Attached primes and annihilators of top local cohomology modules defined by a pair of ideals

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Communicated by V. Lyubashenko

Abstract. Assume that $R$ is a complete Noetherian local ring and $M$ is a non-zero finitely generated $R$-module of dimension $n = \dim(M) \geq 1$. It is shown that any non-empty subset $T$ of $\text{Assh}(M)$ can be expressed as the set of attached primes of the top local cohomology modules $H_{I,J}^n(M)$ for some proper ideals $I, J$ of $R$. Moreover, for ideals $I, J = \bigcap_{p \in \text{Att}_R(H_{I,J}^n(M))} p$ and $J'$ of $R$ it is proved that $T = \text{Att}_R(H_{I,J}^n(M)) = \text{Att}_R(H_{I,J'}^n(M))$ if and only if $J' \subseteq J$. Let $H_{I,J}^n(M) \neq 0$. It is shown that there exists $Q \in \text{Supp}(M)$ such that $\dim(R/Q) = 1$ and $H_{I,J}^n(R/p) \neq 0$, for each $p \in \text{Att}_R(H_{I,J}^n(M))$. In addition, we prove that if $I$ and $J$ are two proper ideals of a Noetherian local ring $R$, then $\text{Ann}_R(H_{I,J}^n(M)) = \text{Ann}_R(M/T_R(I, J, M))$, where $T_R(I, J, M)$ is the largest submodule of $M$ with $\text{cd}(I, J, T_R(I, J, M)) < \text{cd}(I, J, M)$, here $\text{cd}(I, J, M)$ is the cohomological dimension of $M$ with respect to $I$ and $J$. This result is a generalization of [1, Theorem 2.3] and [2, Theorem 2.6].

Introduction

Assume that $R$ is a Noetherian ring and $I, J$ are two ideals of $R$ and $M$ is an $R$-module. As a generalization of the usual local cohomology modules, the local cohomology modules with respect to a system of ideals was introduced, in [3]. As a special case of these extended modules, in [13],

2010 MSC: 13D45, 14B15.

Key words and phrases: associated prime ideals, attached prime ideals, top local cohomology modules.
the local cohomology modules with respect to a pair of ideals is defined. To be more precise, let

\[ W(I, J) = \{ p \in \text{Spec}(R) | I^n \subseteq p + J \text{ for some positive integer } n \}. \]

The \((I, J)\)-torsion submodule \( \Gamma_{I,J}(M) \) of \( M \), which consists of all elements \( x \) of \( M \) with \( \text{Supp}(Rx) \subseteq W(I, J) \), is considered. For an integer \( i \), the local cohomology functor \( H_i^{I,J} \) with respect to \((I, J)\) is defined to be the \( i \)-th right derived functor of \( \Gamma_{I,J} \). The \( i \)-th local cohomology module of \( M \) with respect to \((I, J)\) is denoted by \( H_i^{I,J}(M) \). When \( J = 0 \), then \( H_i^{I,J} \) coincides with the usual local cohomology functor \( H_i^{I} \) with the support in the closed subset \( V(I) \).

Recall that for an \( R \)-module \( K \), a prime ideal \( p \) of \( R \) is said to be an attached prime ideal of \( K \) if \( p = \text{Ann}(K/N) \) for some submodule \( N \) of \( K \). The set of attached prime ideals of \( K \) is denoted by \( \text{Att}_R(K) \). When \( K \) has a secondary representation, this definition agrees with the usual definition of attached primes in [12].

Let \( R \) be a Noetherian local ring with maximal ideal \( m \) and \( M \) be a finitely generated \( R \)-module of dimension \( n \). The main theorem in Section 2, shows that if \( R \) is complete with respect to \( m \)-adic topology, then for any non-empty subset \( T \) of \( \text{Assh}(M) \) there exist ideals \( I, J \) of \( R \) such that \( T = \text{Att}_R(H^n_{I,J}(M)) = \text{Att}_R(H^n_{I}(M)) \) which is an another version of Theorem 2.8 in [8]. Moreover we show that for each \( p \in \text{Att}_R(H^n_{I,J}(M)) \) there exists \( Q \in \text{Supp}(M) \) such that \( \dim(R/Q) = 1 \) and \( H^n_{Q}(R/p) \neq 0 \).

