On colouring integers avoiding \( t \)-AP distance-sets

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Abstract. A \( t \)-AP is a sequence of the form \( a, a + d, \ldots, a + (t-1)d \), where \( a, d \in \mathbb{Z} \). Given a finite set \( X \) and positive integers \( d, t, a_1, a_2, \ldots, a_{t-1} \), define \( \nu(X, d) = \left| \left\{ (x, y) : x, y \in X, y > x, y - x = d \right\} \right| \), \( (a_1, a_2, \ldots, a_{t-1}; d) = \) a collection \( X \) s.t. \( \nu(X, d \cdot i) \geq a_i \) for \( 1 \leq i \leq t - 1 \).

In this paper, we investigate the structure of sets with bounded number of pairs with certain gaps. Let \( (t - 1, t - 2, \ldots, 1; d) \) be called a \( t \)-AP distance-set of size at least \( t \). A \( k \)-colouring of integers \( 1, 2, \ldots, n \) is a mapping \( \{1, 2, \ldots, n\} \to \{0, 1, \ldots, k - 1\} \) where \( 0, 1, \ldots, k - 1 \) are colours. Let \( ww(k, t) \) denote the smallest positive integer \( n \) such that every \( k \)-colouring of \( 1, 2, \ldots, n \) contains a monochromatic \( t \)-AP distance-set for some \( d > 0 \). We conjecture that \( ww(2, t) \geq t^2 \) and prove the lower bound for most cases. We also generalize the notion of \( ww(k, t) \) and prove several lower bounds.

1. Introduction

A \( t \)-AP is a sequence of the form \( a, a + d, \ldots, a + (t - 1)d \), where \( a, d \in \mathbb{Z} \). For example, \( 3, 7, 11, 15 \) is a 4-AP with \( a = 3 \) and \( d = 4 \).

Given a finite set \( X \) and positive integers \( d, t, a_1, a_2, \ldots, a_{t-1} \), define

\[
\nu(X, d) = \left| \left\{ (x, y) : x, y \in X, y > x, y - x = d \right\} \right|, \\
(a_1, a_2, \ldots, a_{t-1}; d) = \text{a collection } X \text{ s.t. } \nu(X, d \cdot i) \geq a_i \text{ for } 1 \leq i \leq t - 1.
\]

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The $t$-AP $\{x, x + d, \ldots, x + (t - 1)d\}$ (say $T$) has $\nu(T, d \cdot i) = t - i$ for $1 \leq i \leq t - 1$. On the other hand, a set $(t - 1, t - 2, \ldots, 1; d)$ (say $Y$) has $\nu(Y, d \cdot i) \geq t - i$ for $1 \leq i \leq t - 1$, but not necessarily contains a $t$-AP.

A $k$-colouring of integers $1, 2, \ldots, n$ is a mapping $\{1, 2, \ldots, n\} \rightarrow \{0, 1, \ldots, k - 1\}$ where $0, 1, \ldots, k - 1$ are colours. Let $ww(k, t)$ denote the smallest integer $n$ such that every $k$-colouring of $1, 2, \ldots, n$ contains a monochromatic set $(t - 1, t - 2, \ldots, 1; d)$ for some $d > 0$. Here, $(t - 1, t - 2, \ldots, 1; d)$ is a $t$-AP distance-set of size at least $t$. The existence of $ww(k, t)$ is guaranteed by van der Waerden's theorem [1]. Given positive integers $k$, $t$, and $n$, a good $k$-colouring of $1, 2, \ldots, n$ contains no monochromatic $t$-AP distance set. We call such a good $k$-colouring, a certificate of the lower bound $ww(k, t) > n$. We write a certificate as a sequence of $n$ colours each in $\{0, 1, \ldots, k - 1\}$, where the $i$-th ($i \in \{1, 2, \ldots, n\}$) colour corresponds to the colour of the integer $i$.

A certificate of lower bound $ww(k, t) > n$ that avoids a monochromatic arithmetic progression, may still be invalid, since it may contain a monochromatic distance set. For example, while looking for a certificate of lower bound of $ww(2, 4)$, if the set $X = \{1, 2, 3, 5, 9, 10\}$ (which does not contain a 4-AP) is monochromatic, then the colouring is “bad” as $\nu(X, 1) = 3$, $\nu(X, 2) = 2$, and $\nu(X, 3) = 1$.

