.6.2].

Note that the zero divisor graph $\Gamma(R)$ is a subgraph of $\text{AG}(R)$. In [7], it has been shown that for any reduced ring $R$ that is not an integral domain, $\text{AG}(R) = \Gamma(R)$ if and only if $R$ has exactly two distinct minimal prime ideals [7, Theorem 3.6]. Note that using the proof of this result,
one can establish that for any reduced ring, $\text{AG}(R)$ is complete if and only if, $\Gamma(R)$ is complete if and only if, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set $V$ and edge set $E$. A graph in which each pair of distinct vertices is joined by the edge is called a complete graph. We use $K_n$ to denote the complete graph with $n$ vertices. An $r$-partite graph is one whose vertex set can be partitioned into $r$ subsets so that no edge has both ends in any one subset. A complete $r$-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes $m$ and $n$ is denoted by $K_{m,n}$. If $G = K_{1,n}$ where $n \geq 1$, then $G$ is a star graph. $P_n$ denotes the path of length $n$ for $n \geq 1$. A graph $G$ is said to be unicyclic if it contains a unique cycle. Given a connected graph $G$, we say that a vertex $v$ of $G$ is a cut vertex if $G - v$ is disconnected. For a subset $S$ of vertices of $G$, the induced subgraph of $G$ is the subgraph with vertex set $S$ together with edges whose both ends are in $S$ and is denoted by $< S >$. A block is a maximal connected subgraph of $G$ having no cut vertices. A result of Battle, Harary, Kodama, and Youngs states that the genus of a graph is the sum of the genera of its blocks[6]. For example, the graph $G$ in Figure 1 has two blocks, both isomorphic to $K_{3,3}$, and so has genus 2 [25, C. Wickham].

Throughout this paper, we assume that $R$ is a finite commutative ring with identity, $Z(R)$ its set of zero-divisors and $\text{Nil}(R)$ its set of nilpotent elements, $R^\times$ its group of units, $F_q$ denotes the field with $q$ elements, and $R^* = R - \{0\}$. The following results are useful for further reference in this paper.
Theorem 1. [23, Theorem 3.5.1] Let \((R, m)\) be a finite local ring which is not a field. Then \(\Gamma(R)\) is planar if and only if \(R\) is isomorphic to one of the following 29 rings:

\[
\begin{align*}
\mathbb{Z}_4, & \quad \mathbb{Z}_2[x], \quad \mathbb{Z}_7[x], \quad \mathbb{Z}_9, \quad \mathbb{Z}_8, \quad \mathbb{Z}_2[x], \quad \mathbb{Z}_2[x, y], \\
\mathbb{Z}_4[x], & \quad \mathbb{Z}_4[x], \quad \mathbb{F}_4[x], \quad \mathbb{Z}_3[x], \quad \mathbb{Z}_4[x], \quad \mathbb{Z}_4[x, y], \\
\langle 2x, x^2 \rangle, & \quad \langle x^2 + x + 1 \rangle, \quad \langle x, y \rangle, \quad \langle x^2 + x + 1 \rangle, \quad \langle x^2 + x + 1 \rangle, \\
\langle x^2 - 2, x^4 \rangle, & \quad \langle x^2 - 2, x^4 \rangle, \quad \langle x^3 - 2, x^4 \rangle, \quad \langle x^3, x^2 - 2, x^4 \rangle, \\
\langle x^3 + 2, x^4 \rangle, & \quad \langle x^3 + 2, x^4 \rangle, \quad \langle x^3, x^2 - 2, x^4 \rangle, \quad \langle x^3, x^2 - 2, x^4 \rangle, \\
\langle x^3, y^2 - 2 \rangle, & \quad \langle x^3, y^2 - 2 \rangle, \quad \langle x^3, y^2 - 2 \rangle, \quad \langle x^3, y^2 - 2 \rangle, \\
\langle x^2 - 3, x^3 \rangle, & \quad \langle x^2 - 3, x^3 \rangle, \quad \langle x^2 - 3, x^3 \rangle, \quad \langle x^2 - 3, x^3 \rangle.
\end{align*}
\]

One can have the following theorem from Theorem 3.7 [15].

Theorem 2. [15, Theorem 3.7] Let \(R = F_1 \times \cdots \times F_n\) be a finite ring, where each \(F_i\) is a field and \(n \geq 2\). Then \(\Gamma(R)\) is planar if and only if \(R\) is isomorphic to one of the following rings:

\[
\mathbb{Z}_2 \times F, \quad \mathbb{Z}_3 \times F, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3,
\]

where \(F\) is a finite field.

Theorem 3. [23, Theorem 3.5.2] Let \((R, m)\) be a finite local ring which is not a field. Then \(g(\Gamma(R)) = 1\) if and only if \(R\) is isomorphic to one of the following 17 rings:

\[
\begin{align*}
\mathbb{Z}_{49}, & \quad \mathbb{Z}_9[x], \quad \mathbb{Z}_2[x], \quad \mathbb{Z}_2[x, y], \quad \mathbb{Z}_4[x], \quad \mathbb{Z}_4[x], \quad \mathbb{Z}_4[x], \\
\mathbb{Z}_8[x], & \quad \mathbb{F}_8[x], \quad \mathbb{Z}_4[x], \quad \mathbb{F}_8[x], \quad \mathbb{Z}_4[x], \quad \mathbb{Z}_4[x], \quad \mathbb{Z}_4[x], \\
\langle x^2, 2x \rangle, & \quad \langle x^3, x, y \rangle, \quad \langle x^3, y^2 \rangle, \quad \langle x^2, 2x \rangle, \quad \langle x^3, x^2 - 2, x^4 \rangle, \\
\langle x^3 + x + 1 \rangle, & \quad \langle x^3, x^2 - 2, x^4 \rangle, \quad \langle x^3, x^2 - 2, x^4 \rangle, \quad \langle x^3, x^2 - 2, x^4 \rangle, \\
\langle x^3 - 2, x^5 \rangle, & \quad \langle x^3 - 2, x^5 \rangle, \quad \langle x^3 - 2, x^5 \rangle, \quad \langle x^3 - 2, x^5 \rangle, \\
\langle x^2 - 2, x^5 \rangle, & \quad \langle x^2 - 2, x^5 \rangle, \quad \langle x^2 - 2, x^5 \rangle, \quad \langle x^2 - 2, x^5 \rangle.
\end{align*}
\]

Theorem 4. [11, 12, Theorem 6.3] Let \(G\) be a connected graph. Then \(G\) is a split graph if and only if \(G\) contains no induced subgraph isomorphic to \(2K_2, C_4, C_5\).
Theorem 5. [3, Lemma 2.12] Let $R$ be a finite commutative ring. If $\Gamma(R)$ has exactly one vertex adjacent to every other vertex and no other adjacent vertices, then either $R \cong \mathbb{Z}_2 \times F$, where $F$ is a finite field with $|F| \geq 3$, or $R$ is local with maximal ideal $m$ satisfying $\frac{R}{m} \cong \mathbb{Z}_2$, $m^3 = 0$ and $|m^2| \leq 2$. Thus $|\Gamma(R)|$ is either $p^n$ or $2^n - 1$ for some prime $p$ and integer $n \geq 1$.

