Abstract. Let $R$ be a ring and $M$ be a right $R$-module. We say a submodule $S$ of $M$ is a (weak) Goldie-Rad-supplement of a submodule $N$ in $M$, if $M = N + S$, $(N \cap S \leq \text{Rad}(M))$ $N \cap S \leq \text{Rad}(S)$ and $N^\beta \ast S$, and $M$ is called amply (weakly) Goldie-Rad-supplemented if every submodule of $M$ has ample (weak) Goldie-Rad-supplements in $M$. In this paper we study various properties of such modules. We show that every distributive projective weakly Goldie-Rad-Supplemented module is amply weakly Goldie-Rad-Supplemented. We also show that if $M$ is amply (weakly) Goldie-Rad-supplemented and satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules, then $M$ is Artinian.

Introduction

Throughout this article, all rings are associative with unity and $R$ denotes such a ring. All modules are unital right $R$-modules unless indicated otherwise. Let $M$ be an $R$-module. $N \leq M$ will mean $N$ is a submodule of $M$. $\text{End}(M)$ and $\text{Rad}(M)$ will denote the ring of endomorphisms of $M$ and the Jacobson radical of $M$, respectively. The notions which are not explained here will be found in [6].

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Recall that a submodule $S$ of $M$ is called small in $M$ (notation $S \ll M$) if $M \neq S + T$ for any proper submodule $T$ of $M$. A module $H$ is called hollow if every proper submodule of $H$ is small in $H$. Let $N$ and $L$ be submodules of $M$. Then $N$ is called a supplement of $L$ in $M$ if $N + L = M$ and $N$ is minimal with respect to this property, or equivalently, $N$ is a supplement of $L$ in $M$ if $M = N + L$ and $N \cap L \ll N$. $N$ is said to be a supplement submodule of $M$ if $N$ is a supplement of some submodule of $M$. Recall from [3] that $M$ is called a supplemented module if any submodule of $M$ has a supplement in $M$. $M$ is called an amply supplemented module if for any two submodule $A$ and $B$ of $M$ with $A + B = M$, $B$ contains a supplement of $A$. $M$ is called a weakly supplemented module if for each submodule $A$ of $M$ there exists a submodule $B$ of $M$ such that $M = A + B$ and $A \cap B \ll M$. Let $K, N \subseteq M$. $K$ is a (weak) Rad-supplement of $N$ in $M$, if $M = N + K$ and $(N \cap K \leq \text{Rad}(M))$ $N \cap K \leq \text{Rad}(K)$ (in this case $K$ is a (weak) generalized supplement of $N$ (see, [5])). $K$ is said to be a (weak) Rad-supplement submodule of $M$ if $K$ is a (weak) Rad-supplement of some submodule of $M$ (in this case $K$ is a generalized (weakly) supplement submodule (see, [5])). A module $M$ is called (weakly) Rad-supplemented if every submodule of $M$ has a (weak) Rad-supplement (in this case $M$ is a generalized (weakly) supplemented module (see, [5])).

In [2], the authors introduced a new class of modules namely Goldie*-Supplemented by defining and studying the $\beta^*$ relation as the following:

Let $X, Y \subseteq M$. $X$ and $Y$ are $\beta^*$ equivalent, $X \beta^* Y$, provided $\frac{X+Y}{X} \ll \frac{M}{X}$ and $\frac{X+Y}{Y} \ll \frac{M}{Y}$. After this work, Talebi et. al. [4] defined and studied the $\beta^{**}$ relation and investigated some properties of this relation. In [4], this $\beta^{**}$ relation was defined as the following:

Let $X, Y \subseteq M$. $X$ and $Y$ are $\beta^{**}$ equivalent, $X \beta^{**} Y$, provided $\frac{X+Y}{X} \leq \frac{\text{Rad}(M)+X}{X}$ and $\frac{X+Y}{Y} \leq \frac{\text{Rad}(M)+Y}{Y}$.

Based on definition of $\beta^{**}$ relation they introduced a new class of modules namely Goldie-Rad-supplemented. A module $M$ is called Goldie-Rad-supplemented if for any submodule $N$ of $M$, there exists a Rad-supplement submodule $D$ of $M$ such that $N \beta^{**} D$.