In this paper, we perform computer experiments to observe the patterns of certificates of $ww(k, t) > n$. We conjecture that $ww(2, t) \geq t^2$ and prove the lower bound for most cases. We also generalize the notion of $ww(k, t)$ and provide several lower bounds.

2. Some values and bounds

With a primitive computer search algorithm, we have computed the following values and bounds of $ww(k, t)$. Theorem 1 gives a lower bound for $ww(k, t)$. A computed lower bound is presented only if it improves over the bound given by Theorem 1.

**Theorem 1.** Given $k \geq 2, t \geq 3$, if $t \leq 2k + 1$, then

$$ww(k, t) > k(t - 1)(t - 2).$$

**Proof.** Let $n = k(t - 1)(t - 2)$ and consider the colouring

$$f : \{1, 2, \ldots, n\} \rightarrow \{0, 1, \ldots, k - 1\}.$$

Let $X_i = \{x \in X : f(x) = i\}$. Take the certificate

$$(0^{t-1}1^{t-1} \cdots (k - 1)^{t-1})^{t-2}.$$
We show that for each $d$, there exists $j$ with $1 \leq j \leq t - 1$ such that $\nu(X_i, d \cdot j) < t - j$ for each $i \in \{0, 1, \ldots, k - 1\}$. The largest difference between two monochromatic numbers in the certificate is

$$n - (k - 1)(t - 1) - 1 = (t - 1)(k(t - 2) - k - 1) - 1 < k(t - 1)(t - 3).$$

Since the existence of a monochromatic set $(t - 1, t - 2, \ldots, 1; d) \in X$ requires $\nu(X_i, d \cdot (t - 1)) \geq 1$, we have $d < k(t - 3)$. We have the following cases:

(a) $1 \leq d \leq k - 1$: Take $x, y \in \{1, 2, \ldots, n\}$ such that $y = x + d(t - 1)$. But by our choice of $d$, we have $f(y) = (f(x) + d) \mod k \neq f(x)$, that is, $x$ and $y$ cannot be monochromatic. So, $\nu(X_i, (t - 1) \cdot d) = 0 < 1$ for each $i \in \{0, 1, \ldots, k - 1\}$.

(b) $k - 1 \leq d \leq t - 3$: Take $x, y \in \{1, 2, \ldots, n\}$ such that $y = x + d(t - a)$ where $a$ is such that $(t - 3)(t - a) \leq (t - 1)(k - 1)$ and $k(t - a) \geq k$, which

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For Lemma 1.

Conjecture 1. For $t \geq 3$, $ww(2, t) \geq t^2$.

Lemma 1. For $t \geq 3$ and $t \neq 2^u$ with $u \geq 2$, $ww(2, t) \geq t^2$.

Proof. Let $t = 2^u + v$ with $1 \leq v \leq 2^u - 1$. Let $n = t^2 - 1$ and $X = \{1, 2, \ldots, n\}$, and consider the colouring $f : X \rightarrow \{0, 1\}$. Let $m = n - 1 - (t - 1) = q \cdot 2^u + r$ with $0 \leq r \leq 2^u - 1$.

Now, take the certificate

$$
\begin{cases}
01^{t-1}(0^21^2)^{q/2}0^r, & \text{if } q \equiv 0 \pmod{2}; \\
01^{t-1}(0^21^2)^{|q/2|}0^21^r, & \text{if } q \equiv 1 \pmod{2}.
\end{cases}
$$

This gives us a bound

$$
t - \frac{(t-1)(k-1)}{t-3} \leq a \leq t - k.
$$

Such an $a$ exists since

$$
\frac{(t-1)(k-1)}{t-3} = k - 1 + \frac{2(k-1)}{t-3} \geq k - 1 + \frac{2(k-1)}{2k+1} - 3 \geq k.
$$

(c) $d = t - 2$: In each block of $t - 1$ colours, there is a pair of integers at distance $t - 2$, and there are $t - 2$ such blocks for each colour. So, $\nu(X, 1 \cdot d) = \nu(X, t - 2) = t - 2 < t - 1$ for each $i \in \{0, 1, \ldots, k - 1\}$.

