A criterion of elementary divisor domain for distributive domains

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Abstract. In this paper we introduce the notion of the neat range one for Bezout duo-domains. We show that a distributive Bezout domain is an elementary divisor domain if and only if it is a duo-domain of neat range one.

A problem of describing elementary divisor rings is classical and far from its completion. The most full history of this problem and close to it problems can be found in [4]. In the case of commutative rings there are many developments on this problem in the case of noncommutative rings it is little investigated and fragmented. A general picture is far from its full description.

Among these results are should especially note a result of [5] which shows that a distributive elementary divisor domain is a duo-domain. Tuiganbaev extended this result in case of a distributive ring [3].

In this paper we give a criterion when a distributive domain is an elementary divisor domain.

We start with necessary definitions and facts. Under a ring $R$ we understand an associative ring with 1, and $1 \neq 0$. We say that matrices $A$ and $B$ over a ring $R$ are equivalent if exist invertible matrices $P$ and $Q$ of appropriate sizes such that $B = PAQ$. The fact that matrices $A$ and $B$ are equivalent is denoted by $A \sim B$. If for a matrix $A$ there exists a diagonal matrix $D = (d_i)$ such that $A \sim D$ and $Rd_{i+1}R \subseteq d_iR \cap Rd_i$ for every $i$


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then we say that the matrix $A$ has a canonical diagonal reduction. A ring $R$ is an elementary divisor ring if every matrix over $R$ has a canonical diagonal reduction. If over a ring $R$ every $1 \times 2$ ($2 \times 1$) matrix has a canonical diagonal reduction then $R$ called a right (left) Hermite ring.

A ring which is both a right and left Hermite ring is called an Hermite ring. We note that a right Hermite ring is a right Bezout ring that is a ring in which every finitely generated right ideal is principal [1], [4].

A ring $R$ is called clean if every element of $R$ is the sum of an idempotent and a unit. A ring $R$ is called an exchange ring if for every element $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$, $1 - e \in (1 - a)R$. [2].

A ring $R$ is called a ring of stable range one if for every $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $(a + bt)R = R$.

A ring $R$ is called right (left) distributive if every lattice right (left) ideal of ring $R$ is distributive. A distributive ring is a ring which is both right and left distributive ring [3].

A right (left) quasi-duo ring is a ring in which every a right (left) maximal ideal is ideal. In the case of distributive right (left) Bezout rings a connection with right (left) quasi-duo rings is established by the following theorem.

**Theorem 1.** [3] The following properties are equivalent for a Bezout ring $R$.

1) $R$ is a distributive ring.
2) $R$ is a quasi-duo ring.
3) From the condition $aR + bR = R$ it follows that $Ra + Rb = R$ for every elements $a, b \in R$.
4) From the condition $Ra + Rb = R$ it follows that $aR + bR = R$ for every elements $a, b \in R$.

**Theorem 2.** [5] Any distributive elementary divisor domain is a duo-domain.

**Definition 1.** We say that a duo-ring $R$ has neat range one if for every $a, b \in R$ such that $aR + bR = R$ there exists an element $t \in R$ such that $aR/(a + bt)R$ is a clean ring.

We note that every duo-ring of stable range one is a ring of neat range one.

The following two theorems are the main result of this paper.

**Theorem 3.** Any Bezout duo-domain is an elementary divisor domain if and only if it is a domain of neat range one.
Theorem 4. Any distributive Bezout domain is an elementary divisor domain if and only if it is a duo-domain of neat range one.

Theorem 3 is a consequence of Theorem 5 and Proposition 4.

Theorem 4 is a consequence of Theorems 2 and 3.

We prove the following result which will be useful in the forthcoming research. Recall that a row \((a_1, \ldots, a_n)\) of elements of a ring \(R\) is called unimodular if \(a_1R + \ldots + a_nR = R\).

**Proposition 1.** Let \(R\) be a right Hermite ring, then every unimodular row \((a_1, \ldots, a_n)\) with elements of the ring \(R\) can be completed to an invertible matrix.

**Proof.** Since \(R\) is a right Hermite ring and \(a_1R + \ldots + a_nR = R\), then

\[
(a_1, \ldots, a_n)P = (1, 0 \ldots 0)
\]

for some matrix \(P\) of order \(n\) over the ring \(R\). Note that

\[
P^{-1} = (p_{ij}).
\]

From equality (1) we have

\[
(a_1, \ldots, a_n) = (1, 0 \ldots 0)P^{-1},
\]

then \(a_1 = p_{11}, \ldots, a_n = p_{1n}\) and hence the row \((a_1, \ldots, a_n)\) is the first row invertible matrix \(P^{-1}\). The proposition is proved.