Let $M$ be an $R$-module. We say a submodule $S$ is a (weak) Goldie-Rad-supplement of a submodule $N$ in $M$, if $M = N + S$, $(N \cap S \leq \text{Rad}(M))$ $N \cap S \leq \text{Rad}(S)$ and $N \beta^{**} S$. We say that $M$ is weakly Goldie-Rad-supplemented if every submodule of $M$ has a weak Goldie-Rad-supplement in $M$. We say that a submodule $N$ of $M$ has ample (weak) Goldie-Rad-supplements in $M$ if, for every $L \subseteq M$ with $N + L = M$, there exists a (weak) Goldie-Rad-supplement $S$ of $N$ with $S \subseteq L$. We say that $M$ is
amply (weakly) Goldie-Rad-supplemented if every submodule of $M$ has ample (weak) Goldie-Rad-supplements in $M$.

We prove that every distributive projective weakly Goldie-Rad-supplemented module is amply weakly Goldie-Rad-supplemented. We show that if $M$ is an amply (weakly) Goldie-Rad-supplemented module and satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules, then $M$ is Artinian. In addition, let $M$ be a radical module ($\text{Rad}(M) = M$). Then $M$ is Artinian if and only if $M$ is an amply (weakly) Goldie-Rad-supplemented module and satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules. Moreover, we also show that the class of amply (weakly) Goldie-Rad-supplemented modules is closed under supplement submodules and homomorphic images.

**Lemma 1.** ([6, 41.1]) Let $M$ be a module and $K$ be a supplement submodule of $M$. Then $K \cap \text{Rad}(M) = \text{Rad}(K)$.

**Theorem 1.** ([1, Theorem 5]) Let $R$ be any ring and $M$ be a module. Then $\text{Rad}(M)$ is Artinian if and only if $M$ satisfies DCC on small submodules.

1. Amply (weakly) Goldie-Rad-supplemented modules

In this section, we discuss the concept of amply (weakly) Goldie-Rad-supplemented modules and we give some properties of such modules.

**Proposition 1.** Every amply (weakly) Goldie-Rad-supplemented module is a (weakly) Goldie-Rad-supplemented module.

*Proof.* Let $M$ be an amply (weakly) Goldie-Rad-supplemented module and $N$ be a submodule of $M$. Then $N + M = M$. Since $M$ is amply (weakly) Goldie-Rad-supplemented, $M$ contains a (weak) Goldie-Rad-supplement $S$ of $N$. So $S$ is a (weak) Goldie-Rad-supplement of $N$ in $M$. Hence $M$ is (weakly) Goldie-Rad-supplemented. \qed

**Example 1.** An hollow radical module $M$ ($\text{Rad}(M) = M$) is amply Goldie-Rad-supplemented.

**Lemma 2.** Let $M$ be an $R$-module and $L \leq N \leq M$. If $S$ is a (weak) Goldie-Rad-supplement of $N$ in $M$, then $(S + L)/L$ is a (weak) Goldie-Rad-supplement of $N/L$ in $M/L$. 
Proof. By the proof of [5, Proposition 2.6 (1)], \((S + L)/L\) is a (weak) Rad-supplement of \(N/L\) in \(M/L\). By [4, Proposition 2.3 (1)], \(\frac{N}{L} \otimes \beta^{**} \left(\frac{S + L}{L}\right)\). Hence \((S + L)/L\) is a (weak) Goldie-Rad-supplement of \(N/L\) in \(M/L\). \(\square\)

**Proposition 2.** Every factor module of an amply (weakly) Goldie-Rad-supplemented module is amply (weakly) Goldie-Rad-supplemented.