(d) $(t - 1) \leq d \leq (k - 1)(t - 1)$: Take $x, y \in \{1, 2, \ldots, n\}$ such that $y = x + d = x + q(t - 1) + r$ where $1 \leq q \leq k - 1$ and $0 \leq r \leq t - 2$. Suppose $x = q_x(t - 1) + r_x$ with $0 \leq r_x \leq t - 2$. Then

$$
f(x) = q_x \mod k;
$$

$$
f(y) = (f(x) + q + [(r + r_x)/(t - 1)]) \mod k.
$$

If $r > 0$, then $q \leq k - 2$, which implies $q + [(r + r_x)/(t - 1)] \leq (k - 2) + 1 = k - 1$. If $r = 0$, then $q \leq k - 1$, which implies $q + [(0 + r_x)/(t - 1)] \leq (k - 1) + 0 = k - 1$. Therefore, $f(y) \neq f(x)$; and $x$ and $y$ cannot be monochromatic. So, $\nu(X, d \cdot 1) = 0 < t - 1$ for each $i \in \{0, 1, \ldots, k - 1\}$.

Since $t \leq 2k + 1$, we have

$$
d < k(t - 3) = kt - 3k = (kt - k - 2k) \leq (kt - k - (t - 1)) = (k - 1)(t - 1).
$$

Hence, we are done and there is no monochromatic $t$-AP distance set in $X$. \qed
We need to show that for each \( d \), there exists \( j \) with \( 1 \leq j \leq t-1 \) such that \( \nu(X, d \cdot j) < t-j \). Since the existence of a monochromatic set \((t-1, t-2, \ldots, 1; d)\) in \( X \) requires \( \nu(X, d \cdot (t-1)) \geq 1 \), we have \( d(t-1) < t^2 - 1 \), that is, \( 1 \leq d \leq t \). Let \( X_i = \{ x \in X : f(x) = i \} \).

Suppose \( q \equiv 0 \pmod{2} \). Then we have the following two cases:

(a) \( d \equiv 1 \pmod{2} \): Take \( x, y \in X \) such that \( y = x + d \cdot 2^u \).

(a1) \( v + 1 \leq x < y \leq n - r \): Since \( f(x) \in \{0, 1\} \) and \( f(x + 1 \cdot 2^u) = (f(x) + 1) \pmod{2} \neq f(x) \), we have \( f(y) = f(x + d \cdot 2^u) \neq f(x) \). So, two monochromatic integers cannot both be in \( \{ v + 1, v + 2, \ldots, n - r \} \).

(a2) \( 2 \leq x \leq v \) and \( y \leq n - r \): Since \( f(x) = 1 \) and \( f(x + 1 \cdot 2^u) = 1 = f(x) \), we have \( f(y) = (x + d \cdot 2^u) = f(x) \). So, there are exactly \( v - 1 \) pairs of integers with colour 1 at distance \( d \cdot 2^u \).

(a3) \( x = 1 \) and \( y \leq n - r \): Since \( f(x) = 0 \), \( f(x + 1 \cdot 2^u) = 1 \neq f(x) \), and \( 1 + 2^u > v \), using case (i) we have 0 pair of integers with colour 0 at distance \( d \cdot 2^u \).

(a4) \( x \geq 1 \) and \( n - r + 1 \leq y \leq n \): Since \( f(y) = 0 \) and \( r < 2^u \), we have \( f(y - 1 \cdot 2^u) = 1 \), which implies \( f(y - d \cdot 2^u) = 1 \neq f(y) \). That is, adding \( r \) trailing zeros does not change the number of monochromatic pairs at distance \( d \cdot 2^u \).

Therefore, for each \( i \in \{0, 1\} \), we have \( \nu(X_i, d \cdot 2^u) \leq v - 1 < v = t - 2^u \).

(b) \( d \equiv 0 \pmod{2} \): Let \( d = 2^w \cdot d_o \), with \( d_o \) being an odd number and \( w \geq 1 \). Then \( \nu(X_i, d \cdot 2^{u-w}) = \nu(X_i, d_o \cdot 2^u) \leq v - 1 < v = t - 2^u \) (by case (i)) for each \( i \in \{0, 1\} \).

