On the Fitting ideals of a multiplication module

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Abstract. In this paper, we characterize all finitely generated multiplication $R$-modules whose the first nonzero Fitting ideal of them is contained in only finitely many maximal ideals. Also, we prove that a finitely generated multiplication $R$-module $M$ is faithful if and only if $M$ is a projective of constant rank one $R$-module.

Introduction

Let $R$ be a commutative ring with identity and $M$ be a finitely generated $R$-module. For a set $\{x_1, \ldots, x_n\}$ of generators of $M$ there is an exact sequence $0 \rightarrow N \rightarrow R^n \xrightarrow{\varphi} M \rightarrow 0$ where $R^n$ is a free $R$-module with the set $\{e_1, \ldots, e_n\}$ of basis, the $R$-homomorphism $\varphi$ is defined by $\varphi(e_j) = x_j$ and $N$ is the kernel of $\varphi$. Let $N$ be generated by $u_\lambda = a_1\lambda e_1 + \ldots + a_n\lambda e_n$, with $\lambda$ in some index set $\Lambda$. Let $\text{Fitt}_i(M)$ be an ideal of $R$ generated by the minors of size $n - i$ of the matrix

$$
\begin{pmatrix}
\vdots & a_1\lambda & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & a_n\lambda & \vdots 
\end{pmatrix}.
$$

For $i > n$, $\text{Fitt}_i(M)$ is defined by $R$, and for $i < 0$, $\text{Fitt}_i(M)$ is defined as the zero ideal. It is known that $\text{Fitt}_i(M)$ is the invariant ideal determined


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by $M$, that is, it is determined uniquely by $M$ and it does not depend on the choice of the set of generators of $M$ [8]. The ideal $\text{Fitt}_i(M)$ will be called the $i$-th Fitting ideal of the module $M$. It follows from the definition of $\text{Fitt}_i(M)$ that $\text{Fitt}_i(M) \subseteq \text{Fitt}_{i+1}(M)$. Moreover, it is shown that $\text{Fitt}_0(M) \subseteq \text{ann}(M)$ and $(\text{ann}(M))^n \subseteq \text{Fitt}_0(M)$ ($M$ is generated by $n$ elements) and $\text{Fitt}_i(M)_P = \text{Fitt}_i(M_P)$, for every prime ideal $P$ of $R$ [6].

The most important Fitting ideal of $M$ is the first of the $\text{Fitt}_j(M)$ that is nonzero. We shall denote this Fitting ideal by $I(M)$. Note that if $I(M)$ contains a nonzerodivisor then $I(M)_P = I(M)_P$ for every prime ideal $P$ of $R$. An element of $R$ is called regular if it is a nonzerodivisor and an ideal of $R$ is regular if it contains a regular element. Assume that $T(M)$, the torsion submodule of $M$, be the submodule of $M$ consisting of all elements of $M$ that are annihilated by a regular element of $R$. An $R$-module $M$ is a torsion module if $T(M) = 0$. Fitting ideals are strong tools to identify properties of modules and sometimes to characterize modules. For example Buchsbaum and Eisenbud have shown in [2] that for a finitely generated $R$-module $M$, $I(M) = R$ if and only if $M$ is a projective of constant rank module. A lemma of Lipman asserts that if $R$ is a local ring and $M = R^m/K$ and $I(M)$ is the $(m - q)$th Fitting ideal of $M$ then $I(M)$ is a regular principal ideal if and only if $K$ is finitely generated free and $M/T(M)$ is free of rank $m - q$ ([11]). Finally it is shown in [9] that if $M$ is a finitely generated module over a Noetherian local UFD $(R, P)$ then $I(M) = P$ if and only if

1. $M$ is isomorphic to $R^n/\langle(a_1, \ldots, a_n)^I\rangle$, where $P = \langle a_1, \ldots, a_n \rangle$ and $n$ is a positive integer if $M$ is torsionfree, and
2. $M$ is isomorphic to $R^n \oplus R/P$, for some positive integer $n$ if $M$ is not torsionfree.

Multiplication modules, first were defined by A. Barnard in [1].

