Associative words in the symmetric group
of degree three

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Abstract. Let $G$ be a group. An element $w(x, y)$ of the absolutely free group on free generators $x, y$ is called an associative word in $G$ if the equality $w(w(g_1, g_2), g_3) = w(g_1, w(g_2, g_3))$ holds for all $g_1, g_2 \in G$. In this paper we determine all associative words in the symmetric group on three letters.

1. Introduction

Let $G$ be a group and let $F = F(x, y)$ be the absolutely free group on free generators $x, y$. Let $V = V(G)$ be the subgroup of $F$ consisting of all words $w$ such that $w(g_1, g_2) = 1$ for all $g_1, g_2 \in G$. An element $w \in F$ is said to be associative in $G$ if the equality

$$w(w(g_1, g_2), g_3) = w(g_1, w(g_2, g_3))$$

holds for all elements $g_1, g_2, g_3 \in G$. The words $1, x, y, xy$ and $yx$ are, of course, associative (trivial words) for any group. It is known that in the absolutely free group ([6,7]) and in the class of all abelian groups ([4]) there are no other associative words. In other groups $(G; \cdot)$ such nontrivial word $w$ can exist, however. Moreover, in some free nilpotent groups there are nontrivial associative words $w(x, y) = x \circ y$ such that $(G; \circ)$ is a group and the group operation $x \cdot y$ can be expressed as a

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word in the group \((G;\circ)\): see [1,2,5,9]. In this paper we are looking for the associative words in the symmetric group of degree three which is metabelian but not nilpotent. We show that each associative word in \(S_3\) is equivalent \(modulo\) \(V(S_3)\) to one of the five words mentioned above or to one of \(x^3, y^3, x^4, y^4, [x, y]^{x+y}, [y, x]^{x+y}\).

2. Preliminaries

We use standard notations:

\[ x^{-1}yx = yx, \quad [y, x] = y^{-1}x^{-1}yx, \quad [y, x]^{-1} = [x, y] \quad x^\beta y = (x^y)^\beta, \]

\[ x^{\alpha + \beta}y^{z+0} = x^{\alpha}(x^\beta y)x^z \]

for arbitrary group elements \(x, y, z\) and all integers \(\alpha, \beta\).

Let us recall the following simple facts about the identities in \(S_3\).

(i) The relations

\[ [xy, z] = [x, z]^y[y, z], \quad [x, yz] = [x, z][x, y]^z \]

are identities in any group.

(ii) The commutator subgroup \(S'_3\) of \(S_3\) consists of all even permutations and the square of each element from \(S_3\) is in \(S'_3\).

This yields

(iii) For all products \(C\) of commutators the equality

\[ C^{(1-x)(1+x)} = 1 \]

is an identity in \(S_3\).

(iv) The equalities

\[ x^6 = [y, x]^3 = 1, \quad [[y, x], [u, v]] = 1, \quad [x^2, [y, z]] = 1, \]

\[ [y^2, x] = [y, x]^{y+1}, \quad [y^3, x] = [y, x]^{y-1}, \quad [y^4, x] = [y, x]^{-y-1} \]

are identities in the group \(S_3\).

From (ii) and (iv) one can derived

(v) The equality

\[ [y, x]^{xy} = [y, x]^{-1-x-y} \]

is an identity in \(S_3\).
The following consequence of Corollary 2 in [8] plays very important role in our considerations.

(vi) If for some \( A, B, C \in \mathbb{Z}_3 \) the equality

\[
[y, x]^{A+Bx+Cy} = 1,
\]
holds for all \( x, y \in S_3 \), then \( A = B = C = 0 \).

**Proposition 2.1.** Any 2-word in \( S_3 \) is equivalent \( (\text{mod } V) \) to some word of the form

\[
w(x, y) = x^\alpha y^\beta [y, x]^{A+Bx+Cy}
\]  

(2.1)

where \( \alpha, \beta \in \mathbb{Z}_6 \) and \( A, B, C \in \mathbb{Z}_3 \).