Theorem 6. [3, Theorem 2.10] Let $R$ be a finite commutative ring. If $\Gamma(R)$ is complete, then either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R$ is local with char $R = p$ or $p^2$, and $|\Gamma(R)| = p^n - 1$, where $p$ is prime and $n \geq 1$.

Theorem 7. [3, Theorem 2.13] Let $R$ be a finite commutative ring with $|\Gamma(R)| \geq 4$. Then $\Gamma(R)$ is a star graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a finite field. In particular, if $\Gamma(R)$ is a star graph, then $\Gamma(R) = p^n$, for some prime $p$ and integer $n \geq 0$. Conversely, each star graph of order $p^n$ can be realized as $\Gamma(R)$.

Theorem 8. [7, Theorem 3.6] Let $R$ be a reduced commutative ring that is not an integral domain. Then $\text{AG}(R) = \Gamma(R)$ if and only if $R$ has exactly two distinct minimal prime ideals.

Theorem 9. [7, Theorem 3.10] Let $R$ be a nonreduced commutative ring with $|\text{Nil}(R)^*| \geq 2$ and let $\text{AG}_N(R)$ be the (induced) subgraph of $\text{AG}(R)$ with vertices $\text{Nil}(R)^*$. Then $\text{AG}_N(R)$ is complete.

2. Basic properties of annihilator graph

In this section, we state some basic observations of the annihilator graph. Especially, we identify the annihilator ideal graph of small order and in particular we list out all local rings $R$ with $|R| \leq 7$, for which the annihilator graph $\text{AG}(R)$ is complete.

Remark 1. Let $R = F_1 \times F_2$ where $F_1$ and $F_2$ are finite fields. Then $R$ is reduced with exactly two distinct minimal prime ideals. Hence, by Theorem 8, $\text{AG}(R) \cong \Gamma(R) = K_{|F_1^*|,|F_2^*|}$.

Remark 2. Let $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$ be a local ring that is not a field and $|Z(R)^*| \geq 3$. By Theorem 9, $\text{AG}(R)$ is complete and hence $\text{gr}(\text{AG}(R)) = 3$. On the other hand, let $R$ be a finite commutative ring with identity but not a field. Since $R$ is finite, $R \cong R_1 \times \cdots \times R_n$, where each $R_i$ is a local ring. If $n \geq 3$, then $(1,0,\ldots,0) - (0,1,0,\ldots,0) - (0,0,1,0,\ldots,0) - (1,0,\ldots,0)$ is a cycle in $\text{AG}(R)$ and hence $\text{gr}(\text{AG}(R)) = 3$. 
Remark 3. Note that $\Gamma(R)$ is a subgraph of $AG(R)$. D.F. Anderson et al., [2] gave all zero-divisor graphs of order $\leq 4$. Using this, we give in Table 1, all commutative local rings $R$ for which $|Z(R)^*| \leq 4$ and $AG(R)$ is complete. One can note that there are only two rings $\mathbb{Z}_2 \times \mathbb{F}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ with $|Z(R)^*| \leq 4$ which are not mentioned in Table 1 since $\Gamma(\mathbb{Z}_2 \times \mathbb{F}_4) = K_{1,3}$ and $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3) = K_{2,2}$ (refer Remark 1).

S. P. Redmond [13, 14] has given all local commutative rings $R$ with $|Z(R)^*| \leq 7$. Using the list given in [13, 14], Table 2 provides all finite commutative local rings, for which $6 \leq |Z(R)^*| \leq 7$ and $AG(R)$ is complete.

| $|Z(R)^*|$ | Local Ring $R$ | $AG(R)$ |
|-----------|----------------|----------|
| 6         | $\mathbb{Z}_{49}$, $\mathbb{Z}_7[x]/(x^2)$ | $K_6$    |
| 7         | $\mathbb{Z}_{16}$, $\mathbb{Z}_2[x]/(x^4)$, $\mathbb{Z}_4[x]/(x^2+2)$, $\mathbb{Z}_8[x]/(2x+2+x^2)$ | $K_7$    |
| 7         | $\mathbb{Z}_4[x]/(x^4-2,2x^2,2x)$, $\mathbb{Z}_2[x,y]/(x^2,x,y,2y)$, $\mathbb{Z}_8[x]/(2x,x,y)$ | $K_7$    |
| 7         | $\mathbb{Z}_4[x]/(x^4,2x^2,2x)$, $\mathbb{Z}_2[x,y]/(x^2-2,2x,y,2y)$, $\mathbb{Z}_4[x]/(x^2+2x)$ | $K_7$    |
| 7         | $\mathbb{Z}_4[x]/(x^2,xy,2x,y,x^2)$, $\mathbb{Z}_2[x,y]/(xy,x^2-2,2x,y)$, $\mathbb{Z}_4[x]/(x^2+2x)$ | $K_7$    |
| 7         | $\mathbb{Z}_4[x,y]/(x^4,y^2-xy)$, $\mathbb{Z}_2[x,y]/(x^2,y^2-xy,xy-2,2x,2y)$, $\mathbb{Z}_4[x]/(x^2+2x)$ | $K_7$    |
| 7         | $\mathbb{Z}_4[x,y]/(x^4,y^2-xy-2,2x,2y)$, $\mathbb{Z}_2[x,y]/(x^2,y^2)$, $\mathbb{Z}_4[x]/(x^2)$ | $K_7$    |
| 7         | $\mathbb{Z}_4[x,y,2]/(x,y,z)^2$, $\mathbb{Z}_2[x,y]/(x^4,y^2,2x,2y)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^3+x+1)$ | $K_7$    |
Theorem 10. Let $R$ be a finite commutative ring with identity that is not a field. Then $AG(R)$ is a tree if and only if $R$ is isomorphic to one of the following 5 rings:

$$Z_4, \frac{Z_2[x]}{(x^2)}, Z_9, \frac{Z_3[x]}{(x^2)} \text{ or } Z_2 \times F,$$

where $F$ is a finite field.

