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Some properties
of the nilradical and non-nilradical graphs
over finite commutative ring $\mathbb{Z}_n$

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Abstract. Let $\mathbb{Z}_n$ be the finite commutative ring of residue
classes modulo $n$ with identity and $\Gamma(\mathbb{Z}_n)$ be its zero-divisor graph.
In this paper, we investigate some properties of nilradical graph,
denoted by $N(\mathbb{Z}_n)$ and non-nilradical graph, denoted by $\Omega(\mathbb{Z}_n)$ of
$\Gamma(\mathbb{Z}_n)$. In particular, we determine the Chromatic number and
Energy of $N(\mathbb{Z}_n)$ and $\Omega(\mathbb{Z}_n)$ for a positive integer $n$. In addition,
we have found the conditions in which $N(\mathbb{Z}_n)$ and $\Omega(\mathbb{Z}_n)$ graphs are
planar. We have also given MATLAB coding of our calculations.

Introduction

The concept of zero-divisor graph was introduced by I. beck in 1988 but
the most common definition of zero-divisor graph given by D. F. Anderson
and P. S. Livingston in 1999 is as follows: “Let $R$ be a commutative ring
(with 1) and let $Z(R)$ be its set of zero-divisors. We associate a simple
graph $\Gamma(R)$ to $R$ with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero
zero-divisors of $R$, and for distinct $x, y \in Z(R)^*$, the vertices $x$ and $y$ are
adjacent if and only if $xy = 0$. Thus, $\Gamma(R)$ is the empty graph if and only
if $R$ is an integral domain.” We have derived some results for the ring $\mathbb{Z}_n$.

A complete graph is a graph (without loops and multiple edges) in
which every vertex is adjacent to any other vertices of the graph. A graph
in which all vertices have the same degree is said to be a regular graph.
A complete bipartite graph is a graph whose vertices can be divided into

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non-nilradical graph, chromatic number, planar graph, energy of a graph.
two sets such that every vertex in one set is connected to every vertex in
the other, and no vertex is connected to any other vertices in the same set.
A star graph is a complete bipartite graph in which at least one of the two
vertex sets contains only one vertex. That one vertex is called the center of
the star graph. A vertex of a graph is isolated if there is no edge incident
on it. A graph is almost connected if there exists a path between any two
non-isolated vertices. A proper coloring of a graph \( Z_n \) is a function that
assigns a color to each vertex such that no any two adjacent vertices have
the same color. The chromatic number of \( Z_n \), denoted by \( \chi(Z_n) \), is the
smallest number of colors required for proper coloring. A planar graph is
a graph that can be embedded in the plane, i.e, it can be drawn on the
plane in such a way that its edges intersect only at their endpoints and
we will repeatedly use Kuratowski’s theorem, which states that a graph is planar if and only if it does not contain a subdivision of \( K_5 \) or \( K_{3,3} \).
The energy of a graph is the sum of absolute value of all eigenvalues of the
adjacency matrix. The adjacency matrix corresponding to a zero divisor
graph is defined as \( A = [a_{i,j}] \), where \( a_{i,j} = 1 \), if \( v_i \) \& \( v_j \) represent zero
divisor, i.e., \( v_i.v_j = 0 \) and \( a_{i,j} = 0 \) otherwise, where \( v_i \) and \( v_j \) are vertices
of the graph.

Nilradical and non-nilradical graphs

**Definition 1.1.** The nilradical graph of \( Z_n \), denoted by \( N(Z_n) \), is the
graph whose vertices are the nonzero nilpotent elements of \( Z_n \) and any
two vertices are connected by an edge if and only if their product is 0.

**Definition 1.2.** The non-nilradical graph of \( Z_n \), denoted by \( \Omega(Z_n) \), is
the graph whose vertices are the non-nilpotent zero-divisors of \( Z_n \) and
any two vertices are connected by an edge if and only if their product
is 0.