**Proof.** It is enough to apply Hall’s classical collection process from [3]. \( \square \)

**Proposition 2.2.** The word

\[
w(x, y) = x^\alpha y^\beta [y, x]^{A+Bx+Cy}
\]

is associative in \( S_3 \), then \( \alpha, \beta \in \{0, 1, 3, 4\} \).

**Proof.** By putting \( y = z = 1 \) and \( x = y = 1 \) into (2.2) we get

\[
x^{\alpha^2} = x^\alpha, \text{ and } x^{\beta^2} = x^\beta
\]  

(2.2)

and therefore \( \alpha(\alpha - 1) \equiv \beta(\beta - 1) \equiv 0 \) (mod 6). \( \square \)

**Proposition 2.3.** If \( w(x, y) \) is associative in a group \( G \), then the word \( u(x, y) = w(y, x) \) is also associative in \( G \).

**Proof.** We have

\[
u(u(x, y), z) = w(z, w(y, x)) = w(w(z, y), x) = u(x, u(y, z)).
\]  

\( \square \)

3. Associative words

First of all we show that for some pairs \( (\alpha, \beta) \) no word of the form

(2.1)

is associative in \( S_3 \). It what follows we shall always assume that \( A, B, C \in \mathbb{Z}_3 \) and sometimes we write \( \gamma(s, t) \) instead of \( A + Bs + Ct \).

**Theorem 3.1.** There are no associative words in the group \( S_3 \) which are of the form

\[
x^\alpha y^\beta [y, x]^{A+Bx+Cy},
\]  

(3.1)

where \( (\alpha, \beta) \) is one of the following pairs

\((1, 3), (3, 1), (1, 4), (4, 1), (3, 4), (4, 3), (3, 3)\)
Proof. Case $\alpha = 1, \beta = 3$.
Let us begin with an auxiliary result

\[
\begin{align*}
\omega(x, y)^3 &= x^3 y^3[x, x]^\gamma x y^3[y, x]^\gamma x y^3[y, x]^\gamma \\
&= x^3 y^3 x^2[y, x]^\gamma y^3[y, x]^\gamma [y, x]^\gamma x y^3[y, x]^\gamma \\
&= x^3 y^3 x^2[y, x]^\gamma [y, x]^\gamma y^3[x, x]^\gamma x y^3[y, x]^\gamma \\
&= x^3 x^3[y, x]^{x+1}[y, x]^\gamma [y, x]^\gamma (y-1) y^2[y, x] x y^3[y, x]^\gamma \\
&= x^3 x^3[y, x]^{x+1}[y, x]^\gamma [y, x]^\gamma (y-1) y^2[y, x] x y^3[y, x]^\gamma.
\end{align*}
\]

We have thus established

\[
(x y^3[y, x]^\gamma)^3 = x^3 y^3[y, x]^{-1+x-y+(1-x-y)\gamma}.
\] (3.2)

Further we have

\[
\begin{align*}
L &= w(w(1, y), z) = w(y^3, z) = y^3 z^3[z, y^3]^\gamma[y, z] = y^3 z^3[z, y]^\gamma (y-1) \gamma[y, z], \\
R &= w(1, w(y, z)) = w(y, z)^3 = (y z^3[z, y]^\gamma[y, z])^3 \\
&= y z^3[y, z^3] y z^3[z, y]^\gamma[y z-1] \gamma[y, z] = y z^3[z, y]^{-1-z+y+(y-1) \gamma[y, z]}.
\end{align*}
\]

Thus $L = R$ is equivalent to the equality

\[
[z, y]^{-1-A-B+C} + (1+A-C)y + (-1-A+B+C)z = 1.
\]