Proof. Since $R$ is finite, $R \cong R_1 \times \cdots \times R_n$, where each $R_i$ is a local ring. Suppose $AG(R)$ is a tree. In view Remark 2, $n \leq 2$.

Suppose $R \cong R_1 \times R_2$. If $Z(R_1)^* \neq \{0\}$, then there exist $x_1, y_1 \in Z(R_1)^*$ such that $x_1 y_1 = 0$ and $|R_1^*| \geq 2$. Let $x = (0, 1), y = (x_1, 0), z = (y_1, 1)$ and $w = (1, 0)$. Then $x, y, z, w \in Z(R)^*$ and $(x_1, 1) \in \text{ann}(zw)$ where as $(x_1, 1)$ is neither in $\text{ann}(z)$ nor in $\text{ann}(w)$. Hence $x - y - z - w - x$ is a cycle in $AG(R)$, a contradiction. Hence $R_1$ and $R_2$ are fields and so $AG(R) \cong K_{|R_1|-1,|R_2|-1}$. Since $AG(R)$ is tree, $|R_1| = 2$ or $|R_2| = 2$ and so $R \cong Z_2 \times F$, where $F$ is a finite field.

Suppose $R \cong R_1$. Since $R$ is not a field, $Z(R)^* \neq \{0\}$. Here $R = R_1$ is a local ring and so $AG(R)$ is complete. Hence by the assumption viz., $AG(R)$ is a tree, we get that $|Z(R)^*| \leq 2$. Hence $R \cong Z_4, \frac{Z_2[x]}{(x^2)}, Z_9$, or $\frac{Z_3[x]}{(x^2)}$.

The converse can be ascertained from Table 1 and Remark 1. $\square$

Theorem 11. Let $R$ be a finite commutative ring with identity that is not a field. Then $AG(R)$ is unicyclic if and only if $R$ is isomorphic to one of the following 8 rings:

$$Z_8, \frac{Z_2[x]}{(x^3)}, \frac{Z_4[x]}{(2x, x^2 - 2)}, \frac{\mathbb{F}_4[x]}{(x^2)}, \frac{Z_4[x]}{(x^2 + x + 1)}, \frac{Z_4[x]}{(x, 2)^2}, \frac{Z_2[x, y]}{(x, y)^2} \text{ or } Z_3 \times Z_3.$$

Proof. Sufficient part follows from Table 1 and Remark 1.

Conversely, assume that $AG(R)$ contains a unique cycle of length $\ell \geq 3$. Since $R$ is finite, $R \cong R_1 \times \cdots \times R_n$, where each $R_i$ is a local ring. Suppose $n \geq 3$. Let $x_1 = (1, 0, 0, \ldots, 0), x_2 = (0, 1, 0, \ldots, 0), x_3 = (0, 0, 1, 0, \ldots, 0), y_1 = (0, 1, 1, 0, \ldots, 0), y_2 = (1, 0, 1, 0, \ldots, 0)$. Then $x_1, x_2, x_3, y_1, y_2 \in Z(R)^*$ and $x_1 - x_2 - x_3 - x_1$ as well as $x_1 - y_1 - y_2 - x_2 - x_1$ are two distinct cycles in $AG(R)$, a contradiction. Hence $n \leq 2$.

Case 1. Suppose $n = 2$. If $Z(R_1) \neq \{0\}$, then there exist $x, y \in Z(R_1)^*$ such that $xy = 0$. Since $R_1$ is local, $R_1$ contains at least one unit $u_1 \in R_1^\times$.
apart from the identity. Then \((x, 0) - (y, 1) - (1, 0) - (0, 1) - (x, 0)\) and 
\((x, 0) - (y, 1) - (u_1, 0) - (0, 1) - (x, 0)\) are cycles in \(AG(R)\), a contradiction. 
Hence \(R_1\) and \(R_2\) are fields and so \(AG(R) \cong K_{|R_1|,|R_2|} - 1\). Since \(AG(R)\) is 
unicyclic, \(R_1 \cong \mathbb{Z}_3\) and \(R_2 \cong \mathbb{Z}_3\).

**Case 2.** Suppose \(n = 1\). Here \(R\) is a local ring but not a field. By 
Theorem 9, \(AG(R)\) is complete. Since \(AG(R)\) is unicyclic, \(|Z(R)\) = 3 \(\) 
and by Table 1, \(R\) is isomorphic to one of the following rings: \(\mathbb{Z}_8, \mathbb{Z}_2[x]_{(x^3)}, \mathbb{Z}_4[x]_{(2x^2)}, \mathbb{F}_4[x], \mathbb{Z}_4[x]_{(2x)}, \mathbb{Z}_4[x]_{(x^2 + x + 1)}, (x, 2)^2, (x, y)^2\).

Note that any complete graph is a split graph with any single vertex 
as an independent set and all the other vertices induce a clique. Hence, if \(R\) is a local ring, then \(AG(R)\) is a split graph. Now, we characterize all 
nonlocal rings \(R\) for which \(AG(R)\) is a split graph.

**Theorem 12.** Let \(R\) be a finite commutative nonlocal ring with identity 
and \(|Z(R)\) = 3 \(\). Then \(AG(R)\) is a split graph if and only if \(R \cong \mathbb{Z}_2 \times F\), 
where \(F\) is a finite field.

**Proof.** Suppose \(R = \mathbb{Z}_2 \times F\), where \(F\) is a finite field. By Remark 1, 
\(AG(R) = K_{1,|F|}\) and hence \(AG(R)\) is a split graph.

Conversely, suppose \(AG(R)\) is a split graph. Since \(R\) is finite, \(R \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2\) where each \(\mathbb{Z}_2\) is local for \(1 \leq i \leq n\) and \(n \geq 2\). If \(n \geq 3\), 
then \((1, 0, 0, \ldots, 0) - (0, 1, 0, \ldots, 0) - (1, 0, 1, 0, \ldots, 0) - (0, 1, 1, 0, \ldots, 0) - (1, 0, 0, \ldots, 0)\) is a cycle of length 4 in \(AG(R)\) and by Theorem 4, \(AG(R)\) is 
not split, a contradiction. Hence \(n = 2\).