Let \( R \) be a Noetherian ring, \( I, J \) be two ideals of \( R \) and \( M \) be a non-zero finitely generated \( R \)-module of dimension \( n \). Let \( cd(I, J, M) \) denote the supremum of all integers \( r \) for which \( H^r_{I,J}(M) \neq 0 \). We call this integer the cohomological dimension of \( M \) with respect to ideals \( I, J \), see [7]. In Section 3, first we define \( T_R(I, J, M) \) the largest submodule of \( M \) such that \( cd(I, J, T_R(I, J, M)) < cd(I, J, M) \) and we show that

\[ T_R(I, J, M) = \Gamma_a(M) = \bigcap_{cd(I,J,R/p_j)=c} N_j, \]

where \( 0 = \bigcap_{j=1}^n N_j \) denotes a reduced primary decomposition of the zero submodule of \( M \), \( N_j \) is a \( p_j \)-primary submodule of \( M \) and \( a = \prod_{cd(I,J,R/p_j)=c} p_j \). We show \( \text{Ann}_R(H^n_{I,J}(M)) = \text{Ann}_R(M/T_R(I, J, M)) \), which is a generalization of [1, Theorem 2.3] and [2, Theorem 2.6] and some applications of this theorem are given.
1. Attached prime ideals of top local cohomology modules

In this section we assume that \((R, \mathfrak{m})\) is local and complete with respect to \(\mathfrak{m}\)-adic topology, \(M\) is a non-zero finitely generated \(R\)-module of dimension \(n \geq 1\) and \(T\) is a non-empty subset of \(\text{Assh}(M)\).

**Definition 1.** Let \(K\) be an \(R\)-module, a prime ideal \(p\) of \(R\) is said to be an attached prime ideal of \(K\) if \(p = \text{Ann}(K/N)\) for some submodule \(N\) of \(K\). The set of attached prime ideals of \(K\) is denoted by \(\text{Att}_R(K)\).

**Lemma 1.** [1, Lemma 3.2] Let \(K\) be an \(R\)-module. Then the set of minimal elements of \(V(\text{Ann}_R(K))\) coincides with that of \(\text{Att}_R(K)\). In particular, 
\[
\sqrt{\text{Ann}_R(K)} = \bigcap_{p \in \text{Att}_R(K)} p.
\]

**Theorem 1.** Let \(M\) be a non-zero finitely generated \(R\)-module of dimension \(n\) and \(T\) be a non-empty subset of \(\text{Assh}(M)\). Then the following statements are true:

(i) If \(T \subseteq \text{Att}_R(H^n_I(M))\) for some ideal \(I\), then \(T = \text{Att}_R(H^n_{I,J}(M))\), where \(J = \bigcap_{p \in T} p\).

(ii) If \(T = \text{Att}_R(H^n_I(M)) = \text{Att}_R(H^n_{I,J}(M))\), where \(I, J\) are ideals of \(R\) and \(J = \sqrt{\text{Ann}_R(H^n_I(M))}\).

**Proof.** (i) By assumption \(T\) is a non-empty subset of \(\text{Assh}(M)\). Set \(J := \bigcap_{p \in T} p\). We show that \(T = \text{Att}_R(H^n_{I,J}(M))\). Assume that \(q \in \text{Att}_R(H^n_{I,J}(M))\). Then by [6, Theorem 2.1] it follows that \(J \subseteq q\). Thus \(p \subseteq q\) for some \(p \in T\). Hence, this fact that \(p, q\) are in \(\text{Assh}(M)\) shows that \(p = q\) and so \(q \in T\). Now, let \(q \in T\). Then \(J \subseteq q\) and also \(q \in \text{Att}_R(H^n_I(M))\). Therefore, \(q \in \text{Att}_R(H^n_{I,J}(M))\) by [6, Theorem 2.1].