Proof. Let \(M\) be an amply (weakly) Goldie-Rad-supplemented module and \(M/K\) be any factor module of \(M\). Let \(N/K \leq M/K\). For \(L/K \leq M/K\), let \(N/K + L/K = M/K\). Then \(N + L = M\). Since \(M\) is an amply (weakly) Goldie-Rad-supplemented module, there exists a (weak) Goldie-Rad-supplement \(S\) of \(N\) with \(S \leq L\). By Lemma 2, \((S + K)/K\) is a (weak) Goldie-Rad-supplement of \(N/K\) in \(M/K\). Since \((S + K)/K \leq L/K\), \(N/K\) has ample (weak) Goldie-Rad-supplements in \(M/K\). Thus \(M/K\) is amply (weakly) Goldie-Rad-supplemented. \(\square\)

**Corollary 1.** Every direct summand of an amply (weakly) Goldie-Rad-supplemented module is amply (weakly) Goldie-Rad-supplemented.

Proof. Let \(M\) be an amply (weakly) Goldie-Rad-supplemented module. Since every direct summand of \(M\) is isomorphic to a factor module of \(M\), then by Proposition 2, every direct summand of \(M\) is amply (weakly) Goldie-Rad-supplemented. \(\square\)

**Corollary 2.** Every homomorphic image of an amply (weakly) Goldie-Rad-supplemented module is amply (weakly) Goldie-Rad-supplemented.

Proof. Let \(M\) be an amply (weakly) Goldie-Rad-supplemented module. Since every homomorphic image of \(M\) is isomorphic to a factor module of \(M\), every homomorphic image of \(M\) is amply (weakly) Goldie-Rad-supplemented by Proposition 2. \(\square\)

Let \(M\) be a module. Then \(M\) is called *distributive* if its lattice of submodules is a distributive lattice, equivalently for submodules \(K, L, N\) of \(M\), \(N + (K \cap L) = (N + K) \cap (N + L)\) or \(N \cap (K + L) = (N \cap K) + (N \cap L)\).

**Proposition 3.** Every supplement submodule of a distributive amply (weakly) Goldie-Rad-supplemented module is amply (weakly) Goldie-Rad-supplemented.

Proof. Let \(M\) be an amply (weakly) Goldie-Rad-supplemented module and \(S\) be any supplement submodule of \(M\). Then there exists a submodule \(N\) of \(M\) such that \(S\) is a supplement of \(N\). Let \(L \leq S\) and \(L + S' = S\)
for $S' \subseteq S$. Then $N + L + S' = M$. Since $M$ is amply (weakly) Goldie-Rad-supplemented, $N + L$ has a (weak) Goldie-Rad-supplement $S''$ in $M$ with $S'' \subseteq S'$.

In this case $(N + L) + S'' = M$, ($(N + L) \cap S'' \subseteq \text{Rad}(M)$) $(N + L) \cap S'' \subseteq \text{Rad}(S'')$ and $(N + L)\beta**S''$. Since $L + S'' \subseteq S$ and $S$ is a supplement of $N$ in $M$, $L + S'' = S$. On the other hand, $L \cap S'' \subseteq (N + L) \cap S'' \subseteq \text{Rad}(S'')$. Now, we show that $L\beta**S''$ in $S$. By Lemma 1, $S \cap \text{Rad}(M) = \text{Rad}(S)$. Therefore, since $(N + L)\beta**S''$,

$$\frac{L + S''}{S''} = \frac{S \cap (L + S'')}{S''} \leq \frac{S \cap (N + L + S'')}{S''} \leq \frac{S \cap (\text{Rad}(M) + S'')}{S''} = \frac{S'' + (S \cap \text{Rad}(M))}{S''} = \frac{S'' + \text{Rad}(S)}{S''},$$

and since $N \cap S \ll S$, $N + L + S'' \subseteq \text{Rad}(M) + N + L$,

$$\frac{L + S''}{L} = \frac{S \cap (L + S'')}{L} \leq \frac{S \cap (L + S'' + N)}{L} \leq \frac{S \cap (\text{Rad}(M) + N + L)}{L} = \frac{L + (S \cap (\text{Rad}(M) + N))}{L} \leq \frac{L + \text{Rad}(S)}{L}.$$

Hence $S''$ is a (weak) Goldie-Rad-supplement of $L$ in $S$. Since $S'' \subseteq S'$, $L$ has ample (weak) Goldie-Rad-supplements in $S$. Thus $S$ is amply (weakly) Goldie-Rad-supplemented.