If \(Z(R_1) \neq \{0\}\), then there exists an element \(x \in Z(R_1)^*\) such that 
\(xy = 0\) for some \(y \in Z(R_1)^*\) and so \((x, 0) - (0, 1) - (1, 0) - (y, 1) - (x, 0)\) 
is a cycle of length 4 in \(AG(R)\), a contradiction. Hence \(R_1, R_2\) are fields 
and so \(AG(R) \cong K_{|R_1|-1,|R_2|-1}\). Since \(AG(R)\) is split, either \(|R_1| = 1 = 1 \) 
or \(|R_2| = 1 = 1\) and so \(R \cong \mathbb{Z}_2 \times F\) where \(F\) is a finite field.

**Theorem 13.** Let \(R\) be a finite commutative ring with identity that is 
not a field. Then

(i) \(gr(AG(R)) = \infty\) if and only if \(R \cong \mathbb{Z}_4, \mathbb{Z}_2[x]_{(x^2)}, \mathbb{Z}_9, \mathbb{Z}_3[x]_{(x^2)}, \mathbb{Z}_2 \times F\), 
where \(F\) is a finite field;

(ii) \(gr(AG(R)) = 4\) if and only if \(R \cong \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2[x]_{(x^2)} \times \mathbb{Z}_2, \mathbb{F}_4 \times \mathbb{F}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2\), 
where \(F_1, F_2\) are finite fields with \(|F_1| \geq 3\) and \(|F_2| \geq 3\);

(iii) \(gr(AG(R)) = 3\) if and only if \(R\) is not isomorphic to the rings in 
(i) and (ii).

**Proof.** (i) Suppose \(gr(AG(R)) = \infty\). Then \(AG(R)\) contains no cycles and 
so \(AG(R)\) is a tree. Remaining part of the proof follows from Theorem 10.
(ii) Suppose \( \text{gr}(AG(R)) = 4 \). By Remark 2, \( R \) cannot be local. Also \(|Z(R)^*| \geq 4 \). Let \( R \cong R_1 \times \cdots \times R_n \) where each \( R_i \) is a local ring. If \( n \geq 3 \), by Remark 2, \( \text{gr}(AG(R)) = 3 \), a contradiction and hence \( n = 2 \).

Suppose \( R \) is not reduced. Then \(|Z(R_i)^*| \neq \{0\} \) for some \( i \). If \(|Z(R_i)^*| \geq 2 \), then there exist \( x, y \in Z(R_i)^* \) such that \( x \neq y \) and \( xy = 0 \). From this, we get that \((x,0) - (y,0) - (0,1) - (x,0)\) is a cycle of length 3 in \( AG(R) \), a contradiction. Thus \(|Z(R_i)^*| \leq 1 \) for \( i = 1, 2 \). If \(|Z(R_i)^*| = 1 \) for \( i = 1, 2 \), then \( R_i \cong Z_4 \) or \( \frac{Z_2[x]}{(x^2)} \). Suppose \( R \cong Z_4 \times Z_4 \), \( (2,0) - (0,1) - (2,2) \) is a cycle in \( AG(R) \). Note that all the remaining cases produce the same \( AG(R) \). Hence in all the cases \( \text{gr}(AG(R)) = 3 \), a contradiction. This shows that \(|Z(R_1)^*| = 1 \) or \(|Z(R_2)^*| = 1 \).

If \(|Z(R_1)^*| = 1 \) and \(|Z(R_2)^*| = 0 \), then \( R_1 \cong Z_4 \) or \( \frac{Z_2[x]}{(x^2)} \) and \( R_2 \) is a field. If \(|R_2| \geq 3 \), then \((2,0) - (2,1) - (2,x) - (2,0)\) for some \( 1 \neq x \in R_2^* \) is a cycle of length three in \( AG(R) \), a contradiction. Hence \(|R_2| = 2 \) and so \( R_2 \cong Z_2 \) and so \( R \cong Z_4 \times Z_2 \) or \( \frac{Z_2[x]}{(x^2)} \times Z_2 \).

Suppose \( R \) is reduced. Then \( R_1 \) and \( R_2 \) are fields and so \( AG(R) \cong K_{|R_1|-1, |R_2|-1} \). Since \( \text{gr}(AG(R)) = 4 \), \(|R_1| \geq 3 \) and \(|R_2| \geq 3 \).

Converse part of (ii) is trivial.

Part (iii) now follows directly from the above two cases. \( \Box \)

**Corollary 1.** Let \( R \) be a finite commutative ring with identity that is not a field. Then \( AG(R) \) is a complete bipartite graph if and only if \( R \) is isomorphic to one of the following rings:

\[
Z_9, \quad \frac{Z_3[x]}{\langle x^2 \rangle}, \quad Z_4 \times Z_2, \quad \frac{Z_2[x]}{\langle x^2 \rangle} \times Z_2 \text{ or } F_1 \times F_2,
\]

where \( F_1 \) and \( F_2 \) are finite fields.

**Proof.** We only need to prove the necessary part. Suppose \( AG(R) \) is a complete bipartite graph. Then \( AG(R) \) does not contain a cycle of odd length and so girth of \( AG(R) \) cannot be odd. By Theorem 13, \( \text{gr}(AG(R)) = 4 \) or \( \text{gr}(AG(R)) = \infty \). By the assumption that \( AG(R) \) is complete bipartite, \( AG(R) \not\cong K_1 \). Hence \( R \) is isomorphic to one of the following rings: \( Z_9, \frac{Z_3[x]}{\langle x^2 \rangle}, Z_4 \times Z_2, \frac{Z_2[x]}{\langle x^2 \rangle} \times Z_2, \text{ or } F_1 \times F_2 \), where \( F_1 \) and \( F_2 \) are fields. \( \Box \)

### 3. Planar annihilator graphs

In this section, we characterize finite commutative rings \( R \) for which \( AG(R) \) is planar. The following are known regarding the genus.
Lemma 1 ([26]). A connected graph $G$ is planar if and only if $G$ contains no subdivision of either $K_5$ or $K_{3,3}$.

Lemma 2 ([26]). Let $n$ be a positive integer and for real number $x$, $\lceil x \rceil$ is the least integer that is the greater than or equal to $x$. Then $g(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$ if $n \geq 3$. In particular, $g(K_n) = 1$ if $n = 5, 6, 7$.

Lemma 3 ([26]). Let $m, n$ be positive integers and for real number $x$, $\lceil x \rceil$ is the least integer that is the greater than or equal to $x$. Then $g(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{4} \rceil$ if $m, n \geq 2$. In particular, $g(K_{4,4}) = g(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$ and $g(K_{5,4}) = g(K_{6,4}) = g(K_{m,4}) = 2$ if $m = 7, 8, 9, 10$.