An $R$-module $M$ is called a multiplication module if for each submodule $N$ of $M$, $N = IM$ for some ideal $I$ of $R$. In this case we can take $I = (N : M)$ [15].

1. **Fitting ideals of multiplication modules**

In this section we study some properties of finitely generated multiplication modules and Fitting ideals of them.

**Proposition 1.** Let $M = M_1 \oplus M_2$ be a finitely generated $R$-module. Then $\text{Fitt}_k(M) = \sum_{i+j=k} \text{Fitt}_i(M_1) \text{Fitt}_j(M_2)$. Particularly $I(M) = I(M_1)I(M_2)$.
Proof. Let $N_i \longrightarrow G_i \longrightarrow M_i \longrightarrow 0$ be exact sequences and $G_i$ be free $R$-modules of rank $r_i$ for $i = 1, 2$. Let $\psi_1$ be the matrix presentation of generators of $N_1$ and $\psi_2$ be the matrix presentation of generators of $N_2$. Thus $N_1 \oplus N_2 \longrightarrow G_1 \oplus G_2 \longrightarrow M \longrightarrow 0$ is an exact sequence and $\psi_1 \oplus \psi_2$ be an ideal of $R$ generated by the minors of size $j$ of matrix presentation of $(\psi_1 \oplus \psi_2)$. Since $\psi_1 \oplus \psi_2 = \left( \begin{array}{cc} \psi_1 & 0 \\ 0 & \psi_2 \end{array} \right)$, hence $\text{Fitt}_i(M) = I_{r_1+r_2-k}((\psi_1 \oplus \psi_2)) = \sum_{i+j=k} Fitt_i(M_1) Fitt_j(M_2)$. \hfill \qed

Theorem 1. Let $R$ be a ring and $M$ be a finitely generated $R$-module. If $\text{Fitt}_0(M) = Q$ be a maximal ideal of $R$ then $M \cong (R/Q)^n$, for some positive integer $n$.

Proof. By [6, Proposition 20-7], $Q = \text{Fitt}_0(M) \subseteq \text{ann}(M)$. So $M$ is a vector space over the field $R/Q$. Hence there exists a positive integer $n$ such that $M \cong (R/Q)^n$. \hfill \qed

The next Proposition asserts the relation between the 0-th Fitting ideal of a module and the 0-th Fitting ideal of it’s submodules.

Proposition 2. Let $M$ be a finitely generated module and $N$ be a submodule of $M$ generated by $k$ elements. Then $\text{Fitt}_0(M)^k \subseteq \text{Fitt}_0(N)$.

Proof. By [6, Proposition 20-7] we have $\text{Fitt}_0(M) \subseteq \text{ann}(M) \subseteq \text{ann}(N)$ and $\text{ann}(N)^k \subseteq \text{Fitt}_0(N)$. Thus $\text{Fitt}_0(M)^k \subseteq \text{Fitt}_0(N)$. \hfill \qed

A Theorem of Barnard [1] asserts that every multiplication module is locally cyclic [1]. Here we give another proof for this result using Fitting ideals.

Lemma 1. Let $M$ be a finitely generated multiplication $R$-module. Then $\text{Fitt}_1(M) = R$.

Proof. Let $M$ be generated by $\{x_1, \ldots, x_n\}$. Consider the exact sequence $0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$, where $\varphi(e_j) = x_j$ and $N = \text{Ker}(\varphi)$. Put $B_i = (Rx_i : M)$, for $i, 1 \leq i \leq n$. For the moment fix $i, 1 \leq i \leq n$. Let $a_{ji} \in B_i$, $1 \leq j \leq n, j \neq i$. Then there exist some $b_{ij} \in R$ such that
$a_{ji}x_j = b_{ij}x_i$, $1 \leq j \leq n, j \neq i$. Consider the matrix

$$
\begin{pmatrix}
  a_{1i} & 0 & \ldots & \ldots & 0 \\
  0 & a_{2i} & 0 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  -b_{11} & -b_{i2} & \ldots & \ldots & -b_{in} \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & a_{ni}
\end{pmatrix}
$$