A module $M$ is said to be $\pi$-projective if, for every two submodules $N, L$ of $M$ with $L + N = M$, there exists $f \in \text{End}(M)$ with $\text{Im}f \leq L$ and $\text{Im}(1 - f) \leq N$ (see, [6]).

**Theorem 2.** Let $M$ be a distributive weakly Goldie-Rad-supplemented and $\pi$-projective module. Then $M$ is an amply weakly Goldie-Rad-supplemented module.

**Proof.** Let $N \subseteq M$ and $L + N = M$ for $L \subseteq M$. Since $M$ is weakly Goldie-Rad-supplemented, there exists a weak Goldie-Rad-supplement $S$ of $N$ in $M$. Then $S + N = M$, $S \cap N \subseteq \text{Rad}(M)$ and $S\beta**N$. Since $M$ is $\pi$-projective, there exists $f \in \text{End}(M)$ such that $f(M) \subseteq L$ and $(1 - f)(M) \subseteq N$. Note that $f(N) \subseteq N$ and $(1 - f)(L) \subseteq L$. Then

$$M = f(M) + (1 - f)(M) \leq f(S + N) + N = f(S) + N.$$

Let $n \in N \cap f(S)$. Then there exists $s \in S$ with $n = f(s)$. In this case $s - n = s - f(s) = (1 - f)(s) \in N$ and then $s \in N$. Hence $s \in N \cap S$ and
\[ N \cap f(S) \leq f(N \cap S). \] Since \( N \cap S \leq \text{Rad}(M) \), \( f(N \cap S) \leq f(\text{Rad}(M)) \). Then
\[
N \cap f(S) \leq f(N \cap S) \leq f(\text{Rad}(M)) \leq \text{Rad}(f(M)) \leq \text{Rad}(M)
\]
Next we show that \( f(S) \beta^{**} N \). Since \( S \beta^{**} N \), \( S + N \leq \text{Rad}(M) + N \) and \( S + N \leq \text{Rad}(M) + S \). Hence
\[
f(S) + N = M = S + N \leq \text{Rad}(M) + N,
\]
and since \( S \cap N \leq \text{Rad}(M) \),
\[
f(S) + N = f(S) + (N \cap M) = f(S) + (N \cap (\text{Rad}(M) + S)) \leq f(S) + \text{Rad}(M).
\]
Hence \( f(S) \) is a weak Goldie-Rad-supplement of \( N \) in \( M \). Since \( f(S) \leq L \), \( N \) has ample weak Goldie-Rad-supplements in \( M \). Thus \( M \) is amply weakly Goldie-Rad-supplemented.

Corollary 3. Every projective distributive weakly Goldie-Rad-supplemented module is an amply weakly Goldie-Rad-supplemented module.

Proof. Since every projective module is \( \pi \)-projective, every projective and distributive weakly Goldie-Rad-supplemented module is an amply weakly Goldie-Rad-supplemented module by Theorem 2.

Corollary 4. Let \( M = \bigoplus_{i=1}^{n} M_i \) be a distributive module and \( M_1, M_2, \ldots, M_n \) be projective modules. Then \( M = \bigoplus_{i=1}^{n} M_i \) is amply weakly Goldie-Rad-supplemented if and only if for every \( 1 \leq i \leq n \), \( M_i \) is amply weakly Goldie-Rad-supplemented.

Proof. "\( \Rightarrow \)" is clear from Corollary 1.
"\( \Leftarrow \)" Since \( M_i \) is amply weakly Goldie-Rad-supplemented, \( M_i \) is weakly Goldie-Rad-supplemented. Let \( U \leq M \) and \( U_i = M_i \cap U \). There exists \( S_i \leq M_i \) such that \( S_i \beta^{**} U_i, S_i + U_i = M_i, S_i \cap U_i \leq \text{Rad}(M_i) \) for \( i = 1, \ldots, n \).