(ii) In view of [8, Theorem 2.8] there exists an ideal \(I\) of \(R\) such that \(T = \text{Att}_R(H^n_I(M))\). Thus by (i) and Lemma 1 the result follows. \(\square\)

**Corollary 1.** Let \(M\) be a non-zero finitely generated \(R\)-module of dimension \(n\) and let \(I_1, I_2, J_1, J_2\) be ideals of \(R\). Then the following statements hold:

(i) If \(T \subseteq \text{Att}_R(H^n_{I_1,J_1}(M)) \cup \text{Att}_R(H^n_{I_2,J_2}(M))\) is a non-empty set and \(J = \bigcap_{p \in T} p\), then \(\text{Att}_R(H^n_{I_1+I_2,J_1}(M)) \cup \text{Att}_R(H^n_{I_1,J_1}(M))\).

(ii) If \(T = \text{Att}_R(H^n_{I_1,J_1}(M)) \cup \text{Att}_R(H^n_{I_2,J_2}(M))\) is a non-empty set and \(J = \bigcap_{p \in T} p\), then \(\text{Att}_R(H^n_{I_1+I_2,J_1}(M)) = \text{Att}_R(H^n_{I_1,J_1}(M)) \cup \text{Att}_R(H^n_{I_2,J_2}(M))\).

(iii) If \(T = \text{Att}_R(H^n_{I_1,J_1}(M)) \cap \text{Att}_R(H^n_{I_2,J_2}(M))\) is a non-empty set and \(J = \bigcap_{p \in T} p\), then \(\text{Att}_R(H^n_{I_1+I_2,J_1}(M)) = \text{Att}_R(H^n_{I_1,J_1}(M)) \cap \text{Att}_R(H^n_{I_2,J_2}(M))\).
Proof. Let $p \in T$ and $p \in \text{Att}_R(H^n_{I,J_1}(M))$. Then $p \in \text{Att}_R(H^n_{I_1}(M))$, by [6, Theorem 2.1]. Thus $\dim R/p = n$ and by Lichtenbaum-Hartshorne Vanishing Theorem $\dim R/(I_1 + p) = 0$. Since $I_1 + p \subseteq I_1 + I_2 + p$, it follows that $\dim R/(I_1 + I_2 + p) = 0$ and so $H^n_{I_1+I_2}(R/p) \neq 0$. Thus $p \in \text{Att}_R(H^n_{I_1+I_2}(M))$, by [9, Theorem A]. Therefore, $T \subseteq \text{Att}_R(H^n_{I_1+I_2}(M))$ and the result follows by Theorem 1(i).

Corollary 2. Let $I, J$ be ideals of $R$ and let $M$ be a non-zero finitely generated $R$-module of dimension $n$. If $T = \text{Att}_R(H^n_I(M))$ and $J' = J + \bigcap_{p \in T} p$, then $\text{Att}_R(H^n_{I,J}(M)) = \text{Att}_R(H^n_{m,J'}(M))$.

Proof. In view of [6, Theorem 2.1] and Lichtenbaum-Hartshorne Vanishing Theorem, we have

$$\text{Att}_R(H^n_{I,J}(M)) = \{p \in \text{Supp}(M) \cap V(J) : \sqrt{I+p} = m\}.$$ 

Let $p \in \text{Att}_R(H^n_{I,J}(M))$. Then $p \in \text{Att}_R(H^n_I(M))$ and $0 \neq H^n_I(R/p) \cong H^n_{I+p}(R/p) \cong H^n_m(R/p)$. Hence, $p \in \text{Att}_R(H^n_{m,J'}(M))$. The proof of the opposite inclusion is similar.