**Proposition 2.** A Hermite duo-ring \(R\) is an elementary divisor ring if for such any elements \(a, b, c \in R\) such that \(aR + bR + cR = R\) there exist elements \(p, q \in R\) such that \((pa)R + (pb + qc)R = R\).

**Proof.** Let \(R\) be an elementary divisor ring. Let \(aR + bR + cR = R\). The matrix \(A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\) has canonical diagonal reduction, i.e., there exists invertible matrices \(P = \begin{pmatrix} p & q \\ * & * \end{pmatrix} \in GL_2(R)\), \(Q \in GL_2(R)\) such that

\[
PAQ = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}.
\]

Hence we get that \((pa)R + (pb + qc)R = R\). The necessity is proved.
In order to prove sufficiency according to [1] it is enough to prove that every matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where $aR + bR + cR = R$ has canonical diagonal reduction. We see that $(pa)R + (pb + qc)R = R$ for some elements $p, q \in R$. Hence $pR + qR = R$, as $R$ is an Hermite ring and the row $(p, q)$, by Proposition 1, is adding to an invertible matrix $P \in GL_2(R)$.

Obviously, the matrix $PA$ has canonical diagonal reduction. The proposition is proved.

**Proposition 3.** Let $R$ be a Bezout duo-domain. For every elements $a, b, c \in R$ such that $aR + bR + cR = R$ the following conditions are equivalent:

1) There exist elements $p, q \in R$ such that $paR + (pb + qc)R = R$;

2) There exist elements $\lambda, u, v \in R$ such that $b + \lambda c = v \cdot u$, where $uR + aR = R$, $vR + cR = R$.

**Proof.** 1) $\Rightarrow$ 2) Let condition 1) be true. Then it follows that $pR + qcR = R$ and hence $pR + cR = R$. Since $R$ is a duo-ring, $Rp + Rc = R$. Hence $vp + jc = 1$ for some elements $v, j \in R$. Then $vpb - b = jcb = ct$ for $t \in R$. Note that since $R$ is a duo-ring, then $t = jc$, where $jc = cj'$.

Then $v(pb + qc) = vpb + vqc = b + ct + vqc = b + ct + ck$, that is $v(pb + qc) - b \in cR$, that is $v(pb + qc) - b = c\lambda$ for some $\lambda \in R$. We note that such an element $k$ exists, since $R$ is a duo-ring. Namely, $vqc = ck$. Hence $vR + cR = R$ and $uR + aR = R$ where $u = pb + qc$. We note that the condition $uR + aR = R$ follows obviously from the condition $paR + (pb + qc)R = R$. Condition 2) is proved.

2) $\Rightarrow$ 1) We assume that exists an element $\lambda \in R$ such that $b + c\lambda = vu$, where $vR + cR = R$ and $uR + aR = R$. Since $vR + cR = R$ then $Rv + Rc = R$ and $pv + jc = 1$ for some elements $p, j \in R$.

We note that $pR + cR = R$. Then $pb = p(vu - c\lambda) = (pv)u - pc\lambda = (1 - jc)u - pc\lambda = u - qc$ for an element $q \in R$. Hence $u = pb + qc$. Therefore, $(pb + qc)R + aR = R$ and $pR + cR = R$. Since $R$ is a Bezout duo-domain, let $pR + qR = dR$, where $p = dp_1$, $q = dq_1$ and $p_1R + q_1R = R$ such that $p_1R + (p_1b + q_1c)R = p_1R + q_1cR$ since $pR + cR = R$ and $p_1R + q_1R = R$ then $p_1R + (p_1b + q_1c)R = R$.

Hence $(p_1b + q_1c)R + aR = R$ and $(p_1b + q_1c)R + p_1R = R$ and hence $p_1aR + (p_1b + q_1c)R = R$. Condition 1) is true.

The proposition is proved.

**Remark 1.** In Proposition 3 we can choose the elements $u$ and $v$ such that $uR + vR = R$. 
Theorem 5. Let \( R \) be a Bezout duo-domain. Then the following conditions are equivalent.

1) \( R \) is an elementary divisor duo-domain;
2) For every elements \( x, y, z \in R \) such that \( xR + yR = R \) there exists an element \( \lambda \in R \) such that \( x + \lambda y = vu \), where \( uR + zR = R \), \( vR + (1-z)R = R \).

**Proof.** 1) \( \Rightarrow \) 2) Let \( R \) be an elementary divisor domain. By Proposition 2, then for every elements \( a, b, c \in R \) such that \( aR + bR + cR = R \) there exist elements \( p, q \in R \) such that \( paR + (pb + qc)R = R \).