Lemma 4 ([26, Euler formula]). If $G$ is a finite connected graph with $n$ vertices, $m$ edges, and genus $g$, then $n - m + f = 2 - 2g$, where $f$ is the number of faces created when $G$ is minimally embedded on a surface of genus $g$.

Theorem 14. Let $(R, m)$ be a finite commutative local ring with identity. Then $AG(R)$ is planar if and only if $R$ is isomorphic to one of the following 13 rings:

\[
\begin{align*}
Z_4, & \quad \frac{Z_2[x]}{\langle x^2 \rangle}, \quad Z_9, \quad \frac{Z_3[x]}{\langle x^2 \rangle}, \quad Z_8, \quad \frac{Z_2[x]}{\langle x^3 \rangle}, \quad \frac{Z_4[x]}{\langle 2x, x^2 - 2 \rangle}, \quad \frac{F_4[x]}{\langle x^2 \rangle}, \\
& \quad \frac{Z_4[x]}{\langle x^2 + x + 1 \rangle}, \quad \frac{Z_4[x]}{\langle x, 2 \rangle^2}, \quad \frac{Z_2[x, y]}{\langle x, y \rangle^2}, \quad Z_{25} \quad \text{or} \quad \frac{Z_5[x]}{\langle x^2 \rangle}.
\end{align*}
\]

Proof. By Lemma 1, $AG(R)$ is planar if and only if $AG(R)$ contains no subdivision of either $K_5$ or $K_{3,3}$. Hence $\lvert Z(R)^* \rvert \leq 4$. Now proof follows from Table 1. \hfill \square

Theorem 15. Let $R = R_1 \times \cdots \times R_n$ be a finite commutative nonlocal ring, where each $R_i$ is a local ring and $n \geq 2$. Then $AG(R)$ is planar if and only if $R$ is isomorphic either of the following rings:

\[
Z_2 \times F, \quad Z_3 \times F, \quad Z_4 \times Z_2, \quad \frac{Z_2[x]}{\langle x^2 \rangle} \times Z_2 \quad \text{or} \quad Z_2 \times Z_2 \times Z_2,
\]

where $F$ is a finite field.

Proof. Note that $AG(Z_2 \times F) = K_{1, \lvert F^* \rvert}$ and $AG(Z_3 \times F) = K_{2, \lvert F^* \rvert}$ and so by Lemma 3, they are planar. Since $AG(Z_4 \times Z_2) = AG(\frac{Z_2[x]}{\langle x^2 \rangle} \times Z_2) \cong K_{2,3}$, they are also planar. As per the embedding given in Figure 2, $AG(Z_2 \times Z_2 \times Z_2)$ is planar.
Conversely assume that \( \text{AG}(R) \) is planar. Suppose \( n \geq 4 \). Let
\[
x_1 = (1, 0, 0, 0, \ldots, 0), \quad x_2 = (0, 1, 0, 0, \ldots, 0), \quad x_3 = (1, 1, 0, 0, \ldots, 0),
\]
\[
y_1 = (0, 0, 1, 0, \ldots, 0), \quad y_2 = (0, 0, 0, 1, 0, \ldots, 0), \quad y_3 = (0, 0, 1, 1, 0, \ldots, 0).
\]
Then \( \Omega = \{x_1, x_2, x_3, y_1, y_2, y_3\} \subseteq Z(R)^{\ast} \) and the subgraph of \( \text{AG}(R) \)
induced by \( \Omega \) contains \( K_{3,3} \) as a subgraph, a contradiction. Hence \( n \leq 3 \).

Case 1. \( n = 2 \). Suppose \( m_i \neq \{0\} \) for all \( i = 1, 2 \). Then \( |R_i| \geq 4 \) and
\( |R_i^{\ast}| \geq 2 \). Let \( a_i, b_i \) be two distinct elements in \( R_i^{\ast} \) other than identity for
\( i = 1, 2 \). Let \( d_1 = (1, 0), d_2 = (a_1, 0), d_3 = (b_1, 0), g_1 = (0, 1), g_2 = (0, a_2),
g_3 = (0, b_2) \in Z(R)^{\ast} \). Then \( \Omega_1 = \{d_1, d_2, d_3, g_1, g_2, g_3\} \subseteq Z(R)^{\ast} \) and
the subgraph of \( \text{AG}(R) \) induced by \( \Omega_1 \) contains \( K_{3,3} \) as a subgraph, a contradiction.
Hence \( m_i = \{0\} \) for some \( i \).

Without loss of generality, we assume that \( m_2 = \{0\} \). Then \( R_2 \) is a field.

Suppose \( m_1 \neq \{0\} \). We claim that \( |m_1^{\ast}| = 1 \). If not, \( |m_1^{\ast}| \geq 2 \) and
\( |R_1^{\ast}| \geq 3 \). Note that \( |R_2| \geq 2 \). For \( a, b \in m_1^{\ast} \) with \( ab = 0 \), and two distinct
units \( u_1, u_2 \in R_1^{\ast} \), let \( z_1 = (a, 1), z_2 = (b, 1), z_3 = (0, 1), w_1 = (1, 0),
w_2 = (u_1, 0), w_3 = (u_2, 0) \in Z(R)^{\ast} \). Then \( \Omega_2 = \{z_1, z_2, z_3, w_1, w_2, w_3\} \subseteq Z(R)^{\ast} \) with \( z_3 w_i = 0 \) in \( R \) for \( i = 1, 2, 3 \). Clearly \( z_2 \in \text{ann}(z_1 w_1), z_2 \notin \text{ann}(z_1) \cup \text{ann}(w_1), z_2 \notin \text{ann}(z_1) \cup \text{ann}(w_2) \) and so \( z_1 \) is
adjacent to both \( w_1 \) and \( w_2 \) in \( \text{AG}(R) \). Further \( z_1 \in \text{ann}(z_2 w_1), z_1 \notin \text{ann}(z_2) \cup \text{ann}(w_1), z_1 \notin \text{ann}(z_2) \cup \text{ann}(w_2) \) and so \( z_2 \) is
adjacent to both \( w_1 \) and \( w_2 \) in \( \text{AG}(R) \). From this, we observe that \( K_{3,3} \) is a subgraph of \( \text{AG}(R) \), a contradiction. Hence \( |m_1^{\ast}| = 1 \) and so \( R_1 \cong \mathbb{Z}_4 \) or
\( \frac{\mathbb{Z}_2[x]}{(x^2)} \).