Theorem 2. Let $M$ be a non-zero finitely generated $R$-module of dimension $n$ and let $I, J = \bigcap_{p \in \text{Att}_R(H^n_I(M))} p$ and $J'$ be ideals of $R$. Then $\text{Att}_R(H^n_{I,J}(M)) = \text{Att}_R(H^n_{I,J'}(M))$ if and only if $J' \subseteq J$.

Proof. Let $\text{Att}_R(H^n_{I,J}(M)) = \text{Att}_R(H^n_{I,J'}(M))$. Then Theorem 1 shows that $\text{Att}_R(H^n_{I,J}(M)) = \text{Att}_R(H^n_{I,J'}(M))$. Hence, by [6, Theorem 2.1]

$$J' \subseteq \bigcap_{p \in \text{Att}_R(H^n_{I,J'}(M))} p = \bigg(\bigcap_{p \in \text{Att}_R(H^n_{I,J}(M))} p\bigg) = \bigcap_{p \in \text{Att}_R(H^n_I(M))} p = J.$$ 

Conversely, if $J' \subseteq J$, then by [6, Theorem 2.1] we have

$$\text{Att}_R(H^n_I(M)) = \text{Att}_R(H^n_{I,J}(M)) \subseteq \text{Att}_R(H^n_{I,J'}(M)) \subseteq \text{Att}_R(H^n_I(M)).$$

So the result follows.

Theorem 3. Let $M$ be a non-zero finitely generated $R$-module of dimension $n$ and let $I, I'$ and $J = \bigcap_{p \in \text{Att}_R(H^n_I(M))} p$ be ideals of $R$ such that $I \subseteq I'$. Then $\text{Att}(H^n_{I,J}(M)) = \text{Att}(H^n_{I',J}(M))$.

Proof. Assume that $p \in \text{Att}(H^n_{I,J}(M))$. Thus [6, Theorem 2.1] shows that $p \in \text{Att}(H^n_I(M))$ and $J \subseteq p$. By assumption and [10, Proposition
Theorem 4. Let $M$ be a finitely generated $R$-module of dimension $n$ and $I, J$ be ideals of $R$ such that $H^n_{I,J}(M) \neq 0$. Then there exists $Q \in \text{Supp}(M)$ such that $\dim(R/Q) = 1$ and $H^n_{Q}(R/p) \neq 0$, for each $p \in \text{Att}_R(H^n_{I,J}(M))$.

Proof. By assumption $T = \text{Att}_R(H^n_{I,J}(M)) \neq \emptyset$. Then in view of [8, Theorem 2.8] we have $T = \text{Att}_R(H^n_{a}(M))$ for some ideal $a$ of $R$. Now, [8, Proposition 2.1] shows that there exists an integer $r$ such that for all $1 \leq i \leq r$ there exists $Q_i \in \text{Supp}(M)$ with $\dim(R/Q_i) = 1$ such that $\bigcap_{p \in T} p \nsubseteq Q_i$. In addition, we may assume that $a = \bigcap_{i=1}^{r-1} Q_i$. Let $p \in \text{Att}_R(H^n_{I,J}(M))$. Then $p \in \text{Att}_R(H^n_{a}(M))$ and $H^n_{\bigcap_{i=1}^{r-1} Q_i}(R/p) \neq 0$.

Now, by setting $b = \bigcap_{i=1}^{r-1} Q_i$ and $c = Q_r$ we have the following long exact sequence

$$ H^n_{b+c}(R/p) \rightarrow H^n_{b}(R/p) \oplus H^n_{c}(R/p) \rightarrow H^n_{br+c}(R/p) \rightarrow 0, $$

where $H^n_{br+c}(R/p) = H^n_{Q_1 \cap \cdots \cap Q_r}(R/p) \neq 0$. So $H^n_{b}(R/p) \oplus H^n_{c}(R/p) \neq 0$. Therefore $H^n_{b}(R/p) \neq 0$ or $H^n_{c}(R/p) \neq 0$. If $H^n_{c}(R/p) \neq 0$ we are done. Otherwise, one can set $b = \bigcap_{i=1}^{r-1} Q_i$ and $c = Q_{r-1}$ and with repeat this method, to get the result.

