On unicyclic graphs of metric dimension 2

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Abstract. A metric basis $S$ of a graph $G$ is the subset of vertices of minimum cardinality such that all other vertices are uniquely determined by their distances to the vertices in $S$. The metric dimension of a graph $G$ is the cardinality of the subset $S$. A unicyclic graph is a graph containing exactly one cycle. The construction of a knitting unicyclic graph is introduced. Using this construction all unicyclic graphs with two main vertices and metric dimensions 2 are characterized.

Introduction

The notion of a metric basis was introduced by L. Blumenthal in [1] for semimetric spaces. F. Harary and R. Melter considered the concepts of a metric basis and metric dimension for simple, connected graphs in [2]. In general case the problem to find a metric basis of a graph is NP-hard [3].

There are three main ways to study metric dimension of graphs. The first way is a characterization of metric dimension of some families of graphs. For example, in [4] metric dimension of trees was considered, in [5] metric dimension of wheels was determined, in [6] for metric dimension of unicyclic graphs some bounds were given. The second is a characterization of metric dimension of constructions of graphs (e.g. [7], [8]). The third way is a description of graphs having a fixed value of metric dimensions. For instance, in [9] it was proved that a graph $G$ has metric dimension 1

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if and only if \( G \) is a chain, the metric dimension of \( G \) equals to \( n - 1 \) if and only if \( G \) is the complete graph on \( n \) vertices and all graphs on \( n \) vertices with metric dimension \( n - 2 \) were characterized.

In this paper we characterize all unicyclic graphs with two main vertices such that their metric dimension equals 2.

1. Metric dimension of unicyclic graphs

We consider only simple, finite, undirected, connected and nontrivial graphs. Let \( G = (V, E) \) be a graph with set of vertices \( V \) and set of edges \( E \). The degree \( \deg_G(v) \) of a vertex \( v \) in \( G \) is the number of edges that incident to \( v \) in \( G \). The path between \( v_1 \) and \( v_2 \) in graph \( G \) is a sequence of vertices and edges \( v_1, e_1, v_2, e_2, \ldots, v_n \), such that any edge \( e_i \) is incident to vertices \( v_i \) and \( v_i + 1 \), \( 1 \leq i \leq n - 1 \). A unicyclic graph is a graph containing exactly one cycle.

The distance between two vertices \( v_1 \) and \( v_2 \) is denoted by \( d_G(v_1, v_2) \) and it equals to the length of the shortest path between \( v_1 \) and \( v_2 \). We denote by \( C_n \) and \( L_n \) the cycle and the path on \( n \) vertices correspondingly. For unspecified notions in graph theory we refer to [10].

A vertex \( u \) of a graph \( G \) is said to resolve two vertices \( v_1 \) and \( v_2 \) of graph \( G \) if the following inequality holds:

\[
d_G(u, v_1) \neq d_G(u, v_2).
\]

An ordered vertex set \( S \) of \( G \) is a resolving set of \( G \) if every two distinct vertices of \( G \) are resolved by some vertex of \( S \). A resolving set also is called a metric generator. A metric basis of \( G \) is a resolving set of minimum cardinality. The metric dimension of \( G \) is the cardinality of its basis. We denote metric dimension of \( G \) by the symbol \( \dim G \).

Let \( \hat{G} = (\hat{V}, \hat{E}) \) be a subgraph of the unicyclic graph \( G = (V, E) \), which is a simple cycle. In other words, \( \hat{G} \) is isomorphic to \( C_m \) for some positive integer \( m \).

**Proposition 1.** Let \( G = (V, E) \) be a unicyclic graph. If metric dimension of \( G \) equals 2 then for any \( v \in V \setminus \hat{V} \) the inequality \( \deg_G(v) \leq 3 \) holds.