Suppose \( |R_2| \geq 3 \). For \( d \in m_1^{\ast} \) with \( d^2 = 0 \) and \( 1 \neq v_1 \in R_1^{\ast} \) and
\( 1 \neq v_2 \in R_2^{\ast} \), let \( s_1 = (d, 1), s_2 = (d, v_2), s_3 = (0, 1), p_1 = (d, 0), p_2 =
\( (v_1, 0), p_3 = (1, 0). \) Then \( \Omega_3 = \{s_1, s_2, s_3, p_1, p_2, p_3\} \subseteq Z(R)^* \) and, \( p_1 s_1 = p_1 s_2 = 0 \) in \( R \) and \( s_3 p_1 = 0 \) in \( R \) for \( i = 1, 2, 3. \) Clearly \( s_2 \in \text{ann}(s_1 p_2), s_2 \notin \text{ann}(s_1) \cup \text{ann}(p_2), s_2 \in \text{ann}(s_1 p_3), s_2 \notin \text{ann}(s_1) \cup \text{ann}(p_3) \) and so \( s_1 \) is adjacent to both \( p_2 \) and \( p_3 \) in \( \text{AG}(R). \) Also \( s_1 \in \text{ann}(s_2 p_2), s_1 \notin \text{ann}(s_2) \cup \text{ann}(p_2), s_1 \in \text{ann}(s_2 p_3), s_1 \notin \text{ann}(s_2) \cup \text{ann}(p_3) \) and so \( s_2 \) is adjacent to both \( p_2 \) and \( p_3 \) in \( \text{AG}(R). \) From this, we get that \( K_{3,3} \) is a subgraph of \( \text{AG}(R), \) a contradiction. Hence \( |R_2| = 2 \) and so \( R_2 \cong \mathbb{Z}_2. \)

Suppose \( m_1 = \{0\}. \) Then \( R_1 \) is a field and by Remark 1, \( \text{AG}(R) \cong K_{|R_1|-1,|R_2|-1}. \) By Lemma 3, \( R \cong \mathbb{Z}_2 \times F \) or \( \mathbb{Z}_3 \times F, \) where \( F \) is a finite field.

![Figure 3. AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3).](image)

Case 2. \( n = 3. \) Suppose \( |R_i| \geq 4 \) for some \( i. \) Without loss of generality, we assume that \( |R_1| \geq 4. \) Let \( u_1, u_2, u_3 \) be three distinct nonzero elements in \( R_1. \) Let \( h_1 = (u_1, 0, 0), h_2 = (u_2, 0, 0), h_3 = (u_3, 0, 0), t_1 = (0, 1, 0), t_2 = (0, 0, 1), t_3 = (0, 1, 1) \in Z(R)^+. \) Then \( h_i t_j = 0 \) for all \( i, j = 1, 2, 3 \) and so \( K_{3,3} \) is a subgraph of \( \text{AG}(R), \) a contradiction. Hence \( |R_i| \leq 3 \) is field for \( i = 1, 2, 3 \) and so \( R_i \cong \mathbb{Z}_2 \) or \( \mathbb{Z}_3 \) for \( i = 1, 2, 3. \) Since \( \Gamma(R) \) is a subgraph of \( \text{AG}(R) \) and \( \text{AG}(R) \) is planar, by Theorem 2, the possibilities for \( R \) are \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) or \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3. \) Note that the edges in dark lines of Figure 3 constitute a subdivision of \( K_{3,3} \) and so by Lemma 1, \( \text{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3) \) is not planar. Thus \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \)

4. Genus of \( \text{AG}(R) \)

In this section, we characterize isomorphism classes of finite commutative rings \( R \) whose \( \text{AG}(R) \) has genus one. First, let us characterize finite commutative local rings \( R \) for which genus of \( \text{AG}(R) \) is one.

**Theorem 16.** Let \((R, \mathfrak{m})\) be a finite commutative local ring with identity that is not a field. Then \( g(\text{AG}(R)) = 1 \) if and only if \( R \) is isomorphic to
one of the following 22 rings:

\[
\begin{align*}
Z_4[x] &\langle x^4, x^3 + x^2 - 2 \rangle, & Z_2[x] &\langle x^4, x^2 - 2 \rangle, & Z_2[x] &\langle x^3 - 2, x^4 \rangle, \\
Z_2[x] &\langle x^4 \rangle, & Z_4[x] &\langle x^3, x^2 - 2x \rangle, & Z_4[x] &\langle x^3, xy, y^2 - x^2 \rangle, \\
Z_4[x] &\langle x^3, xy, x^2 - 2, y^2 - 2, y^3 \rangle, & Z_4[x] &\langle x^2, y^2, xy - 2 \rangle, & Z_4[x] &\langle x^2, y^2 \rangle, \\
Z_4[x, y] &\langle x^2, y^2, xy \rangle, & Z_4[x] &\langle x^3, x^2 - 2, xy, y^2 \rangle, & Z_8[x] &\langle x^2 \rangle, \\
Z_4[x, y] &\langle x^3 + x + 1 \rangle, & Z_2[x, y, z] &\langle x^2, 2y, x^2, y^2 \rangle, & F_5[x] &\langle x^2 \rangle.
\end{align*}
\]

**Proof.** By Theorem 9, \(AG(R)\) is complete. By Lemma 2, \(5 \leq |Z(R)^*| \leq 7\). Note that there is no local ring \(R\) with \(|Z(R)^*| = 5\). Now the proof follows from Table 2. \( \Box \)

**Remark 4.** Note that if \(R \cong R_1 \times \cdots \times R_n\) is a commutative ring with identity, where each \((R_i, m_i)\) is a local ring with \(m_i \neq \{0\}\) and \(n \geq 2\), then \(K_{5,6}\) is a subgraph of \(AG(R)\) and hence \(g(AG(R)) \geq 2\). Thus if \( g(AG(R)) = 1\), then one of the components \(R_i\) must be a field.