1. Chromatic number and planarity of nilradical and non-nilradical graphs

**Theorem 1.** If \( p \) and \( q \) are distinct prime numbers and \( n \) is a positive
integer, then

(1) \( \chi(N(Z_n)) = 0 \) if \( n = pq \);
(2) \( \chi(N(Z_n)) = p - 1 \) if \( n = p^2 \);
(3) \( \chi(N(Z_n)) = pq - 1 \) if \( n = p^2q^2 \);
(4) \( \chi(N(Z_n)) = p \) if \( n = p^3 \);
(5) \( \chi(N(Z_n)) = p - 1 \) if \( n = p^2q \).
Proof. (1) Let \( n = pq \), where \( p \) and \( q \) are distinct primes. Then \( N(\mathbb{Z}_n) \) is an empty graph. So, there is no need of any color for coloring the graph. Hence, chromatic number is zero.

(2) Let \( n = p^2 \), where \( p \) is a prime number. If \( p = 2 \), then \( N(\mathbb{Z}_n) \) has only one vertex. This implies the chromatic number is one. If \( p \geq 3 \), then the number of nilpotent elements which are divisible by \( p^2 \) are \((p - 1)\). Also, these \((p - 1)\) nilpotent elements form a complete graph. So, \((p - 1)\) colors are required for coloring the graph and these \((p - 1)\) colors are minimum in numbers. Therefore, chromatic number is \((p - 1)\).

(3) Let \( n = p^2 q^2 \), where \( p \) and \( q \) are prime numbers and \( p \neq q \). Then the nilpotent elements are multiple of \( pq \) and number of nilpotent elements are \( pq - 1 \). Also, these \((pq - 1)\) elements are connected to each other. Thus, \((pq - 1)\) colors are required for coloring the graph. Hence, chromatic number of \( N(\mathbb{Z}_{p^2 q^2}) \) is \((pq - 1)\).

(4) If \( n = p^3 \), where \( p \) is a prime number, then \( N(\mathbb{Z}_n) \) is a complete \( p \)-partite graph with \((p^2 - 1)\) vertices. Therefore, we required \( p \) colors for proper coloring. Hence, chromatic number of \( N(\mathbb{Z}_n) \) is \( p \).

(5) Let \( n = p^2 q \), where \( p \) and \( q \) are distinct prime numbers. Then the nilpotent elements are multiple of \( pq \), and the number of nilpotent elements are \((p - 1)\). These \((p - 1)\) elements are connected to each other and form a complete graph with \((p - 1)\) vertices. Therefore, \((p - 1)\) colors are required for coloring the graph \( N(\mathbb{Z}_{p^2 q}) \). Hence, chromatic number of \( N(\mathbb{Z}_{p^2 q}) \) is \((p - 1)\).

\[ \square \]

Theorem 2. Let \( p \) and \( q \) be two distinct prime numbers and \( n \) a positive integer. Then

1. \( \chi(\Omega(\mathbb{Z}_n)) = m \) if \( n = p_1 p_2 p_3 \ldots p_m \), \( m \geq 1 \), where \( p_1, p_2, \ldots, p_m \) are distinct primes;
2. \( \chi(\Omega(\mathbb{Z}_n)) = 0 \) if \( n = p^2 \);
3. \( \chi(\Omega(\mathbb{Z}_n)) = 0 \) if \( n = p^3 \);
4. \( \chi(\Omega(\mathbb{Z}_n)) = 2 \) if \( n = p^2 q \), for \( q = 2 \) or \( 3 \).

Proof. (1) Let \( n = p_1 p_2 p_3 \ldots p_m \), for some positive integer \( m \), such that all \( p_i \) are distinct prime numbers. Then \( \Omega(\mathbb{Z}_n) \) is equal to \( \Gamma(\mathbb{Z}_n) \) and since \( \Gamma(\mathbb{Z}_n) \) is \( m \)-partite graph, therefore \( \Omega(\mathbb{Z}_n) \) is also \( m \)-partite graph. In this case, \( m \) distinct colors are needed for proper coloring of the graph \( \Omega(\mathbb{Z}_n) \). Thus, Chromatic number of graph \( \Omega(\mathbb{Z}_n) \) is \( m \).

(2) Let \( n = p^2 \), where \( p \) is a prime number. Then clearly \( \Omega(\mathbb{Z}_n) \) is an empty graph. Hence, there is no need of any color for coloring the graph \( \Omega(\mathbb{Z}_n) \). Hence, chromatic number is zero.
(3) Let \( n = p^3 \), where \( p \) is a prime number. Then \( \Omega(\mathbb{Z}_n) \) is an empty graph. Hence, there is no need of any color for coloring the graph \( \Omega(\mathbb{Z}_n) \). So, chromatic number is zero.