By (vi) we have $-1-A-B+C = 0, 1+A-C = 0$ and $-1-A+B+C = 0$, which has the solution $B = 0, C = A + 1$. Therefore the associative word (3.1) has to be of the form $w(x, y) = x y^3[y, x] C^{-1} + C y$. Let us put $z = x$ into the associative low (1.1). We get

\[
\begin{align*}
L &= w(w(x, y), x) = x^3 y^3[y, x]^\gamma(x, y) w(x, y) x^3[y, x]^\gamma(x, y) x^3[y, x]^\gamma(x, y) x^3[y, x]^\gamma(x, y), \\
x^4 y^3[y, x]^{x+y}[y, x]^{x y(x, y)} [y, x]^{(1-x+y) \gamma(x, y)} [y, x]^{(1-x) \gamma(x, y)} x^{(1-x) \gamma(x, y)},
\end{align*}
\]

which in the case $B = 0, A = C - 1$ gives

\[
\begin{align*}
L &= x^4 y^3[y, x]^{x+y} x (C-1+C y) + (1-x+y)(C-1+C x) + (1-x)((C+C x)-1)(C-1+C y) \\
x^4 y^3[y, x]^{x+y} x (C-1+C y) + (1-x+y)(C-1+C x) + (x-1)(C-1+C y).
\end{align*}
\]

After some calculations we get

\[
L = x^4 y^3[y, x]^{C+C(x+1)} y.
\]
Similarly we have

\[ R = w(x, w(y, x)) = xw(y, x)^3[yx^3[x, y]]^\gamma(y,x), x]^\gamma(x,xy) \]
\[ = xw(y, x)^3[yx^3, x]^\gamma(x,xy)[x, y]^\gamma(y,x), x]^\gamma(x,xy) \]
\[ = xw(y, x)^3[y, x]x^\gamma(x,xy)[y, x]^{(1-x)}\gamma(y,x)^\gamma(x,xy), x] \]

which in the case \( B = 0 \) and \( A = C - 1 \) implies

\[ R = xw(y, x)^3[y, x]^{x(C-1+Cxy)[y, x]^{(1-x)}(C+Cx-1)(C+1+Cxy)} \]
\[ = xw(y, x)^3[y, x]^{-C+Cx+(1+C)y}. \]

Now from the equality \( L = R \) we obtain

\[ w(x, y)^3 = x^3y^3[y, x]^{-x}. \]

We get a contradiction, because formula (3.2) for \( \gamma = C - 1 + Cy \) gives

\[ w(x, y)^3 = x^3y^3[y, x]^{C+1-x+(C+1)y}. \]

Case \( \alpha = 1, \beta = 4. \)

We have

\[ L = w(w(1, y), z) = w(y^4, z) = y^4z^4[y, y]^\gamma(1,z) = y^4z^4[y, y]^{-y+1}\gamma(1,z). \]

Similarly

\[ R = w(1, w(y, z)) = (y^4z^4[y, y]^\gamma(y,z)yz^4[z, y]^\gamma(y,z))^2 \]
\[ = (y^4z^4[y, y]^\gamma(y,z)yz^4[z, y]^\gamma(y,z))^2 = y^4z^4[z, y]^{-y+1}\gamma(1,z). \]

Hence \( L = R \) yields \([z, y]^{1+z} = 1 \) which, by (vi), is not an identity in \( S_3 \).

Case \( \alpha = 3, \beta = 4. \)

We have

\[ L = w(w(1, y), z) = w(y^4, z) = z^4[y, y]^\gamma(1,z) = z^4[y, y]^{-y+1}\gamma(1,z). \]

Since \( y^2 \) commutes both \( z^4 \) and \([z, y]^\gamma(y,z), \) we can use of the previous case. We obtain

\[ R = w(1, w(y, z)) = w(y, z)^4 = (y^2y^4[z, y]^\gamma(y,z)y^2yz^4[z, y]^\gamma(y,z))^2 \]
\[ = y^2(y^4z^4[z, y]^\gamma(y,z)y^4z^4[z, y]^\gamma(y,z))^2 = z^4[z, y]^{-(z+1)-y+1}\gamma(y,z). \]
Thus the condition $L = R$ yields the equality

$$[z, y]^{z+1} = 1,$$

which is not an identity in $S_3$.