**Proof.** Assume that \( w \) is a vertex of \( G \) such that \( \deg_G(w) \geq 4 \). Then there are 4 vertices \( u_1, u_2, u_3, u_4 \) such that the distance from any of these vertices to \( w \) equals 1 (see Figure 1). Then all pairs of vertices \( u_i, u_j \),
1 \leq i < j \leq 4 are resolved by some set $S$ of vertices that consists of more than three vertices. Therefore, $\dim(G) > 2$. 

In this paper we consider only graphs $G = (V, E)$ such that the inequality $\deg_G(v) \leq 3$ holds for all $v \in V$.

![Figure 1. Graph $G$.](image)

A vertex $u \in V \setminus \hat{V}$ of the graph $G$ is said to be projected in the vertex $w \in \hat{V}$ if for any vertex $q \in \hat{V}$ the inequality

$$d_G(u, w) < d_G(u, q)$$

holds. A vertex with degree 3 from $\hat{G}$, in which the vertices that have degree 3 and are located outside the cycle are projected, is called a main vertex. For example, the graph $G$ on Figure 1 has a unique main vertex $w$.

We need the following lemma from [11].

**Lemma 1** ([11]). Let $G = (V, E)$ be a unicyclic graph and $\dim(G) = 2$. Then there exist at most two main vertices in the graph $G$.

A vertex of degree at least 3 in a graph $G$ will be called an exterior vertex of $G$. Any end-vertex $u$ of $G$ is said to be a terminal vertex of an exterior vertex $v$ of the graph $G$ if for every other exterior vertex $w$ of $G$ the inequality $d_G(u, v) < d_G(u, w)$ holds. An exterior vertex $v$ will be called two-leaf if there exist two different terminal vertices of the vertex $v$. An exterior vertex $v$ will be called one-leaf if there exist exactly one terminal vertex of the vertex $v$. For example, the vertex $z_1$ is two-leaf and the vertex $w$ is one-leaf on Figure 3.

**Lemma 2.** Let $G = (V, E)$ be a unicyclic graph and $\dim(G) = 2$. A vertex $v \in \hat{V}$ with degree 3 is a main vertex of the graph $G$ if and only if $v$ is not a one-leaf vertex.
Proof. Let \( v \in \hat{V} \) with degree 3 be a main vertex. Then there is an exterior vertex \( w \in V \setminus \hat{V} \) that is projected in \( v \). Moreover, \( v \) is a vertex of a cycle \( G_1 \). Hence, \( v \) is not a one-leaf vertex. Note, that \( v \) is not a two-leaf vertex also.

Assume now, that \( u \in \hat{V} \) is an one-leaf vertex. Then there is one terminal end-vertex \( z \in V \setminus \hat{V} \) of the vertex \( v \) projected in \( v \). Hence there is no vertices with degree 3 projected in \( u \). Therefore, the vertex \( u \) is not a main vertex of graph \( G \).

Lemma 3. Let \( G = (V, E) \) be a unicyclic graph and \( \dim(G) = 2 \). Then for any main vertex \( v \) of \( G \) there exists exactly one two-leaf vertex that projected in \( v \).

The proof of this lemma directly follows from the proof of Lemma 1 in [11].

2. Metric dimension of knitting of unicyclic graphs

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be graphs. Fix vertices \( v_1 \in V_1 \) and \( v_2 \in V_2 \). A graph \( G \) is built from \( G_1 \) and \( G_2 \) by gluing along the vertices \( v_1 \) and \( v_2 \) if \( G = (V, E) \) has the set of vertices \( V = V_1 \cup (V_2 \setminus v_2) \) and the set of edges \( E = E_1 \cup E_2 \) (a vertex \( v_2 \) is replaced by \( v_1 \) for all edges of \( G_2 \)). Roughly speaking, we identify vertices \( v_1 \) and \( v_2 \) of graphs \( G_1 \) and \( G_2 \) (see Figure 2).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{gluing_graphs.png}
\caption{Gluing of two graphs.}
\end{figure}

Definition 1. A unicyclic graph \( G \) is said to be a basic graph if following conditions hold:
(A) for any vertex \( v \) from \( G \) \( \deg_G(v) \leq 3 \);
(B) for any main vertex \( v \) of \( G \) there exists exactly one two-leaf vertex projected in \( v \);
(C) \( \hat{G} \) has exactly two main vertices;
(D) \( \hat{G} \) has only main vertices with degree more than two.