**Theorem 17.** Let \(R = R_1 \times \cdots \times R_n\) be a finite commutative nonlocal ring, where each \(R_i\) is a local ring and \(n \geq 2\). Then \(g(AG(R)) = 1\) if and only if \(R\) is isomorphic to one of the following 7 rings:

\[
F_4 \times F_4, \quad F_4 \times Z_5, \quad Z_5 \times Z_5, \quad F_4 \times Z_7, \quad Z_4 \times Z_3, \quad \frac{Z_2[x]}{\langle x^2 \rangle} \times Z_3 \quad \text{or} \quad Z_2 \times Z_2 \times Z_3.
\]

**Proof.** Assume that \(g(AG(R)) = 1\).

Suppose \(n \geq 4\). Let

\[
\begin{align*}
x_1 &= (1, 1, 0, 0, \ldots, 0), & x_2 &= (0, 1, 0, 1, \ldots, 1), & x_3 &= (0, 0, 1, 0, \ldots, 0), \\
x_4 &= (0, 1, 1, 0, \ldots, 0), & x_5 &= (1, 0, 1, \ldots, 1), & x_6 &= (1, 1, 0, 1, \ldots, 1), \\
x_7 &= (1, 1, 1, 0, \ldots, 0), & x_8 &= (1, 0, 1, 0, \ldots, 0), & x_9 &= (0, 0, 0, 1, \ldots, 1), \\
x_{10} &= (1, 0, 0, 1, \ldots, 1), & x_{11} &= (0, 1, 1, \ldots, 1).
\end{align*}
\]

Then \(\Omega = \{x_1, \ldots, x_{11}\} \subseteq Z(R)^*\) with \(x_1x_3 = x_2x_3 = x_3x_6 = x_1x_9 = x_7x_9 = x_8x_9 = 0\). Clearly \(x_5 \in \text{ann}(x_1x_4), x_5 \notin \text{ann}(x_1) \cup \text{ann}(x_4), \)
\(x_{11} \in \text{ann}(x_1x_5), x_{11} \notin \text{ann}(x_1) \cup \text{ann}(x_5), x_5 \in \text{ann}(x_2x_4), x_5 \notin \text{ann}(x_2) \cup \text{ann}(x_4)\),
ann(x_4), x_7 \in \text{ann}(x_2x_5), x_7 \notin \text{ann}(x_2) \cup \text{ann}(x_5), x_4 \in \text{ann}(x_5x_6), x_4 \notin \text{ann}(x_5) \cup \text{ann}(x_6), x_5 \in \text{ann}(x_6x_4), x_5 \notin \text{ann}(x_6) \cup \text{ann}(x_4), x_{11} \in \text{ann}(x_1x_{10}), x_{11} \notin \text{ann}(x_1) \cup \text{ann}(x_{10}), x_5 \in \text{ann}(x_1x_{11}), x_5 \notin \text{ann}(x_1) \cup \text{ann}(x_{11}), x_{11} \in \text{ann}(x_8x_{10}), x_{11} \notin \text{ann}(x_8) \cup \text{ann}(x_{10}), x_6 \in \text{ann}(x_{11}x_8), x_6 \notin \text{ann}(x_{11}) \cup \text{ann}(x_8), x_{10} \in \text{ann}(x_{11}x_7), x_{10} \notin \text{ann}(x_{11}) \cup \text{ann}(x_7), x_{11} \in \text{ann}(x_{10}x_7) \text{ and } x_{11} \notin \text{ann}(x_{10}) \cup \text{ann}(x_7). \text{ From all the above observations, Note that the subgraph induced by } \Omega \text{ in } AG(R) \text{ contains } G \text{ given in Figure 1 as a subgraph and so } g(AG(R)) \geq 2. \text{ Hence } n \leq 3.

Case 1. n = 3. Suppose R_1 \text{ and } R_2 \text{ are not fields. Then as mentioned in Remark 4, } K_{5,6} \text{ is a subgraph of } AG(R) \text{ and so } g(AG(R)) \geq 2, \text{ a contradiction. Hence at least two of the components } R_i, (1 \leq i \leq 3) \text{ must be fields. Without loss of generality, let us assume that } R_2 \text{ and } R_3 \text{ are fields. Then } |R_2| \geq 2 \text{ and } |R_3| \geq 2. \text{ Suppose } R_1 \text{ is not a field. Then } |m_1| \geq 2 \text{ and } |R_1^*| \geq 2. \text{ For } z \in m_1^* \text{ with } \text{ann}(z) = m_1 \text{ and } u \in R_1^* \text{ and } 1 \neq u, \text{ let } x_1 = (z, 1, 0), x_2 = (0, 1, 0), x_3 = (z, 0, 0), x_4 = (1, 0, 0), x_5 = (u, 0, 0), x_6 = (0, 0, 1), x_7 = (0, 1, 1), x_8 = (z, 1, 1), x_9 = (1, 0, 1), x_{10} = (z, 0, 1), x_{11} = (u, 0, 1) \text{ and } \Omega_1 = \{x_1, \ldots, x_{11}\}. \text{ Without much difficulty, one can check that the subgraph induced by } \Omega_1 \text{ in } AG(R) \text{ contains } G \text{ given in Figure 1 and so } g(AG(R)) \geq 2, \text{ a contradiction. Hence } R_1 \text{ must be a field. Since } AG(R) \text{ is non-planar, by Theorem 15, } R \neq Z_2 \times Z_2 \times Z_2.

Suppose at least two of the components } R_i(i \leq i \leq 3) \text{ contain at least three elements. Without loss of generality, assume that } |R_2| \geq 3 \text{ and } |R_3| \geq 3. \text{ For } 1 \neq a \in R_3^* \text{ and } 1 \neq b \in R_2^*, \text{ let } x_1 = (1, 1, 0), x_2 = (0, 1, 0), x_3 = (0, 0, 1), x_4 = (0, 0, a), x_5 = (1, 0, 1), x_6 = (0, 1, 0), x_7 = (1, b, 0), x_8 = (1, 0, 0), x_9 = (0, 1, a), x_{10} = (0, 1, 1), x_{11} = (0, b, 1) \text{ and } \Omega_2 = \{x_1, \ldots, x_{11}\}. \text{ Then the subgraph of } AG(R) \text{ induced by } \Omega_2 \text{ contains } G \text{ in Figure 1 and so } g(AG(R)) \geq 2, \text{ a contradiction. Hence two of the components } R_i \text{ must be } Z_2. \text{ Without loss of generality, we assume that } R_1 = R_2 = Z_2 \text{ and } |R_3| \geq 3.

Suppose } |R_3| \geq 4. \text{ Let } u \text{ and } v \text{ be two distinct non-zero elements in } R_3^* \text{ other than identity. Let } x_1 = (1, 0, 0), x_2 = (1, 0, u), x_3 = (0, 1, 1), x_4 = (0, 1, v), x_5 = (0, 1, u), x_6 = (1, 0, v), x_7 = (1, 1, 0), x_8 = (0, 1, 0), x_9 = (0, 0, 1), x_{10} = (1, 0, u), x_{11} = (1, 0, v) \text{ and } \Omega_3 = \{x_1, \ldots, x_{11}\}. \text{ Then the subgraph of } AG(R) \text{ induced by } \Omega_3 \text{ contains } G \text{ given in Figure 1 and so } g(AG(R)) \geq 2, \text{ a contradiction. Hence } R_3 = Z_3 \text{ and so } R \cong Z_2 \times Z_2 \times Z_3.