Case $\alpha = 3, \beta = 3$. We have

$$L = w(w(x, 1), z) = x^3 z^3[z, x^3]^{\gamma(x, z)} = x^3 z^3[z, x]^{(x-1)(A+Bx+Cz)},$$

$$R = w(x, w(1, z)) = x^3 z^3[z^3, x]^{\gamma(x, z)} = x^3 z^3[z, x]^{(z-1)(A+Bx+Cz)}.$$

Thus the equality $L = R$ implies, in view of (vi),

$$C - B \equiv A - C + B \equiv B - A - C \equiv 0 \pmod{3}$$

which yields $A = 0$ and $B - C = 0$. So every word of the form $w(x, y) = x^3 y^3 [x, y]^{B(x+y)}$ satisfies the equation $w(w(x, 1), z) = w(x, w(1, z))$ but none of them is associative. Indeed, for such words we have

$$L = w(w(1, y), z) = w(y^3, z) = y^3 z^3[z^3, y^3]^{B(y+z)} = y^3 z^3[z, y]^{B(y-1)(z-1)(y+z)} = y^3 z^3,$$

$$R = w(1, w(y, z)) = w(y, z)^3 = y^3 z^3[y^3, z, y]^{B(y+z)} y^3 z^3[z, y]^{B(y+z)} = y^3 z^3[z, y]^{B(y+z)},$$

$$= y^3 z^3[z, y]^{B(y+z)} [z, y]^{(z-1)(y-1)} [z, y]^{-B(y+z)} = y^3 z^3[z, y]^{(B+1)(y+z)}.$$

Thus $L = R$ implies the equation $[z, y]^{y+z} = 1$, which is not an identity in $S_3$.

Now by Proposition 2.3 we know that if the word $w(x, y)$ of the form (2.1) is associative in $S_3$, then $w(y, x)$ is also associative in $S_3$. Since

$$w(y, x) = y^i x^j [x, y]^{A+Bx+Cz} = x^j y^i [y, x]^{A'+B'x+C'y}$$

for some $A', B', C' \in \mathbb{Z}_3$ the proof of Theorem 3.1 is complete. \hfill $\Box$

In the following lemmas we consider the cases of pairs $(\alpha, \beta)$ for which there exist associative words in $S_3$.

**Lemma 3.2.** The word

$$w(x, y) = x[y, x]^{A+Bx+Cy} = x[y, x]^{\gamma(x, y)} \quad (3.3)$$

is associative in $S_3$ if and only if $A = B = C = 0$. 

Proof. Using the identities (ii), (iv) and (v) we have

\[ w(x, w(y, y)) = x[y, x]^{-1}(x, y), \]
\[ w(x, w(y, y)) = x[y, x]^{-1}(x, y), \]
\[ = x[y, x]^{-1}(x, y) \]
\[ \times x^{-1}(y, x) \gamma(x, y) \gamma(x, y). \]

Taking into account (iii) we see that if \( w \) is associative, then

\[ [y, x]^{-1}(A + Bx + Cy + (1 - y)(A - C + Bx)^2) = 1, \]

which, by (vi), ensures the following system of congruences

\[
\begin{cases}
A + (A - C)^2 + B^2 + 2(A - C)B & \equiv 0 \pmod{3}, \\
B + 2(A - C)B + 2(A - C)B & \equiv 0 \pmod{3}, \\
C - (A - C)^2 - B^2 + 2(A - C)B & \equiv 0 \pmod{3}.
\end{cases}
\]

The solution of the system are four triples \((A, B, C)\) of the form \((0, 0, 0), (2, 2, 0), (2, 0, 1)\) and \((0, 1, 1)\). In order to exclude the last three cases we put \( y = x \) into (3.3). Then we get

\[ L = w(w(x, x), z) = x[z, x]^{-1}(x, z) \]
\[ R = w(x, w(x, z)) = x[z, x]^{-1}(x, z) \gamma(x, x) \]
\[ = x[z, x]^{-1}(x, z) \gamma(x, x). \]

Thus the condition \( L = R \) together with (iii) gives the equality

\[ [z, x]^{-1}(A - B - C)(A + Bx + Cz) = [z, x]^{-1}(A + Bx + Cz). \]

The equality is, by (vi), an identity in \( S_3 \) if and only if the triples \((A, B, C)\) satisfies the following system of congruences

\[
\begin{cases}
(A - B - C)(B - A - C) & \equiv A \pmod{3}, \\
(A - B - C)(A - B - C) & \equiv B \pmod{3}, \\
(A - B - C)(B - A - C) & \equiv C \pmod{3}.
\end{cases}
\]

The proof of the lemma is complete, because none of the triples \((2, 2, 0), (2, 0, 1)\) and \((0, 1, 1)\) do satisfy the system.