The case \( q \equiv 1 \pmod{2} \) is similar.

\[ \square \]

**Lemma 2.** Suppose \( t = 2^u \) for some \( u \geq 2 \) and \( t-1 \) is prime. Then for each \( r \in \{1, 2, \ldots, u\} \) and for each \( d_o \in \{1, 3, \ldots, 2^{u-r} - 1\} \), there exists \( s \in \{1, 2, 3, \ldots, d_o - 1\} \) such that \( d_o \) divides \( s(t-1) - 2^{u-r} \).

**Proof.** Since \( t-1 \) is prime, we have \( \gcd(t-1, d_o) = 1 \), and hence the linear congruence \((t-1)s \equiv 2^{u-r} \pmod{d_o}\) has a solution. Extended Euclid Algorithm yields \( x, y \in \mathbb{Z} \) such that \((t-1) \cdot x + d_o \cdot y = 1\). Then \( s = (x \cdot 2^{u-r}) \mod d_o \). Since \( d_o \mid x \) and \( d_o \mid 2^{u-r} \), we have \( s \neq 0 \). Hence \( s \in \{1, 2, \ldots, d_o - 1\} \).

\[ \square \]

**Lemma 3.** If \( t = 2^u \) for \( u \geq 2 \) with \( t-1 \) prime, then \( \omega(2, t) \geq t^2 \).

**Proof.** Take the certificate \( 0(1^{t-1}0^{t-1})^{t/2}1^{t-2} \). We have the following two cases:

(a) \( d \equiv 1 \pmod{2} \): Take \( x, y \in X \) such that \( y = x + d \cdot (t-1) \).
(a1) $2 \leq x \leq t^2 - t + 1 = n - (t - 2)$: Since $f(x) \in \{0, 1\}$ and $f(x + 1 \cdot (t - 1)) = (f(x) + 1) \mod 2 \neq f(x)$, we have $f(y) = f(x + d \cdot (t - 1)) \neq f(x)$. So, two monochromatic integers cannot both be in $\{2, 3, \ldots, n - (t - 2)\}$.

(a2) $x = 1$ and $y \leq n - (t - 2)$: Since $f(x) = 0$, $f(x + 1 \cdot (t - 1)) = 1 \neq f(x)$, using case (i) we have 0 pair of integers with colour 0 at distance $d \cdot (t - 1)$.

(a3) $x \geq 1$ and $n - (t - 2) + 1 \leq y \leq n$: Since $f(y) = 0$, we have $f(y - 1 \cdot (t - 1)) = 1$, which implies $f(y - d \cdot (t - 1)) = 1 \neq f(y)$. That is, adding $t - 2$ trailing ones does not change the number of monochromatic pairs at distance $d \cdot (t - 1)$.

Therefore, for each $i \in \{0, 1\}$, we have $\nu(X_i, d \cdot (t - 1)) = 0 < 1$.

(b) $d \equiv 0 \pmod{2}$: For $d \in \{2, 4, \ldots, t\}$ and $j \in \{1, 2, \ldots, t - 1\}$,

$$2 \leq d \cdot j \leq t(t - 1) = t^2 - t.$$

For a given $d \in \{2, 4, \ldots, t\}$, we show that there exists $(j, w)$ with $j \in \{1, 2, \ldots, t-1\}$ and $w \in \{1, 3, 5, \ldots, t-1\}$ such that $d \cdot j = w(t-1) - 1$. In that case, since $d \cdot j \leq t(t-1)$, we have $w \leq t$; and also since $d \cdot j$ is even, $w(t-1)$ is odd, which implies $w$ is odd, that is, $w \in \{1, 3, 5, \ldots, t-1\}$.

Let $d = 2^r d_o$ with $1 \leq r \leq u$ and odd number $d_o \in \{1, 3, \ldots, 2^{u-r}-1\}$ (since $d \leq t = 2^u$). For a $w$ to exist and satisfy $d \cdot j = w(t-1) - 1$, we need

$$2^r d_o \cdot j = w(t-1) - 1 = wt - (w + 1),$$

that is, $2^r$ divides $w + 1$ (since $2^r$ divides $wt = w2^u$). Let $w = s \cdot 2^r - 1$ with $s \in \{1, 2, \ldots, d_o - 1\}$. The chosen $s$ requires to satisfy that $d_o$ divides $(w(t-1) - 1)/2^r = s(t-1) - 2^{u-r}$. By Lemma 2, such an $s$ exists.