Corollary 3. Let $M$ be a finitely generated $R$-module of dimension $n \geq 1$ and $I, J$ be ideals of $R$ such that $H^n_{I,J}(M) \neq 0$. Then there exists $Q \in \text{Supp}(M)$ such that $\dim(R/Q) = 1$ and $\text{Att}_R(H^n_{Q,J}(M)) \neq \emptyset$.

2. Annihilators of top local cohomology modules

In this section $(R, m)$ is a Noetherian local ring with maximal ideal $m$ and $I, J$ are two proper ideals of $R$.

Let $M$ be a non-zero finitely generated $R$-module and let $\text{cd}(I, J, M)$ denote the supremum of all integers $r$ for which $H^r_{I,J}(M) \neq 0$. We call this integer the cohomological dimension of $M$ with respect to ideals $I, J$. When $J = 0$, we have $\text{cd}(I, 0, M) = \text{cd}(I, M)$ which is just the supremum of all integers $r$ for which $H^r_I(M) \neq 0$. In [7, Corollary 3.3] a characterization of $\text{cd}(I, J, M)$ is provided.
Lemma 2. [7, Proposition 3.2] Let $M$ and $N$ be two finitely generated $R$-modules such that $\text{Supp}(N) \subseteq \text{Supp}(M)$. Then $\text{cd}(I, J, N) \leq \text{cd}(I, J, M)$.

Definition 2. Let $M$ be a non-zero finitely generated $R$-module of cohomological dimension $c$. We denote by $T_R(I, J, M)$ the largest submodule of $M$ such that $\text{cd}(I, J, T_R(I, J, M)) < \text{cd}(I, J, M)$.

It is easy to check that $T_R(I, J, M) = \cup \{N : N \subseteq M, \text{cd}(I, J, N) < \text{cd}(I, J, M)\}$. When $J = 0$, this definition coincides with that of [1, Definition 2.2].

Lemma 3. Let $M$ be a non-zero finitely generated $R$-module of dimension $n$ such that $n = \text{cd}(I, J, M)$. Then $T_R(m, M) \subseteq T_R(I, M) \subseteq T_R(I, J, M)$.

Proof. For the first inclusion let $x \notin T_R(I, M)$. Then $\text{cd}(I, J, Rx) = n$ and so $H^n_I(Rx) \neq 0$. Thus $\dim(Rx) = n$. Hence, by Grothendieck’s Vanishing Theorem $H^n_m(Rx) \neq 0$ and $x \notin T_R(m, M)$. Let $x \notin T_R(I, J, M)$. Then $\text{cd}(I, J, Rx) = n$ and so $H^n_n(I, J, Rx) \neq 0$. Thus $\text{Att}_R(H^n_n(I, J, Rx)) \neq \emptyset$ and $\text{Att}_R(H^n_n(I, J, Rx)) \neq \emptyset$ by [6, Theorem 2.1]. Hence, $H^n_n(I, J, Rx) \neq 0$ and $\text{cd}(I, J, Rx) = n$. Therefore, $x \notin T_R(I, M)$.

Theorem 5. Let $M$ be a non-zero finitely generated $R$-module with cohomological dimension $c = \text{cd}(I, J, M)$. Then

$$T_R(I, J, M) = \Gamma_a(M) = \bigcap_{\text{cd}(I, J, R/p_j) = c} N_j.$$  

Here $0 = \bigcap_{j=1}^n N_j$ is a reduced primary decomposition of the zero submodule of $M$, $N_j$ is a $p_j$-primary submodule of $M$ and $a = \prod_{\text{cd}(I, J, R/p_j) \neq c} p_j$.