By Proposition 2.3 we have also
Corollary 3.3. The word

\[ y[y, x]^{A + Bx + Cy} \]

satisfies the associativity low if and only if \( A = B = C = 0 \).

Lemma 3.4. The word

\[ w(x, y) = x y[y, x]^{A + Bx + Cy} \]

is associative in \( S_3 \) if and only if \( B = C = A = 0 \) or \( A - 1 = B = C = 0 \).

Proof. We have

\[ w(w(x, y), z) = x y[y, x]^{\gamma(x,y)}z[z, x y[y, x]^{\gamma(x,y)}]^{\gamma(x,y)} \]

\[ = x y z[y, x]^{\gamma(x,y)+(z-1)\gamma(x,y)\gamma(xy,z)}[z, x]^{\gamma(xy,z)}[z, y]^{\gamma(xy,z)} \]

and

\[ w(x, w(y, z)) = x y z[y, x]^{\gamma(x,y)}[y z[z, y]^{\gamma(y,z)}]^{\gamma(x,yz)} \]

\[ = x y z[y, x]^{\gamma(x,yz)}[z, x]^{\gamma(xy,z)}[z, y]^{\gamma(y,z)+(x-1)\gamma(y,z)\gamma(xy,z)} \]

Hence we get

\[ (w(x, w(y, z)))^{-1} w(w(x, y), z) \quad (3.4) \]

\[ = [y, x]^{(1-z}\{C y + \gamma(x,y)\gamma(xy,z)\}[z, x]^{(1-y)}\{A\}[z, y]^{(1-z)}\{B y + \gamma(y,z)\gamma(xy,z)\} \]

By putting \( z = y \) into (3.4) we obtain

\[ (w(x, w(y, y)))^{-1} w(w(x, y), y) = \]

\[ [y, x]^{(1-y}\{A - C y + (A + B x + C y)(A + B x y + C y)\}, \]

which in view of (iii) and (v) can be rewritten as

\[ [y, x]^{(1-y}\{(A - C)^2 - (A - C) - B^2\}. \]

Now we put \( y = x \) into (3.4). This gives

\[ w(x, w(x, z))^{-1} w(w(x, x), z) = \]

\[ [z, x]^{(1-x}\{(A - B)^2 - (A - B) - C^2\}} \]
In view of (vi) if the word \( w(x, y) \) is associative in \( S_3 \), then the following system of congruences
\[
\begin{align*}
(A - C)^2 - (A - C) - B^2 & \equiv 0 \pmod{3}, \\
(A - B)^2 - (A - B) - C^2 & \equiv 0 \pmod{3}.
\end{align*}
\]
has to satisfy. The solution of the system is \( B = C = 0 \) and \( A = 0 \) or \( A = 1 \). Since the words \( xy \) and \( yx \) are associative, Lemma 3.4 follows.

Lemma 3.5. The 2-word
\[
w(x, y) = x^3[y, x]^{A+Bx+Cy}
\] (3.5)
is associative in \( S_3 \) if and only if \( A = B = C = 0 \).