(4) Let \( n = p^2 q \), where \( p \) and \( q \) are distinct prime numbers. Then multiple of \( p \), \( p^2 \) and \( q^2 \) are not adjacent to themselves. But the vertices which are multiple of \( p^2 \) are adjacent to those vertices which are multiple of \( q \) and not adjacent with multiple of \( p \). Similarly, elements which are multiple of \( q \) are not adjacent with multiple of \( p \). Thus, there are two disjoint sets of vertices which are adjacent from one set to other but not adjacent to each other in a set. Therefore, two colors are required for coloring the \( \Omega(\mathbb{Z}_n) \) graph and also we can use one color from them for isolated vertices. Hence, chromatic number is two for \( \Omega(\mathbb{Z}_n) \), when \( n = p^2 q \), where \( p, q \) are distinct prime numbers.

**Theorem 3.** If \( p \) and \( q \) are distinct prime numbers and \( n \) is a positive integer, then

1. \( N(\mathbb{Z}_n) \) is planar, where \( n = pq \);
2. \( N(\mathbb{Z}_n) \) is planar for \( p \leq 5 \) and non-planar for \( p > 5 \), where \( n = p^2 \);
3. \( N(\mathbb{Z}_n) \) is planar for \( p \leq 5 \) and \( q \) is any prime number, where \( n = p^2 q \);
4. \( N(\mathbb{Z}_n) \) is planar, if \( p < 5 \) and non-planar for \( p \geq 5 \), where \( n = p^3 \);
5. \( N(\mathbb{Z}_n) \) is planar, where \( n = 4k, \gcd(2, k) = 1, p^2 \not| k \) for any prime \( p \) and \( k \) is any positive integer;
6. \( N(\mathbb{Z}_n) \) is planar, where \( n = 9k, \gcd(3, k) = 1, p^2 \not| k \) for any prime \( p \) and \( k \) is any positive integer.

**Proof.**

(1) If \( n = pq \), where \( p \) and \( q \) are distinct prime numbers, then \( N(\mathbb{Z}_n) \) is an empty graph. Therefore, \( N(\mathbb{Z}_n) \) graph is a planar graph.

(2) If \( n = p^2 \), where \( p \) is a prime number, then the nilpotent elements of \( (\mathbb{Z}_n) \) are multiple of \( p \). So, there are \((p - 1)\) nilpotent elements which form a complete graph with \((p - 1)\) vertices and all vertices are adjacent to each other. If \( p = 2 \), then \( N(\mathbb{Z}_n) \) has only one vertex and when \( p = 3 \), then \( N(\mathbb{Z}_n) \) has two vertices. In this case, \( N(\mathbb{Z}_n) \) is a planar graph. If \( p = 5 \), then \( N(\mathbb{Z}_n) \) is a complete graph with 4 vertices and all vertices are adjacent to each other. Therefore, \( N(\mathbb{Z}_n) \) is a planar graph.

For \( p > 5 \), \( N(\mathbb{Z}_n) \) graph contains \( K_{3,3} \) or \( K_5 \) as a proper subgraph. Hence, \( N(\mathbb{Z}_n) \) is not a planar graph for \( p > 5 \).

(3) If \( n = p^2 q \), where \( p \) and \( q \) are distinct prime numbers, then \( N(\mathbb{Z}_n) \) is a complete graph with \((p - 1)\) vertices. Thus, \( N(\mathbb{Z}_n) \) is a planar graph only when \( p \leq 5 \) and \( q \) is any prime, \( p \neq q \), otherwise \( N(\mathbb{Z}_n) \) contains
as a subgraph which is not planar and therefore \( N(Z_n) \) is a planar if \( p \leq 5 \).

(4) If \( n = p^3 \), where \( p \) is any prime, then \( N(Z_n) \) is a complete \( p \)-partite graph with \( (p^2 - 1) \) vertices. Therefore, \( N(Z_n) \) is planar for \( p < 5 \) and non-planar for \( p \geq 5 \).