Since each columns of this matrix belongs to $N$, so we have

$$
\begin{vmatrix}
  a_{1i} & 0 & \ldots & 0 \\
  0 & a_{2i} & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & a_{ni}
\end{vmatrix} \in \text{Fitt}_1(M).
$$

This implies that $B_i^{n-1} \subseteq \text{Fitt}_1(M)$, for all $i, 1 \leq i \leq n$. Hence $B_1^{n-1} + \cdots + B_n^{n-1} \subseteq \text{Fitt}_1(M)$. So by [13, 2.25],

$$
\sqrt{B_1^{n-1} + \cdots + B_n^{n-1}} = \sqrt{B_1^{n-1}} + \cdots + \sqrt{B_n^{n-1}} 
$$

$$
= \sqrt{B_1} + \cdots + \sqrt{B_n} \subseteq \sqrt{\text{Fitt}_1(M)}.
$$

Since $(B_1 + \cdots + B_n)M = M$, by [14, Corollary], $R = B_1 + \cdots + B_n + \text{ann}(M)$. on the other hand we have $\text{ann}(M) \subseteq B_i = (Rx_i : M)$. Thus $R = B_1 + \cdots + B_n$. Therefore $\sqrt{\text{Fitt}_1(M)} = R$. Thus $\text{Fitt}_1(M) = R$. 

**Theorem 2.** Let $M$ be a finitely generated multiplication module over a ring $R$. Then $M$ is locally cyclic.

**Proof.** Let $M$ be a finitely generated multiplication module over a local ring $(R, P)$. Let $\{x_1, \ldots, x_n\}$ be a minimal generator set for $M$. Consider the exact sequence $0 \to N \to R^n \xrightarrow{\varphi} M \to 0$, where $\varphi$ is defined by $\varphi(e_j) = x_j$ and $N$ is the kernel of $\varphi$. Let $N$ be generated by $u_i = a_{1i}e_1 + \cdots + a_{ni}e_n$, with $i$ in some index set $I$. Since $\{x_1, \ldots, x_n\}$ is a minimal generator set for $M$, hence $a_{ij} \in P$, for all $i,j$. Thus $\text{Fitt}_{n-1}(M) \subseteq P$. On the other hand by Lemma 1, we have $\text{Fitt}_{n-1}(M) = R$, for $n \geq 2$. Hence $n = 1$. This means that $M$ is cyclic. 

**Proposition 3.** Let $M$ be a finitely generated multiplication $R$-module. Then $\text{Fitt}_0(M) = \text{ann}(M)$. 
Proof. By 1, \(M_Q\) is cyclic. Thus \(\text{Fitt}_0(M_Q) = \text{ann}(M_Q)\), for every prime ideal \(Q\) of \(R\). Since \(M\) is finitely generated, hence \(\text{ann}(M_Q) = \text{ann}(M)_Q\). Thus by [6, Corollary 20.5], \(\text{Fitt}_0(M)_Q = \text{ann}(M)_Q\), for every prime ideal \(Q\) of \(R\). Therefore \(\text{Fitt}_0(M) = \text{ann}(M)\).

Lemma 2. Let \(R\) be an integral domain and \(M\) be an \(R\)-module. Then \(T(M_P) = T(M)_P\), for every prime ideal \(P\) of \(R\).

Proof. Let \(P\) be a prime ideal of \(R\). It is easily seen that \(T(M)_P \subseteq T(M_P)\). Now let \(R\) be an integral domain. Assume that \(x_1 \in T(M_P)\). Thus there exists a regular element \(\frac{r}{s} \in R_P\) such that \(\frac{r}{s} x_1 = 0\). So there exists an element \(t \in R \setminus P\) such that \(tx = 0\). Thus \(r(tx) = 0\). It is sufficient to prove that \(r\) is a regular element of \(R\). Then \(tx \in T(M)\) and consequently \(x_1 = \frac{tx}{t} \in T(M)_P\). To prove the regularity of \(r\), assume that \(ar = 0\), for some element \(a\) of \(R\). Thus \(\frac{ra}{s} = 0\), in \(R_P\). Since \(\frac{r}{s}\) is a regular element of \(R_P\), \(\frac{a}{1} = 0\). Thus there exists an element \(b \in R \setminus P\) such that \(ab = 0\). Since \(R\) is an integral domain and \(b \in R \setminus P\), hence \(a = 0\) and we are done.