Proof. First we show the equality $\Gamma_a(M) = \bigcap_{\text{cd}(I, J, R/p_j) = c} N_j$. To do this, the inclusion $\bigcap_{\text{cd}(I, J, R/p_j) = c} N_j \subseteq \Gamma_a(M)$ follows easily by the proof of [11, Theorem 6.8(ii)]. In order to prove the opposite inclusion, suppose, the contrary is true. Then there exists $x \in \Gamma_a(M)$ such that $x \notin \bigcap_{\text{cd}(I, J, R/p_j) = c} N_j$. Thus there exists an integer $t$ such that $x \notin N_t$ and $\text{cd}(I, J, R/p_t) = c$. Now, as $x \in \Gamma_a(M)$, it follows that there is an integer $s \geq 1$ such that $a^s x = 0$, and so $a^s x \subseteq N_t$. Because of $x \notin N_t$ and $N_t$ is a $p_t$-primary submodule, it yields that $a \subseteq p_t$. Hence, there is an integer $j$ such that $p_j \subseteq p_t$ and $\text{cd}(I, J, R/p_j) \leq c - 1$. Therefore, in view of Lemma 2, we have

$$\text{cd}(I, J, R/p_t) \leq \text{cd}(I, J, R/p_j) \leq c - 1,$$
which is a contradiction. Now, we show that $T_R(I, J, M) = \Gamma_a(M)$. Let $x \in T_R(I, J, M)$. Then in view of Lemma 2, $\text{cd}(I, J, Rx) \leq c - 1$. Let $p$ be a minimal prime ideal of $\text{Ann}_R(Rx)$, it follows that $\text{cd}(I, J, R/p) \leq c - 1$. So

$$a \subseteq \bigcap_{\text{cd}(I, J, R/p) \leq c - 1} p_j \subseteq \bigcap_{p \in \text{Ass}_R(Rx)} p = \sqrt{\text{Ann}_R(Rx)}.$$ 

Thus there exists an integer $k \geq 1$ such that $a^k \subseteq \text{Ann}_R(Rx)$. Hence, $a^k x = 0$ and $x \in \Gamma_a(T_R(I, J, M))$. Thus $T_R(I, J, M) = \Gamma_a(T_R(I, J, M))$. Now, we have $T_R(I, J, M) = \Gamma_a(T_R(I, J, M)) \subseteq \Gamma_a(M)$. We show that $\Gamma_a(M) \subseteq T_R(I, J, M)$, to do this, we show $\text{cd}(I, J, \Gamma_a(M)) \leq c - 1$. Let $p \in \text{Supp}(\Gamma_a(M))$. Then $a \subseteq p$ and there exists $p_j \subseteq p$ such that $\text{cd}(I, J, R/p_j) \leq c - 1$. Thus by Lemma 2, $\text{cd}(I, J, R/p) \leq c - 1$. Hence, $\text{cd}(I, J, \Gamma_a(M)) \leq c - 1$ by [7, Theorem 3.1]. Therefore, $T_R(I, J, M) = \Gamma_a(M)$.

**Corollary 4.** Let $M$ be a non-zero finitely generated $R$-module of dimension $n$ with cohomological dimension $c = \text{cd}(I, J, M)$. Then the following statements are true:

(i) $\text{Ass}_R(T_R(I, J, M)) = \{ p \in \text{Ass}_R(M) : \text{cd}(I, J, R/p) \leq c - 1 \}$,

(ii) $\text{Ass}_R(M/T_R(I, J, M)) = \{ p \in \text{Ass}_R(M) : \text{cd}(I, J, R/p) = c \}$. If $n = c$, then $\text{Ass}_R(M/T_R(I, J, M)) = \text{Att}_R(H^n_{I,J}(M))$.