(5) If \( n = 4k \), and \( p^2 \nmid k \), for a prime \( p \) and \( k \) is any positive integer, then \( N(Z_n) \) has only one vertex, hence \( N(Z_n) \) graph is a planar graph.

(6) If \( n = 9k \), \( p^2 \nmid k \), for all prime \( p \) and \( k \) is any positive integer, then \( N(Z_n) \) has two vertices which are adjacent to each other. Thus, \( N(Z_n) \) is a planar graph.

Theorem 4. If \( p \) and \( q \) are distinct prime numbers and \( n \) is a positive integer, then

1. \( \Omega(Z_n) \) is not planar, for \( n = pq \), (specially \( p \geq 5 \) and \( q \geq 3 \));
2. \( \Omega(Z_n) \) is planar, for \( n = p^2 \);
3. \( \Omega(Z_n) \) is planar, for \( n = p^3 \);
4. \( \Omega(Z_n) \) is planar for \( k \leq 6 \) and non-planar for all \( k > 6 \), where \( n = 4k \), \( \gcd(2, k) = 1 \) and \( p^2 \nmid k \), for a prime \( p \) and \( k \) is any positive integer;
5. \( \Omega(Z_n) \) is a planar for \( k \leq 4 \) and non-planar for all \( k > 4 \), where \( n = 9k \), \( \gcd(3, k) = 1 \) and \( p^2 \nmid k \), for a prime \( p \) and \( k \) is any positive integer;
6. \( \Omega(Z_n) \) is planar for \( q = 2 \) and \( 3 \), and \( p \) is any prime number, where \( n = p^2q \).

Proof. (1) Let \( n = pq \), such that \( p \) and \( q \) are distinct primes. Then clearly \( \Omega(Z_n) \) is a bi-partite graph. If, we take \( n = pq \) where \( p = 2 \) and \( q \) is any prime number, then \( \Omega(Z_n) \) is a star graph. We know that star graph is a planar graph. Hence, \( \Omega(Z_n) \) is a planar graph in this case. If \( p = 3 \) and \( q \) is any prime number, then \( \Omega(Z_n) \) is a complete bi-partite graph, which is a planar graph.

(2) Let \( n = p^2 \), where \( p \) is any prime number. Then, there are no non-nilpotent elements of \( Z_n \) in \( \Omega(Z_n) \). Therefore, \( \Omega(Z_n) \) is an empty graph. Hence, \( \Omega(Z_n) \) is a planar graph.

(3) Let \( n = p^3 \), where \( p \) is any prime number. Then, there is no non-nilpotent element of \( Z_n \) in \( \Omega(Z_n) \). Therefore, \( \Omega(Z_n) \) is an empty graph. Hence, \( \Omega(Z_n) \) is a planar graph.

(4) Let \( n = 4k \), where \( p^2 \nmid k \), for a prime \( p \) and \( k \) is any positive integer. Then, \( \Omega(Z_n) \) is planar for \( k \leq 6 \). If we take \( k \) is any prime number,
then \( \Omega(Z_n) \) is always complete bi-partite graph. We know that complete bi-partite graph is planar graph. Therefore, \( \Omega(Z_n) \) is the planar graph for the prime \( k \). On the other hand, if \( k > 6 \), then \( \Omega(Z_n) \) graph contains \( K_{3,3} \) or \( K_5 \) as a subgraph. Thus, for \( k > 6 \), \( \Omega(Z_n) \) graph is not a planar.

(5) Let \( n = 9k \), where \( p^2 \nmid k \), for a prime \( p \) and \( k \) is any positive integer. Then \( \Omega(Z_n) \) is a planar graph for \( k \leq 4 \). For \( k \geq 5 \), \( \Omega(Z_n) \) graph contains \( K_{3,3} \) as a subgraph. Thus, \( \Omega(Z_n) \) is non-planar.