Proof. Clearly, the word \( x^3 \) is associative in the group \( S_3 \). We have
\[
R = w(x, w(1, z)) = x^3,
\]
\[
L = w(w(x, 1)z) = w(x^3, z) = x^3[z, x]^{\gamma(x,z)} = x^3[z, x]^{(x-1)(A+Bx+Cz)}
\]
\[
[z, x]^{(-A+B-C)+(A-B-C)x+Cz}.
\]
So the equality \( R = L \) is equivalent to the conditions \( C = 0 \) and \( A = B \). Further we have
\[
w(w(x, x), z) = x^3[z, x^3]^{A+Bx+Cz} = x^3[z, x]^{(x-1)(A+Bx+Cz)},
\]
\[
w(x, w(x, z)) = x^3[x^3[z, x]^{A+Bx+Cz}, x]^{A+Bx+Cx}
\]
\[
= x^3[z, x]^{(A-B-C)(x-1)(A-B+Cz)}.
\]
Hence the equality \( w(w(x, x), z) = w(x, w(x, z)) \) after using (v) and (vi), yields the system of equalities
\[
\begin{align*}
(A - B - C)(B - A - C) & \equiv 2A + B - C \pmod{3}, \\
(A - B - C)^2 & \equiv A - B - C \pmod{3}, \\
C(A - B - C) & \equiv C \pmod{3}.
\end{align*}
\]
The system has four solutions for \((A, B, C)\): (0,0,0),(1,0,0),(1,1,0) and (2,2,0). We check that the last three triple do not produce associative words of the form \( w(x, y) = x^3[y, x]^\gamma(x, y) \). To do this let us calculate
\[
w(x, y)^3 = x^3[y, x]^{\gamma(x,y)}(x^3[y, x]^{\gamma(x,y)}x^3[y, x]^{\gamma(x,y)})
\]
\[
= x^3[y, x]^{\gamma(x,y)}[y, x]^{x\gamma(x,y)}[y, x]^{\gamma(x,y)} = x^3[y, x]^{(x-1)\gamma(x,y)}
\]
Taking this into account we get

\[
L(A, B, C) = w(w(x, y), y) = w(x, y)^3[y, w(x, y)] = x^3[y, x]^{(x-1)\gamma(x,y) + (1-y)\gamma(x,y)},
\]

\[
R(A, B, C) = w((x, w(y, y))) = x^3[y^3, x]^{\gamma(x,y)} = x^3[y, x]^{(y-1)\gamma(x,y)}
\]

Now it easy to check the following equalities

\[
L(1, 0, 0) = x^3[y, x]^{x-y}, R(1, 0, 0) = x^3[y, x]^{y-1}
\]

\[
L(1, 1, 0) = x^3[y, x]^{x+y}, R(1, 1, 0) = x^3[y, x]^{x+1}
\]

\[
L(2, 2, 0) = x^3[y, x]^{x-y}, R(2, 2, 0) = x^3[y, x]^{x-1}.
\]

The proof is thus complete.

\[\square\]

**Lemma 3.6.** The 2-word

\[w(x, y) = x^4[y, x]^{A+Bx+Cy}\]

is associative in \(S_3\) if and only if \(A = B = C = 0\).

**Proof.** We put \(z = 1\) into the associativity law and we make use of the formulas (i), (ii), (iii) and (iv). We have

\[
L = w(w(x, y), 1) = w(x, y)^4 = (x^4[y, x]^{A+Bx+Cy}
\]

\[
R = w(x, w(y, 1)) = w(x, y^4) = x^4[y^4, x]^{\gamma(x,y)} = x^4[y, x]^{-(y+1)\gamma(x,y)}
\]

\[
= x^4[y, x]^{(B-A-C)+y(B-A-C)}.
\]

Therefore the equality \(L = R\) ensures \(B = 0\) and \(A = C\). Taking this into account we get

\[
w(w(x, x), z) = w(x^4, z) = x^4[z, x]^{\gamma(1,z)} = x^4[z, x]^{-(x+1)(A+Az)} = x^4
\]

\[
w(x, w(x, z)) = x^4[x^4[z, x]^{\gamma(x,z)}, x]^{\gamma(1,z)} = x^4[z, x]^{A(1-x)},
\]

which shows that \(A = B = C = 0\) and Lemma 3.6 follows. \[\square\]

**Lemma 3.7.** The word

\[w(x, y) = x^Ay^4[y, x]^{A+Bx+Cy}\]

is associative in \(S_3\) if and only if \(A = B = C = 0\).
Proof. We have

\[ L = w(w(1, y), z) = w(y^4, z) = y^4z^4[y^4, z]^\gamma(1, z) \]

\[ = y^4z^4[z, y]^{-y(1+(A+B+C)z)} = y^4z^4[z, y]^{-(C-A-B)+(C-A-B)y} \]

and

\[ R = w(1, w(y, z)) = w(y, z)^4 = y^4z^4[z, y]^{A+B+y+C} \]