It can be observed that for a given $d \in \{2, 4, \ldots, t\}$, if $d \cdot j_1 = w_1 \cdot (t - 1) - 1$ for some $j_1 \in \{1, 2, 3, \ldots, t-1\}$ and $w_1 \in \{1, 3, 5, \ldots, t-1\}$, then

$$d \cdot j_2 = w_2 \cdot (t - 1) + 1,$$

with $j_1 + j_2 = t - 1$ and $w_1 + w_2 = d$. We claim that $\nu(X_1, d \cdot t_i) < t - j_i$ for at least one $i \in \{1, 2\}$. If $\nu(X_1, d \cdot j_1) < t - j_1$, then we are done. Suppose

$$\nu(X_1, d \cdot j_1) = \nu(X_1, w_1 (t - 1) - 1) = t - \frac{w_1 - 1}{2} \geq t - j_1,$$

which implies $t/2 + (w_1 - 1)/2 \leq j_1$. Now,

$$\nu(X_1, d \cdot j_2) = \nu(X_1, w_2 (t - 1) + 1) = \frac{t}{2} - \frac{w_2 - 1}{2} = \frac{t}{2} - \frac{d - w_1 - 1}{2} = \frac{t}{2} + \frac{w_1 - 1}{2} + \frac{d}{2} - j_1 + 1 - \frac{d}{2} = t - j_2 + 1 - \frac{d}{2} < t - j_2.$$
Similarly, we can show that \( \nu(X_0, d \cdot j) < t - j \) for some \( j \in \{1, 2, \ldots, t - 1\} \).

\[ \square \]

3. Generalized distance-sets

Here we consider variants of \( \text{ww}(k, t) \) with different variations of parameters in a distance set.

Let \( \text{gww}(k, t; a_1, a_2, \ldots, a_{t-1}) \) (with \( a_i \geq 1 \)) denote the smallest positive integer \( n \) such that any \( k \)-colouring of \( 1, 2, \ldots, n \) contains monochromatic set \( (a_1, a_2, \ldots, a_{t-1}; d) \) for some \( d > 0 \). In this definition,

\[ \text{gww}(k, t; t - 1, t - 2, \ldots, 1) = \text{ww}(k, t). \]

Observation 1. Let us write \( \text{gww}(2, t; a_1, a_2, \ldots, a_{t-1}) \) as \( \text{gww}(2, t, r) \), where \( a_i = r \) for \( 1 \leq i \leq t - 1 \). It is trivial that \( \text{gww}(2, t, t - 1) \geq \text{ww}(2, t) \).

Table 2 contains a few computed values of \( \text{gww}(2, t, r) \).

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Lemma 4. For \( u \geq 1 \) and \( 1 \leq v \leq 2^u \),

\[ \text{gww}(2, 2^u + v, 1) \geq (2^u + v - 1)2^{u+1} + 1. \]

Proof. Consider \( t = 2^u + v \) (\( t \geq 5 \)) and let \( n = (2^u + v - 1)2^{u+1} = (t-1)2^{u+1} \) and \( X = \{1, 2, \ldots, n\} \). Consider the colouring \( f : X \to \{0, 1\} \) and take the certificate \((0^{2^u}1^{2^u})^{t-1}\).
Let \( X_i = \{ x \in X : f(x) = i \} \). We claim that this 2-colouring of \( X \) does not contain a monochromatic set \((1,1,\ldots,1;d)\) for any \( d > 0 \), that is, for each \( d \) with \( 1 \leq d < 2^{u+1} \) and for each \( i \in \{0,1\} \), there exists \( j \in \{1,2,\ldots,t-1\} \) such that \( \nu(X_i,d \cdot j) = 0 \).

(a) \( d \equiv 1 \ (\text{mod } 2) \): Take \( x, y \in X \) such that \( y = x + d \cdot 2^u \). Since \( d \) is odd, if \( f(x) = 0 \), then \( f(x + d \cdot 2^u) = 1 \) and vice-versa. Hence, \( \nu(X_i,d \cdot 2^u) = 0 \) for each \( i \in \{0,1\} \).