Theorem 3. Let \(M\) be a finitely generated multiplication module over an integral domain \(R\). Then \(M\) is a torsionfree \(R\)-module or \(M\) is a torsion \(R\)-module.

Proof. Let \(M = \langle x_1, \ldots, x_n \rangle\) be a finitely generated nontorsionfree multiplication module over an integral domain \(R\). So \(T(M) \neq 0\). Since \(M\) is a multiplication module, hence there exists an ideal \(I\) of \(R\) such that \(T(M) = IM\). Let \(0 \neq a \in I\) be arbitrary. For \(1 \leq i \leq n\), there exist some \(0 \neq r_i\) such that \(r_iax_i = 0\). So \(r_1 \ldots r_n a \in \text{ann}(M)\). Since \(r_1 \ldots r_n\) is a regular element so \(M = T(M)\).

Theorem 4. Let \(M\) be a finitely generated multiplication module over an integral domain \(R\). Then the following conditions are equivalent.

1) \(M\) is a torsionfree \(R\)-module.

2) \(M\) is a projective of constant rank one \(R\)-module.

Proof. (1 \(\implies\) 2) Let \(M\) be a finitely generated torsionfree multiplication module over an integral domain \(R\). So \(\text{ann}(M) = 0\). Thus \(M_P \cong R_P / \text{ann}(M)_P = R_P / \text{ann}(M_P) = R_P\), for every prime ideal \(P\) of \(R\). Hence by [3, 5&3, Theorem 2] \(M\) is a projective of constant rank one \(R\)-module.
(2 $\implies$ 1) Since $M$ is a projective of constant rank one $R$-module, hence for every prime ideal $p$ of $R$, we have $M_p = R_p$. So $T(M)_p = T(M_p) = 0$, for every prime ideal $p$ of $R$. Therefore $T(M) = 0$. □

**Corollary 1.** Let $M$ be a finitely generated multiplication module over an integral domain $R$. Then $M$ is a projective of constant rank $R$-module or $M$ is a torsion $R$-module.

*Proof. By Theorem 3 and Theorem 4.* □

**Theorem 5.** Let $M$ be a finitely generated multiplication $R$-module. If $I(M)$ is contained in only finitely many maximal ideals of $R$, then $M$ is cyclic.

*Proof. If $\text{ann}(M) = 0$, then by Proposition 3, $\text{Fitt}_0(M) = 0$. So by Lemma 1, $I(M) = R$ that is not contained in any maximal ideal of $R$. Thus $\text{ann}(M) \neq 0$. Then by Proposition 3, $I(M) = \text{Fitt}_0(M) = \text{ann}(M)$. Since there exist only finitely many maximal ideals of $R$ containing $I(M)$, hence by [1, Lemma 10], $M$ is a cyclic $R$-module.* □

**Corollary 2.** Let $M$ be a finitely generated multiplication module. Let $I(M) = P_1^{n_1} \cdots P_k^{n_k}$, for some maximal ideals $P_i$ of $R$ and for some positive integers $n_i$, $1 \leq i \leq k$. Then $M \cong R/P_1^{n_1} \oplus \cdots \oplus R/P_k^{n_k}$.

*Proof. Since $I(M)$ is contained in only finitely many maximal ideals $P_1$, $\ldots$, $P_n$, hence by Theorem 5, $M \cong R/P_1^{n_1} \cdots P_k^{n_k} \cong R/P_1^{n_1} \oplus \cdots \oplus R/P_k^{n_k}$. □

**Corollary 3.** Let $M$ be a finitely generated multiplication module. Then $M$ is a faithful $R$-module if and only if $M$ is an injective of constant rank one $R$-module.