Hence \( C = 0 \) and \( A = B \). Taking this into account we check

\[ L = w(w(x, y), x) = x^2y^4[y, x]^{\gamma(x, y)}[x, x^4y^4[y, x]^{\gamma(x, y)}]^{\gamma(1, x)} \]

\[ = x^2y^4[y, x]^{\gamma(x, y)}[y, x]^{(y+1)(A(x+1))}[y, x]^{(1-x)A(x+1)}\gamma(x, y) \]

\[ = x^2y^4[y, x]^{A(1+x)} \]

and also

\[ R = w(x, w(y, x)) = x^4y^4x^4[y, x]^{\gamma(y, x)}[y^4x^4[y, y]^{\gamma(y, x)}]^{\gamma(x, 1)} \]

\[ x^4y^4x^4[y, y]^{\gamma(y, x)}[x, x]^{(x-1)\gamma(x, 1)}[x, y]^{\gamma(1, y)} \]

\[ = x^2y^4[y, x]^{A(1+y)}. \]

By (vi) \( L = R \) if and only if \( A = 0 \). Clearly, \( x^4y^4 \) is associative word in \( S_3 \). The proof is thus completed. \( \square \)

Lemma 3.8. If the word

\[ w(x, y) = [y, x]^{A+Bx+Cy} = [y, x]^{\gamma(x, y)} \]

is associative, then \( A = B - C = 0 \). Conversely, the word

\[ w(x, y) = [y, x]^{B(x+y)} \]

(3.6)

satisfies the associativity law for all \( B \in Z_3 \).

Proof. Using the identities (i), (ii),(ii) and (iv) we have

\[ L = w(w(x, y), z)) = [z, [y, x]^{\gamma(x, y)}]^{\gamma(1, z)} = [y, x]^{(1-z)(A+Bx+Cy)(A+B+Cz)} \]

and similarly

\[ R = w(x, w(y, z)) = [w(y, z), x]^{\gamma(x, 1)} \]

\[ = [z, y]^{(x-1)(A+B+y+Cz)(A+B+C)}. \]
Thus if \( w \) is an associative word in \( S_3 \), then in the case \( y = x \), we get

\[
[z, x]^{(x-1)(A-B+C)(A-B+Cz)} = 1,
\]

because of (iii) and (v). Similarly, in the case \( z = y \) we obtain the equation

\[
\]

Now (3.7), (3.8) and (vi) imply the system of congruences

\[
\begin{align*}
(A + C - B)^2 & \equiv 0 \pmod{3}, \\
(A + C - B)(A - B - C) & \equiv 0 \pmod{3}, \\
(A + C - B)C & \equiv 0 \pmod{3}, \\
(A + B - C)^2 & \equiv 0 \pmod{3}, \\
(A + B - C)B & \equiv 0 \pmod{3}, \\
(A + B - C)(B + C - A) & \equiv 0 \pmod{3},
\end{align*}
\]

which have the solution \( A = B - C = 0 \).

Conversely, we check that the word \( w(x, y) = [y, x]^{Bx+By} \) is associative. Indeed, by (ii) and (iii) we have

\[
w(w(x, y), z) = [z, [y, x]^{B(x+y)}]^{B(1+z)} = [y, x]^{B^2(1-z)(1+z)(x+y)} = 1
\]

and

\[
w(x, w(y, z)) = [[z, y]^{B(z+y)}, x]^{B(1+x)} = [z, y]^{B^2(y+z)(x-1)(x+1)} = 1,
\]

as required.

We have thus established our main result

\textbf{Theorem 3.9.} There are precisely (modulo \( V(S_3) \)) twelve associative words in the group \( S_3 \). Namely \( 1, x, x^3, x^4, y, y^3, y^4, xy, yx, x^4y^4, [y, x]^{x+y} \) and \([x, y]^{x+y}\).

\textbf{References}


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