We obtain Condition 2 of Proposition 3 to the elements \( a = z, b = x, c = y(1-z) \).

It is complicated to prove the fact that from Condition 2) of our theorem we obtain the condition that for every \( a, b, c \in R \) such that \( R \) there exist elements \( p, q \in R \) such that \( paR + (pb + qc)R = R \)

2) \( \Rightarrow \) 1) Put \( x = b_1, y = c_1, z = d_1d \). By Condition 2) of our theorem, there exists an element \( \lambda_1 \in R \) such that \( b_1 + c_1 \lambda_1 = vu_1 \) where \( u_1 R + (1-d_1)R = R \), \( vR + d_1 dR = R \).

We have that \( vR + d_1 R = R \) and \( vR + dR = R \). Remark that \( vR + cR = vR + d_1 cR = vR + cR \) as \( xR = bR + cR = \lambda c \). Since \( (1-d_1)dR = R \) and also the fact that \( u_1 R + (1-d_1)R = R \), then \( u_1 R + aR = R \). We show that \( u = u_1 d \) hence \( uR + aR = R \). Let \( \lambda \in R \) be such that \( c_1 \lambda_1 = \lambda c_1 \).

The theorem is proved. \( \Box \)

**Proposition 4.** Let \( R \) be a Bezout duo-domain and \( c \in R \setminus \{0\} \). Then \( \overline{R} = R/cR \) is a clean ring if and only if for every element \( a \in R \) there exist elements \( v, u \) such that \( c = vu \) where \( uR + aR = R \), \( vR + (1-a)R = R \), \( uR + vR = R \).

**Proof.** Let \( R \) be a clean ring. According to [2], \( R \) is an exchange ring. Let \( \overline{a} = a + cR \). Then there exists an idempotent \( \overline{e} \in \overline{R} \) such that \( \overline{e} \in \overline{aR} \), \( 1-\overline{e} \in (1-\overline{a})\overline{R} \). Since \( \overline{e} \in \overline{aR} \), \( e - ap = cs \) for elements \( p, s \in R \). Similarly, \( 1 - e - (1-a)\alpha = c\beta \) for elements \( \alpha, \beta \in R \). Since \( \overline{e^2} = \overline{e} \), then \( e(1-e) = ct \)
for an element \( t \in R \). Let \( eR + cR = dR \). Hence \( e = de_0, c = dc_0 \) for elements \( e_0, c_0 \in R \) such that \( e_0R + c_0R = R \), hence \( e_0(1 - e) = c_0t \) and \( e + c_0j \equiv 1 \) for every element \( j \in R \).

Denote that \( v = d, u = c_0 \) we have \( c = vu \). Since \( e = 1 - c_0j \), then \( uR + eR = R \). Since \( e = ap + cs \), then \( uR + aR = R \). We show that \( vR + (1 - a)R = R \). As \( 1 - e + (1 - a)\alpha = c\beta \) and \( e = de_0, c = dc_0 \) hence \( 1 - de_0 + (1 - a)\alpha = dc_0\beta \) and this means that \( d(e_0 + c_0\beta) + (1 - a)\alpha = 1 \), thus \( dR + (1 - a)R = R \) that is \( vR + (1 - a)R = R \). The necessity is proved.

Let \( c = vu \), where \( uR + aR = R, vR + (1 - a)R = R \). Let \( \bar{u} = u + cR, \bar{v} = v + cR \). From the equality \( uR + vR = R \) we have \( ur + vs = 1 \) for some elements \( r, s \in R \). Hence \( vur + v^2s = v \) and \( u^2r + uvs = u \) and this means that \( \bar{v}^2\bar{s} = \bar{v}, \bar{u}^2\bar{r} = \bar{u} \).

Let \( \bar{v}s = \bar{v} \), it is obvious that \( \bar{v}^2 = \bar{e} \) and \( \bar{1} - \bar{e} = \bar{u} \). Since \( uR + aR = R \), we have \( ux + ay = 1 \) for elements \( x, y \in R \). Hence \( vux + vay = v, vuxs + vays = vs \).

Let \( va = av' \) for some element \( v' \). Hence \( vuxs + av'y = vs \) and this means that \( \bar{a}v'y\bar{s} = \bar{v} \cdot \bar{s} \) that is \( \bar{a}j = \bar{e} \) for \( j \in R \) that is \( \bar{e} \in \bar{a}R \). Similarly, from the equality \( vR + (1 - a)R = R \) it follows that \( \bar{1} - \bar{e} \in (\bar{1} - \bar{a})R \).

According to [2], \( \bar{R} \) is a clean ring. The proposition is proved. \( \square \)

References


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