*Proof. By Lemma 1, $\text{Fitt}_1(M) = R$ and by Proposition 3, $\text{Fitt}_0(M) = \text{ann}(M) = 0$. So $I(M) = R$. Thus by [2, Lemma 1], $M$ is a projective of constant rank $R$-module. On the other hand by Theorem 2, $M$ is locally cyclic. Hence $M$ is a projective of constant rank one $R$-module. Now Let $M$ be a projective of constant rank $R$-module. So $\text{ann}(M_P) = \text{ann}(M)_P = 0$. Thus $\text{ann}(M) = 0$. □

Note that if $M$ is a projective $R$-module then the converse of the previous lemma is not true always. See the following Lemma.
Lemma 3. Let $M$ be a finitely generated multiplication module. If $I(M) = \langle e \rangle$, where $e$ is an idempotent element of $R$, then $M$ is a projective $R$-module.

Proof. If $I(M) = \text{Fitt}_0(M)$, then by Proposition 3, $\text{Fitt}_0(M) = \text{ann}(M) = \langle e \rangle$. So by [4, Theorem 2.1], $M$ is a projective $R$-module. If $\text{Fitt}_0(M) = 0$, then by Lemma 1, $\text{Fitt}_1(M) = R$. So by [2, Lemma 1], $M$ is a projective $R$-module.

Theorem 6. Let $M$ be a finitely generated multiplication module. If $e$ is an idempotent element of $R$ such that $\text{ann}(M) \subsetneq \langle e \rangle \subsetneq R$, then $M \cong eM \oplus \frac{M}{eM}$.

Proof. Since $\langle e \rangle \neq R$, hence $eM \neq M$. It is easily seen that $\frac{M}{eM}$ is a multiplication module and we have $\text{ann}(\frac{M}{eM}) = (eM : M) \supseteq \langle e \rangle$. Let $r \in (eM : M)$ and $m \in M$. Thus $rm \in eM$. So there exists an element $m' \in M$, such that $rm = em'$. Hence $rem = e^2m' = em' = rm$. Therefore $(re-r)m = 0$. If $re-r \in \text{ann}(M) \subseteq \langle e \rangle$. Thus $r \in \langle e \rangle$. So $\text{ann}(\frac{M}{eM}) = \langle e \rangle$.

By Lemma 3, $\frac{M}{eM}$ is a projective $R$-module. So $M \cong eM \oplus \frac{M}{eM}$.

Theorem 7. Let $M$ be a finitely generated multiplication $R$-module. If $I(M)$ is a primary ideal of $R$ then $M$ is an indecomposable $R$-module.

Proof. Let $M = N \oplus K$, for some $R$-submodules $N$ and $K$ of $M$. Assume that $\pi_1, \pi_2 : M = N \oplus K \longrightarrow M$ be defined by $\pi_1(n + k) = n$ and $\pi_2(n + k) = k$, for $n \in N$ and $k \in K$. Since $M$ is a finitely generated multiplication module, so by [5, Theorem 3], there exist $0 \neq r_1$ and $0 \neq r_2$ in $R$ such that $\pi_1(m) = r_1m$ and $\pi_2(m) = r_2m$, for every $m \in M$. Since $\pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 = 0$, hence $r_1r_2M = 0$. So $r_1r_2 \in \text{ann}(M) = \text{Fitt}_0(M)$. Since $I(M) = \text{Fitt}_0(M) = \text{ann}(M)$ is a primary ideal of $R$, hence $r_1^{n_1} \in \text{ann}(M)$ or $r_2^{n_2} \in \text{ann}(M)$, for some positive integers $n_1$ and $n_2$. If $r_1^{n_1} \in \text{ann}(M)$, then $\pi_1(m) = r_1^{n_1}(m) = r_1^{n_1}m = 0$, for every element $m \in M$. Therefore $N = 0$. Similarly if $r_2^{n_2} \in \text{ann}(M)$, then $\pi_2(m) = r_2^{n_2}(m) = r_2^{n_2}m = 0$, for every element $m \in M$. Thus $K = 0$.

References


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