**Proof.** By Theorem 5, $T_R(I, J, M) = \Gamma_a(M)$, where $\prod_{\text{cd}(I, J, R/p) \leq c - 1} p_j = a$. So by [4, Section 2.1, Proposition 10] we have

$$\text{Ass}_R(T_R(I, J, M)) = \text{Ass}_R(M) \cap V(a).$$

Now (i) follows from Lemma 2.

In order to show (ii), use [5, Exercise 2.1.12] and [6, Theorem 2.1].

**Corollary 5.** Let $M$ be a non-zero finitely generated $R$-module of dimension $n$ such that $n = \text{cd}(I, J, M)$. Then there exists a positive integer $t$ such that $J^t M \subseteq T_R(I, J, M)$.

**Proof.** Let $0 = \bigcap_{j=1}^n N_j$ denote a reduced primary decomposition of the zero submodule of $M$, where $N_j$ is a $p_j$-primary submodule of $M$. In view of Theorem 5, $T_R(I, J, M) = \bigcap_{\text{cd}(I, J, R/p_j) = n} N_j$. If $\text{cd}(I, J, R/p_j) = n$, then $H^n_{J,J}(R/p_j) \neq 0$. Thus $J \subseteq p_j = \sqrt{\text{Ann}_R(M/N_j)}$ by [13, Theorem 4.3]. Hence, there exists a positive integer $t_j$ such that $J^{t_j} M \subseteq N_j$. Let $t = \max\{t_j : \text{cd}(I, J, R/p_j) = n\}$. Then $J^t M \subseteq \bigcap_{\text{cd}(I, J, R/p_j) = n} N_j = T_R(I, J, M)$.
Theorem 6. Let $M$ be a non-zero finitely generated $R$-module of dimension $n = \text{cd}(I, J, M)$. Then

$$\text{Ann}_R(H^n_{I,J}(M)) = \text{Ann}_R(M/T_R(I, J, M)).$$

Proof. By Corollary 5, $J^tM \subseteq T_R(I, J, M)$ for some integer $t \geq 1$ and by [13, Proposition 1.4(8)], $H^n_{I,J}(M) \cong H^n_{I,J}(M)$ for all $i \geq 0$. Then we can assume that $JM \subseteq T_R(I, J, M)$. First we show that $T_R(I, M/JM) = T_R(I, J, M)/JM$. Let $x \in M$ and consider the exact sequence

$$0 \rightarrow Rx \cap JM \rightarrow Rx \rightarrow \frac{Rx}{Rx \cap JM} \rightarrow 0$$

that induce an exact sequence

$$\cdots \rightarrow H^n_{I,J}(Rx \cap JM) \rightarrow H^n_{I,J}(Rx) \rightarrow H^n_{I,J}(\frac{Rx}{Rx \cap JM}) \rightarrow 0. \quad (\ast)$$

If $x + JM \in T_R(I, M/JM)$, then $H^n_{I,J}(Rx/Rx \cap JM) \cong H^n_I(R(x + JM)) = 0$, by [13, Corollary 2.5]. As $Rx \cap JM \subseteq T_R(I, J, M)$ it follows that $H^n_{I,J}(Rx \cap JM) = 0$. Hence, $H^n_{I,J}(Rx) = 0$. So that $x \in T_R(I, J, M)$. If $x \in T_R(I, J, M)$, then $H^n_{I,J}(Rx) = 0$. Thus $H^n_{I,J}(Rx/Rx \cap JM) = H^n_I(R(x + JM)) = 0$ by $(\ast)$. Therefore, $x + JM \in T_R(I, M/JM)$. Now, from the exact sequence

$$0 \rightarrow JM \rightarrow M \rightarrow \frac{M}{JM} \rightarrow 0$$

we have the exact sequence

$$\cdots \rightarrow H^n_{I,J}(JM) \rightarrow H^n_{I,J}(M) \rightarrow H^n_{I,J}(\frac{M}{JM}) \rightarrow 0.$$