Case 2. n = 2. If } R_1 \text{ and } R_2 \text{ are not fields, then as mentioned in Remark 4, } K_{5,6} \text{ is a subgraph of } AG(R) \text{ and so } g(AG(R)) \geq 2, \text{ a contradiction. Hence one of the components } R_i \text{ must be a field. Without loss generality, we assume that } R_2 \text{ is a field and so } |R_2| \geq 2. \text{ By Theorem 15, } R \neq Z_4 \times Z_2 \text{ and } \frac{Z_4[x]}{(x^2)} \times Z_2.
We claim that, if $|m|^2 \geq 2$, then $g(\text{AG}(R)) \geq 2$.

Suppose $|m|^2 = 2$. By the facts given in Table 1, $R_1 \cong \mathbb{Z}_9$ or $\mathbb{Z}_2[x]/(x^3)$ and hence $|R_1^X| = 6$. Let $m^1 = \{a, b\}$, $R_1^X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $y_1 = (a, 1), y_2 = (b, 1), y_3 = (0, 1), x_1 = (a, 0), x_2 = (b, 0), x_3 = (u_1, 0, 0), x_4 = (u_2, 0), x_5 = (u_3, 0), x_6 = (u_4, 0), x_7 = (u_5, 0), x_8 = (u_6, 0)$. Then $\Omega_4 = \{y_1, y_2, y_3, x_1, \ldots, x_8\} \subseteq Z(R)^*$ and the subgraph of $\text{AG}(R)$ induced by $\Omega_4$ contains $K_{3,8}$. By Lemma 3, $g(\text{AG}(R)) \geq 2$, a contradiction.

Suppose $|m|^2 \geq 3$. Then $|R_1^X| \geq 4$. Let $d, e, f \in m^1$ such that $de = df = 0$ and $\{v_1, \ldots, v_4\} \subseteq R_1^X$. Consider $\Omega_5 = \{z_1, \ldots, z_7, w_1, \ldots, w_4\} \subseteq Z(R)^*$, where $z_1 = (d, 0), z_2 = (e, 0), z_3 = (f, 0), z_4 = (v_1, 0), z_5 = (v_2, 0), z_6 = (v_3, 0), z_7 = (v_4, 0), w_1 = (d, 1), w_2 = (e, 1), w_3 = (f, 1), w_4 = (0, 1)$. Then the subgraph induced by $\Omega_5$ of $\text{AG}(R)$ contains $K_{4,7}$ and by Lemma 3, $g(\text{AG}(R)) \geq 2$, a contradiction.

Hence we conclude that $|m| \leq 2$. From this $R_1 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ when $|m| = 2$ and $R_1$ must be a field when $|m| = 1$. By Theorem 15, $R_2 \not\cong \mathbb{Z}_2$ and so $|R_2| \geq 3$.

Suppose $|m| = 2$ and $|R_2| \geq 5$. For $a \in m^1$ with $a^2 = 0$ and $u_1, u_2 \in R_1^X$, $e_j \in R_2^*$, let $x_1 = (a, 0), x_2 = (u_1, 0), x_3 = (u_2, 0), x_4 = (a, e_1), x_5 = (a, e_2), x_6 = (a, e_3), x_7 = (a, e_4), x_8 = (0, e_1), x_9 = (0, e_2), x_{10} = (0, e_3), x_{11} = (0, e_4)$ and $\Omega_6 = \{x_1, \ldots, x_{11}\} \subseteq Z(R)^*$. Then the subgraph induced by $\Omega_6$ of $\text{AG}(R)$ contains $K_{3,8}$ and by Lemma 3, $g(\text{AG}(R)) \geq 2$, a contradiction. Hence $R_2$ is isomorphic to either $\mathbb{Z}_3$ or $\mathbb{F}_4$.

Consider the case that $R_2 \cong \mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$. One can observe from Figure 4 that $K_{3,6}$ is a subgraph of $\text{AG}(R)$. By Lemma 3, $g(K_{3,6}) = 1$ and hence one can fix an embedding of $K_{3,6}$ on the surface of torus. By Euler’s formula, there are 9 faces in the embedding of $K_{3,6}$, say $\{S_1, \ldots, S_9\}$. Let $n_i$ be the length of the face $S_i$. Note that $\sum_{i=1}^{9} n_i = 36$ and $n_i \geq 4$ for

\[ \text{Figure 4. } \text{AG}(\mathbb{Z}_4 \times \mathbb{F}_4) \cong \text{AG}\left(\mathbb{Z}_2[x]/(x^2) \times \mathbb{F}_4\right). \]
every $i$. Thus $n_i = 4$ for every $i$. Let $U = \{(2,1), (2, \omega), (2, \omega^2)\} \subset V(K_{3,6})$. Further, the subgraph $G$ of $AG(R)$ induced by the vertices in $U$ is $K_3$, $E(G) \cap E(K_{3,6}) = \emptyset$. Since $K_3$ cannot be embedded in the torus along with an embedding with only rectangles as faces, one cannot have an embedding of $G$ and $K_{3,6}$ together in the torus. This implies that $g(AG(R)) \geq 2$.

Hence $R_2 \not\cong \mathbb{F}_4$ and so $R$ is isomorphic to either $\mathbb{Z}_4 \times \mathbb{Z}_3$ or $\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_3$.

Suppose $|m_1| = 1$ and in this case both $R_1$ and $R_2$ are fields. Note that $AG(R) \cong K_{|R_1|-1, |R_2|-1}$. Since $g(AG(R)) = 1$ and by Lemma 3, $R$ is isomorphic to one of the following rings $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_5$ or $\mathbb{F}_4 \times \mathbb{Z}_7$.

Converse follows from Remark 1, Lemma 3, Figure 5 and Figure 6. □

**Figure 5.** Embedding of $AG(\mathbb{Z}_4 \times \mathbb{Z}_3) \cong AG(\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_3)$ in $S_1$.

**Figure 6.** Embedding of $AG(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ in $S_1$. 
Acknowledgments

The authors thank the anonymous referees for their comments which improved the presentation of the paper in many places.

References


**Contact Information**

T. Tamizh Chelvam, Department of Mathematics  
K. Selvakumar, Manonmaniam Sundaranar University  
Tirunelveli 627 012, Tamil Nadu, India  
*E-Mail(s):* tamche59@gmail.com, selva_158@yahoo.co.in

Received by the editors: 06.10.2015  
and in final form 17.07.2016.