By [4, Proposition 2.5], \( U \beta^{**} (\Sigma_{i=1}^{n} S_i) \). Moreover, \( U + (\Sigma_{i=1}^{n} S_i) = M \) and \( U \cap (\Sigma_{i=1}^{n} S_i) = \Sigma_{i=1}^{n} (S_i \cap U_i) \leq \Sigma_{i=1}^{n} \text{Rad}(M_i) \leq \text{Rad}(\Sigma_{i=1}^{n} M_i) = \text{Rad}(M) \).

This means that, \( (\Sigma_{i=1}^{n} S_i) \) is a weak Goldie-Rad-supplement of \( U \) in \( M \). Hence \( M \) is weakly Goldie-Rad-supplemented. Since, for every \( 1 \leq i \leq n \), \( M_i \) is projective, \( M = \bigoplus_{i=1}^{n} M_i \) is also projective. Then \( M \) is amply weakly Goldie-Rad-supplemented by Corollary 3.
**Proposition 4.** Let $M$ be an amply (weakly) Goldie-Rad-supplemented module. If $M$ satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules, then $M$ is Artinian.

*Proof.* Let $M$ be an amply (weakly) Goldie-Rad-supplemented module which satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules. Then $\text{Rad}(M)$ is Artinian by Theorem 1. It suffices to show that $M/\text{Rad}(M)$ is Artinian. Let $N$ be any submodule of $M$ containing $\text{Rad}(M)$. Then there exists a (weak) Goldie-Rad-supplement $S$ of $N$ in $M$, i.e., $M = N + S$, $N \cap S \leq \text{Rad}(S) \leq \text{Rad}(M)$ and $N \beta \ast S$. Thus $M/\text{Rad}(M) = (N/\text{Rad}(M)) \oplus ((S + \text{Rad}(M))/\text{Rad}(M))$ and so every submodule of $M/\text{Rad}(M)$ is a direct summand. Therefore $M/\text{Rad}(M)$ is semisimple.

Now suppose that $\text{Rad}(M) \leq N_1 \leq N_2 \leq N_3 \leq \cdots$ is an ascending chain of submodules of $M$. Because $M$ is amply (weakly) Goldie-Rad-supplemented, there exists a descending chain of submodules $S_1 \geq S_2 \geq S_3 \geq \cdots$ such that $S_i$ is a (weak) Goldie-Rad-supplement of $N_i$ in $M$ for each $i \geq 1$. By hypothesis, there exists a positive integer $t$ such that $S_t = S_{t+1} = S_{t+2} = \cdots$. Because $M/\text{Rad}(M) = N_t/\text{Rad}(M) \oplus (S_t + \text{Rad}(M))/\text{Rad}(M)$ for all $i \geq t$, it follows that $N_t = N_{t+1} = \cdots$. Thus $M/\text{Rad}(M)$ is Noetherian and since $M/\text{Rad}(M)$ is semisimple, by [6, 31.3] $M/\text{Rad}(M)$ is Artinian, as desired. \hfill \Box

**Corollary 5.** Let $M$ be a finitely generated amply (weakly) Goldie-Rad-supplemented module. If $M$ satisfies DCC on small submodules, then $M$ is Artinian.

*Proof.* Since $M/\text{Rad}(M)$ is semisimple and $M$ is finitely generated, then by [6, 31.3] $M/\text{Rad}(M)$ is Artinian. Now that $M$ satisfies DCC on small submodules, $\text{Rad}(M)$ is Artinian by Theorem 1. Thus $M$ is Artinian. \hfill \Box

**Corollary 6.** Let $M$ be a radical module ($\text{Rad}(M) = M$). Then $M$ is Artinian if and only if $M$ is an amply (weakly) Goldie-Rad-supplemented module and satisfies DCC on (weak) Goldie-Rad-supplement submodules and on small submodules.

*Proof.* "$\Leftarrow$" is clear by Proposition 4.

"$\Rightarrow$" It suffices to prove that $M$ is amply (weakly) Goldie-Rad-supplemented. It is well known that a module $M$ is Artinian if and only if $M$ is an amply supplemented module and satisfies DCC on supplement submodules and on small submodules. Since an amply supplemented
module is amply Rad-supplemented and for every submodules $N, S$ of $M$, $N^{\beta**}S$, $M$ is amply (weakly) Goldie-Rad-supplemented, as desired. □

References


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