(b) \( d \equiv 0 \ (\text{mod } 2) \): Let \( d = 2^w \cdot d_o \), with \( d_o \) being an odd number and \( w \geq 1 \). Then for each \( i \in \{0,1\} \),

\[
\nu(X_i,d \cdot 2^{u-w}) = \nu(X_i,d_o \cdot 2^u) = 0 \quad \text{(by case (a))}.
\]

So, \( X \) does not contain a monochromatic set \((1,1,\ldots,1;d)\) for any \( d > 0 \).

\[\square\]

**Conjecture 2.** For \( u \geq 1 \) and \( 1 \leq v \leq 2^u \),

\[
gww(2,2^u + v,1) = (2^u + v - 1)2^{u+1} + 1.
\]

**Lemma 5.** For \( u \geq 2 \) and \( 1 \leq v \leq 2^u \),

\[
gww(2,2^u + v,2) \geq (2^u + v - 1)2^{u+1} + 5.
\]

**Proof.** Let \( t = 2^u + v \ (t \geq 5) \), \( n = (2^u + v - 1)2^{u+1} + 4 = (t-1)2^{u+1} + 4 \), and \( X = \{1,2,\ldots,n\} \). Consider the colouring \( f : X \to \{0,1\} \) and take the certificate \( 000(10^{2u-3}11012^{u-3}00)t^{-2}(10^{2u-3}11)(01^{2u-3})011 \). We show that this colouring of \( X \) does not contain a monochromatic set \((2,2,\ldots,2;d)\) for any \( d > 0 \), that is, for each \( d \) with \( 1 \leq d < 2^{u+1} \) and for each \( i \in \{0,1\} \), there exists \( j \in \{1,2,\ldots,t-1\} \) such that \( \nu(X_i,d \cdot j) \leq 1 \).

Note that the largest difference between two integers with colour 0 in the colouring is \((n - 2) - 1 = n - 3 = (t-1) \cdot 2^{u+1} + 1 = p \) (say); and the largest difference between two integers with colour 1 in the colouring is \( n - 4 = (t-1) \cdot 2^{u+1} = p - 1 \).

(a) \( d = 2^{u+1} \): Note that \( d \cdot (t-1) = (t-1)2^{u+1} = n - 4 = p - 1 \). The only pair \((x,y)\) with \( f(x) = f(y) = 0 \) and \( y = x + d \cdot (t-1) \) is \((1,n-3)\) and the only pair \((x,y)\) with \( f(x) = f(y) = 1 \) and \( y = x + d \cdot (t-1) \) is \((4,n)\). Hence, we have \( \nu(X_i,d \cdot (t-1)) = 1 \) for each \( i \in \{0,1\} \).

(b) \( d \equiv 1 \ (\text{mod } 2) \): Write the certificate as

\[
000A_0A_1 \ldots A_{2t-5}, A_{2t-4}C11,
\]

where \( A_i = 10^{2u-3}11 \) if \( i \equiv 0 \ (\text{mod } 2) \), \( A_i = 01^{2u-3}00 \) if \( i \equiv 1 \ (\text{mod } 2) \), and \( C = 01^{2u-3}0 \). Take \( x, y \in \{4,5,\ldots,n-2^u-2\} \) such that \( y = x + d \cdot 2^u \).
for some odd $d < 2^{u+1} + 1$. Suppose $x = 3 + i \cdot 2^u + j$, that is, $f(x)$ is the $j$-th ($1 \leq j \leq 2^u$) bit in $A_i$. Then $y = 3 + (i + d) \cdot 2^u + j$, that is, $f(y)$ is the $j$-th bit in $A_{i+d}$. If $i \equiv 1 \pmod{2}$, then $(i + d) \equiv 0 \pmod{2}$ (since $d$ is odd), and vice-versa. Therefore, $f(x) \neq f(y)$. So, two monochromatic integers at distance $d \cdot 2^u$ cannot both be in $\{4, 5, \ldots, n - 2^u - 2\}$.

Now, take $x, y \in \{4, 5, \ldots, n - 2^u - 2\}$ such that $y = x + d \cdot 2^u$ and $y \in \{n - 2^u - 1, n - 2^u, \ldots, n - 2\}$. Since $|C| < 2^u$, $x$ must be in $\{4, 5, \ldots, n - 2^u - 2\}$. With similar reasoning as above, it can be shown that two monochromatic integers at distance $d \cdot 2^u$ cannot both be in $\{4, 5, \ldots, n - 2^u - 2\}$. Following are the remaining cases:

(b1) If $x = 1$, then $x + d \cdot 2^u = 3 + d \cdot 2^u - 2 = 3 + (d - 1) \cdot 2^u + (2^u - 2) = y$ (say). We have $f(x) = 0$. Again $(2^u - 2)$-th bit in $A_{d-1}$ is also zero since $d$ is odd and $A_{d-1} = 10^{2^u - 3}11$.