(6) Let \( n = p^2q \), where \( p \) and \( q \) are distinct primes. If \( q = 2 \) and \( p \) is any prime number, \( \Omega(Z_n) \) graph is a star graph. Therefore, \( \Omega(Z_n) \) is planar graph. If \( q = 3 \) and \( p \) is any prime number, \( \Omega(Z_n) \) graph is a complete bi-partite graph. Therefore, \( \Omega(Z_n) \) is planar graph. For \( q \geq 5 \) and \( p \) is any prime greater than 2 (and 3), \( \Omega(Z_n) \) graph contains \( K_{3,3} \) or \( K_5 \) as a subgraph. Thus, \( \Omega(Z_n) \) is non-planar.

**Lemma 1.** If \( n = pq \), where \( p \) and \( q \) are primes, then there is no isolated vertex in \( \Omega(Z_n) \) graph.

*Proof.* If \( n = pq \), where \( p \) and \( q \) are distinct primes, \( \Omega(Z_n) \) is a complete bi-partite graph. Hence, there is no isolated vertex. When \( n = p^2 \), for any prime \( p \), there is no vertex in \( \Omega(Z_n) \). Hence, graph is empty. Thus, in this case again we have no isolated vertex.

**Lemma 2.** If \( n = p^3 \), for any prime \( p \), \( \Omega(Z_n) \) graph has no isolated vertex.

*Proof.* If \( n = p^3 \), then zero divisor graph has \( p^2 - 1 \) elements in which all elements are nilpotent and no element is non-nilpotent. Also all nilpotent elements are adjacent with nilpotent elements, but in \( \Omega(Z_n) \), there are no non-nilpotent elements. Thus, \( \Omega(Z_n) \) is an empty graph. Therefore, \( \Omega(Z_n) \) graph has no isolated vertex.

**Observation 1.** If \( n = p^2q \), for \( p \) and \( q \) are distinct prime numbers, \( \Omega(Z_n) \) graph has \((p-1)(q-1)\) isolated vertices.

2. **Energy of nilradical and non-nilradical graphs**

**Theorem 5.** If \( n = p^2 \), for prime \( p \), then \( E(N(Z_n)) \) is \((2p - 4)\) and \( E(\Omega(Z_n)) \) is zero \((E(\Omega(Z_n)) \) is zero also for \( p^3 \)).

*Proof.* When \( n = p^2 \), \( N(Z_n) \) is a complete graph with \( p - 1 \) vertices. Then \( f(\lambda) = |\lambda I_{p-1} - M(N(Z_n))| = (\lambda - 1)p^{-2}(\lambda + p - 2) \) by [2], where \( M \)
is a matrix of order \((p - 1)\). If \(f(\lambda) = 0\), then \(\lambda = 1, 2 - p\). Therefore,
\[
\sum_{i=1}^{p-1} |\lambda_i| = 2p - 4.
\]

When \(n = p^2\), then \(\Omega(Z_n)\) graph is an empty graph. Hence, it has zero energy.

When \(n = p^3\), then \(\Omega(Z_n)\) is an empty graph and hence, it has zero energy.

**Theorem 6.** If \(n = pq\), where \(p\) and \(q\) are distinct primes, then energy of \(\Omega(Z_n)\) is \(2\sqrt{(p - 1)(q - 1)}\) and energy of \(N(Z_n)\) is zero.

**Proof.** Let \(n = pq\), where \(p\) and \(q\) are two distinct prime. Then \(\Omega(Z_n)\) is a bi-partite graph. Also, its eigen polynomial \(f(\lambda) = \lambda M(q) - M(\Omega(Z_n)) = \lambda^p + q - 2 - \lambda^p - q - 4\), where \(M\) is a matrix of order \((p + q - 2)\). Thus, nonzero eigenvalues are \(\pm \sqrt{(p - 1)(q - 1)}\) and so \(E(\Omega(Z_n)) = 2\sqrt{(p - 1)(q - 1)}\). Also, \(N(Z_n)\) graph has no vertices for distinct primes \(p\) and \(q\). Thus, \(E(N(Z_n))\) has no energy. \(\square\)

**Theorem 7.** For \(n = p^2q\), energy of \(N(Z_n)\) is \(2p - 4\), for all distinct primes \(p\) and \(q\).

**Proof.** Same as above Theorem (5). \(\square\)

**Observation 2.** If \(n = p^2q\), then energy of \(\Omega(Z_n)\) is:

1. \(2\sqrt{pq - 2}\), for \(p = 2\) and \(q\) is any prime number;
2. \(2\sqrt{pq + p(q - 2)}\), for \(p = 3\) and \(q\) is any prime number;
3. \(2\sqrt{2pq + 2p(q - 2)}\), for \(p = 5\) and \(q\) is any prime number.