Since $JM \subseteq T_R(I, J, M)$, it follows that $H^n_{I,J}(JM) = 0$ and so we have $H^n_{I,J}(M) \cong H^n_{I,J}(M/JM)$. Thus $H^n_{I,J}(M) \cong H^n_I(M/JM)$. Therefore,

$$\text{Ann}_R(H^n_{I,J}(M)) = \text{Ann}_R(H^n_I(M/JM)) = \text{Ann}_R(\frac{M/JM}{T_R(I, M/JM)}) = \text{Ann}_R(\frac{M/JM}{T_R(I, J, M)/JM}) = \text{Ann}_R(M/T_R(I, J, M)),$$

see [1, Theorem 2.3]. \hfill \Box

Corollary 6. Let $M$ be a non-zero finitely generated $R$-module of dimension $n = \text{cd}(I, J, M)$ and $JM \subseteq T_R(I, M)$. Then $\text{Ann}_R(H^n_I(M)) = \text{Ann}_R(H^n_{I,J}(M))$. 

Proof. By a similar argument to that of Theorem 6, one can show that $T_R(I, M/JM) = T_R(I, M)/JM$. Also, by Lemma 3 we have $T_R(I, M) \subseteq T_R(I, J, M)$. Thus $JM \subseteq T_R(I, J, M)$ and so $H^n_{I,J}(JM) = 0$. Hence, it follows by (*) that $H^n_{I,J}(M) \cong H^n_I(M/JM)$. Therefore,

\[
\text{Ann}_R(H^n_{I,J}(M)) = \text{Ann}_R(H^n_I(M/JM)) = \text{Ann}_R(\frac{M/JM}{T_R(I, M/JM)}) = \text{Ann}_R(\frac{M/JM}{T_R(I, M)/JM}) = \text{Ann}_R(M/T_R(I, M)).
\]

Now, the result follows by [1, Theorem 2.3] and Theorem 6.  

Corollary 7. Let $M$ be a non-zero finitely generated $R$-module of dimension $n = \text{cd}(I, J, M)$. Then the following statements hold:

(i) $\sqrt{\text{Ann}_R(H^n_{I,J}(M))} = \bigcap_{p \in \text{Ass}_R M, \text{cd}(I,J,R/p)=n} p$,
(ii) $V(\text{Ann}_R(H^0_{I,J}(M))) = \text{Supp}(M/T_R(I, J, M))$,
(iii) If $T_R(I, J, M) = 0$, then $\text{Supp}(M) = V(\text{Ann}_R(H^0_{I,J}(M)))$.

Proof. (i) It follows by Lemma 1 and [6, Theorem 2.1].

To prove (ii) use Theorem 6.

(iii) It follows from (ii). 

Corollary 8. Let $d = \dim R = \text{cd}(I, J, R)$. Then the following statements hold:

(i) $\text{cd}(I, J, \text{Ann}_R(H^d_{I,J}(R))) < \dim R$.
(ii) If $d \geq 1$, then $\dim R = \dim R/\text{Ann}_R(H^d_{I,J}(R)) = \dim R/\Gamma_{I,J}(R)$.

Proof. (i) It follows from Theorem 6.

(ii) By [13, Corollary 1.13 (4)], $H^d_{I,J}(R) \cong H^d_{I,J}(R/\Gamma_{I,J}(R))$. So that $\Gamma_{I,J}(R) \subseteq \text{Ann}_R(H^d_{I,J}(R))$.  

Corollary 9. If $R$ is a domain of dim $R = d$ and $H^d_{I,J}(R) \neq 0$, then $\text{Ann}_R(H^d_{I,J}(R)) = 0$.

Proof. It follows by Theorems 5 and 6. 

3. Acknowledgment

The authors are deeply grateful to the S. Babaei and the referee for careful reading of the manuscript, very helpful suggestions and insightful comments.
References


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Received by the editors: 13.03.2017.