(b2) If $x = 2, 3$ (where $f(x) = 0$), then with similar reasoning as above, $f(x + d \cdot 2^u) = 1$.

(b3) If $y = n - 1$ (where $f(y) = 1$), then

$$y - d \cdot 2^u = (t - 1) 2^{u+1} + 3 - d \cdot 2^u$$
$$= 3 + (2t - 3 - d) 2^u + 2^u = x \text{ (say)}.$$ Since $d$ is odd, $2t - 3 - d$ is even, that is, $A_{2t-3-d} = 10^{2^u - 3}11$. The $2^u$-th element in $A_{2t-3-d}$ is one, that is $f(x) = 1$.

(b4) If $y = n$ (where $f(y) = 1$), then

$$y - d \cdot 2^u = (t - 1) 2^{u+1} + 4 - d \cdot 2^u$$
$$= 3 + (2t - 2 - d) 2^u + 1 = x \text{ (say)}.$$ Since $d$ is odd, $2t - 2 - d$ is odd, that is, $A_{2t-2-d} = 01^{2^u - 3}00$. The 1-st element in $A_{2t-2-d}$ is zero, that is $f(x) = 0$.

Hence $\nu(X_i, d \cdot 2^u) \leq 1$ for each $i \in \{0, 1\}$.

(c) Otherwise ($d \neq 2^{u+1}$ and $d \equiv 0 \pmod{2}$): Let $d = 2^w \cdot d_o$, with $d_o$ being an odd number and $w \geq 1$. Then for each $i \in \{0, 1\}$,

$$\nu(X_i, d \cdot 2^{u-w}) = \nu(X_i, d_o \cdot 2^u) \leq 1 \text{ (by case (b))}.$$ Therefore, $X$ does not contain a monochromatic set $(2, 2, \ldots, 2; d)$ for any $d > 0$.

**Conjecture 3.** For $r \geq 2$ and $t \geq 2^r + 1$,

$$gw(t, r) = (t - 1) 2^{\lceil \log_2 (t-1) \rceil + 1} + 2^r + 1.$$
Observation 2. We observe the following experimental results:
(a) Primitive search gives $gww(2, 10, 9) > 186$;
(b) Using the certificate
\[
\begin{cases}
01^{t+1}(0^{t-1}1^{t-1})^{t/2+1}, & \text{if } t \equiv 0 \pmod{2}; \\
01^{t+1}(0^{t-1}1^{t-1})^{[t/2]+1}0^{t-1}, & \text{if } t \equiv 1 \pmod{2},
\end{cases}
\]
we obtain the following lower bounds with $12 \leq t \leq 48$:
\[
\begin{align*}
gww(2, 12, 11) > 168, & \quad gww(2, 14, 13) > 224, \quad gww(2, 17, 16) > 323, \\
gww(2, 18, 17) > 360, & \quad gww(2, 20, 19) > 440, \quad gww(2, 24, 23) > 624, \\
gww(2, 29, 29) > 960, & \quad gww(2, 32, 31) > 1088, \quad gww(2, 33, 32) > 1155, \\
gww(2, 38, 37) > 1520, & \quad gww(2, 42, 41) > 1848, \quad gww(2, 44, 43) > 2024.
\end{align*}
\]
(c) $0^{35}(1^{18}0^{18})21^{20}0^{17}(1^{19}0^{18})51^{19}0^{17}1^{20}0^{10}14$ proves $gww(2, 19, 18) > 399$.
(d) $0^{41}1^{21}0^{21}(1^{22}0^{21})10^{15}$ proves $gww(2, 22, 21) > 528$.

Conjecture 4. For $t \geq 4$, $gww(2, t, t-1) \geq (t+1)^2$.

Conjecture 5. For $t \geq 5$, $gww(2, t, t-1) < 2^{t+1}$.

We do not have enough data to make stronger upper bound conjecture for $gww(2, t, t-1)$, but it may be possible that $gww(2, t, t-1) < t^3$.

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References


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