3. **Computer program**

Now, we offer three algorithms for calculating energy with MATLAB software. These algorithms include several sub-algorithms. It is enough to input \(n\). In the first algorithm at the first stage, we obtain \(M(N(Z_n))\) and plot \(N(Z_n)\) by function \texttt{nil\_radical\_zn2(p)}). At the second stage, we calculate Energy index by using \texttt{energy}.

In the second algorithm at the first stage, we obtain \(\Omega(N(Z_n))\) and plot \(\Omega(Z_n)\) by function \texttt{non\_nil\_radical\_zn2(p)}). At the second stage, we calculate Energy index by using \texttt{energy}.

In third algorithm, we put the value of \(n\) and call above two functions together.
First algorithm

function Nz=nil_radical_zn2(p)
n=p;
M=[];
for i=1:n-1
    for j=1:n-1
        if mod(i*j,n)==0
            M=[M, i ];
            break;
        end
    end
end
M
n=length(M);
for i=0:n-1
    axes(i+1,:)=[cos(2*pi*i/n), sin(2*pi*i/n)];
end
Nz=zeros(n);
hold on
for i=1:n
    plot(axes(i,1),axes(i,2), '∗')
    if mod(M(i)^2,p)==0
        Nz(i,i) = 1;
        plot(axes(i,1),axes(i,2), 'ro')
    end
end
for i=1:n-1
    for j=i+1:n
        if mod(M(i)*M(j),p)==0
            Nz(i,j)=1; Nz(j,i)=1;
            plot(axes([i,j],1),axes([i,j],2));
        end
    end
end
eg=eig(Nz)
E=sum(abs(eg))

Second algorithm

function NNz=non_nil_radical_zn2(p)
n=p;
M=[];
for i=1:n-1
    for j=1:n-1
        if mod(i*j,n)==0
            M=[M, i ];
            break;
        end
    end
end
M
n=length(M);
for i=0:n-1
    axes(i+1,:)=[cos(2*pi*i/n), sin(2*pi*i/n)];
end
Nz=zeros(n);
hold on
for i=1:n
    plot(axes(i,1),axes(i,2), '∗')
    if mod(M(i)^2,p)==0
        Nz(i,i) = 1;
        plot(axes(i,1),axes(i,2), 'ro')
    end
end
for i=1:n-1
    for j=i+1:n
        if mod(M(i)*M(j),p)==0
            Nz(i,j)=1; Nz(j,i)=1;
            plot(axes([i,j],1),axes([i,j],2));
        end
    end
end
eg=eig(Nz)
E=sum(abs(eg))
if mod(i*i,n)==0
    M=[M, i ];
    break;
end
end
end
end
M
n=length(M);
for i=0:n-1
    axes(i+1,:)=[cos(2*pi*i/n), sin(2*pi*i/n)];
end
NNz=zeros(n);
hold on
for i=1:n
    plot(axes(i,1),axes(i,2), '*')
    if mod(M(i)^2,p)==0
        NNz(i,i)=1;
        plot(axes(i,1),axes(i,2), 'ro')
    end
end
end
for i=1:n-1
    for j=i+1:n
        if mod(M(i)*M(j),p)==0
            NNz(i,j)=1; NNz(j,i)=1;
            plot(axes([i,j],1),axes([i,j],2));
        end
    end
end
eg=eig(NNz)
E=sum(abs(eg))

Third algorithm

p=n;
Nz=nil_radical_zn2(p)
figure;
NNz=non_nil_radical_zn2(p)
figure;

All above algorithms are also useful for $p^3$. If we use the formula "if mod(i*j,n)==0" at the place of sixth line in the first algorithm, then it will give fruitful result for $p^3$. 
Table 1. The values of $E(N(Z_n))$ and $E(\Omega(Z_n))$ for $n = 27, 45, 77, 121, 225$ and 343.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$E(N(Z_n))$</th>
<th>$E(\Omega(Z_n))$</th>
</tr>
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References


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