**Definition 2.** Let now \( G_1 \) be a basic graph. Denote the main vertices of \( G_1 \) by \( u \) and \( v \). A unicyclic graph \( G \) is called a *knitting* of the graph \( G_1 \) by chains \( L_1, \ldots, L_r \) if \( G \) is obtained from the graph \( G_1 \) by gluing vertices with degree 2 of its cycle and begins of the chains \( L_1, \ldots, L_r \) such that each vertex with degree 2 of the cycle is glued to the end of exactly one chain and for any one-leaf vertex \( w \) and any adjacent to \( w \) vertex \( a \) the following inequality holds (see Figure 3):

\[
d_G(u, v) + d_G(v, w) + 1 \neq d_G(u, a). \tag{1}
\]

![Figure 3. Knitting condition does not hold.](image)

A unicyclic graph \( G \) is *even*, if \( |\hat{V}| = 2k \) for some positive integer \( k \). A unicyclic graph \( G \) is *odd*, if \( |\hat{V}| = 2k + 1 \) for some positive integer \( k \).

**Proposition 2.** Let \( G = (V, E) \) be an even unicyclic graph with two main vertices \( u \) and \( v \), \(|\hat{V}| = 2k \). If \( \dim G = 2 \), then \( d(u, v) \neq k \).

**Proof.** Assume that \( \dim G = 2 \) and \( d(u, v) = k \). Since \( u \) and \( v \) are main vertices of graph \( G \), a resolving set of \( G \) contains two end-vertices \( z \) and \( l \) projected in \( u \) and \( v \) respectively. Let \( a \) and \( b \) be vertices from the cycle \( \hat{G} \) adjacent to \( u \). Hence, \( d_G(a, u) = d_G(b, u) = 1 \) and \( d_G(a, v) = d_G(b, v) = k - 1 \). Therefore, the set \( \{z, l\} \) is not a resolving set and \( \dim G > 2 \). \( \square \)

**Theorem 1.** An odd unicyclic graph \( G = (V, E) \) with two main vertices has metric dimension 2 if and only if one of the following conditions hold:
1) $G$ is a basic graph;
2) $G$ is a knitting of some basic graph $G_1$.

An even unicyclic graph $G = (V, E)$ with two main vertices $u$ and $v$, $|\hat{V}| = 2k$, has metric dimension 2 if and only if one of the conditions 1) or 2) holds and $d(u, v) \neq k$.

Proof. 1. Assume, that $G = (V, E)$ is an odd basic graph. Then if some vertex $u$ from $\hat{G}$ has degree 3 then $u$ is a main vertex of $G$. Let $u$ and $v$ be main vertices of the graph $G$, $z$ and $l$ be end-vertices projected in $u$ and $v$ respectively. It is not hard to verify that the set $\{z, l\}$ is a resolving set of the graph $G$. Since vertices of a cycle are resolved of two vertices the set $\{z, l\}$ is a metric basic and then $\dim G = 2$. If $G$ is a knitting of some basic graph $G_1$ then from the construction of knitting it follows that a metric basis of $G_1$ is a metric basis of $G$.

2. Let now $G = (V, E)$ be an odd unicyclic graph with two main vertices and metric dimension 2. It follows from Proposition 1, Lemma 3 and Lemma 1 that for the graph $G$ conditions $(A) - (C)$ of Definition 1 hold. If $\hat{G}$ has only main vertices with degree more than two then $G$ is a basic graph. Assume that $\hat{G}$ has not only main vertices with degree 3. From Lemma 2 it follows that all of these vertices are one-leaf ones. Then the graph $G$ can be considered as a knitting of some basic graph. \(\square\)

References


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