On disjoint union of M-graphs

Sergiy Kozerenko

Communicated by V. V. Kirichenko

Abstract. Given a pair \((X, \sigma)\) consisting of a finite tree \(X\) and its vertex self-map \(\sigma\) one can construct the corresponding Markov graph \(\Gamma(X, \sigma)\) which is a digraph that encodes \(\sigma\)-covering relation between edges in \(X\). M-graphs are Markov graphs up to isomorphism. We obtain several sufficient conditions for the disjoint union of M-graphs to be an M-graph and prove that each weak component of M-graph is an M-graph itself.

Introduction

In 1964 Sharkovsky proved the following remarkable theorem.

Theorem 1. [9] If the continuous map \(f : [0, 1] \to [0, 1]\) has a periodic point of period \(n \in \mathbb{N}\), then it also has a periodic point of period \(m \in \mathbb{N}\) for all \(m \geq n\), where

\[
1 \prec 2 \prec 2^2 \prec \cdots \prec 2^n \prec \cdots \prec 7 \cdot 2 \prec 5 \cdot 2 \prec 3 \cdot 2 \prec \cdots \prec 7 \cdot 5 \prec 3
\]

is Sharkovsky’s ordering of \(\mathbb{N}\). Moreover, for every number \(m \in \mathbb{N}\) there exists a continuous map that has a periodic point of period \(m\) but does not have periodic points of periods \(n \in \mathbb{N}\), where \(m \prec n\).

In [10] Straffin proposed a strategy on how to prove Sharkovsky’s theorem using some elegant combinatorial arguments. The cornerstone of his idea is to use directed graphs which naturally arise from orbits of periodic

\textbf{2010 MSC}: 05C20, 37E25, 37E15.

\textbf{Key words and phrases}: tree maps, Markov graphs, Sharkovsky’s theorem.
points. Namely, let \( x \in [0,1] \) be \( n \)-periodic point of a continuous map \( f : [0,1] \to [0,1] \). Consider the orbit \( \text{orb}_f(x) = \{ x, f(x), \ldots, f^{n-1}(x) \} = \{ x_1 < \cdots < x_n \} \) and its natural ordering inherited from the interval. Periodic graph \( G_f(x) \) has the vertex set \( \{1, \ldots, n-1\} \) and the arc set \( \{(i,j) : \min\{f(x_i), f(x_{i+1})\} \leq x_j < \max\{f(x_i), f(x_{i+1})\}\} \). Here each \( 1 \leq i \leq n-1 \) represents the minimal interval \([x_i, x_{i+1}]\) and there is an arc \( i \to j \) in \( G_f(x) \) if \([x_i, x_{i+1}]\) “covers” \([x_j, x_{j+1}]\) under \( f \). Periodic graphs are useful in combinatorial dynamics because of the following result known as \textit{Itinerary lemma}.

**Lemma 1.** [10] Let \( x \in [0,1] \) be some periodic point of a continuous map \( f : [0,1] \to [0,1] \). Suppose that there is a closed walk \( W = \{i_0 \to \cdots \to i_{m-1} \to i_0\} \) of length \( m \) in \( G_f(x) \). Then there exists a periodic point \( y \in [0,1] \) such that \( f^m(y) = y \) and \( f^k(y) \in [x_{i_k}, x_{i_k+1}] \) for all \( 0 \leq k \leq m-1 \). Moreover, if \( W \) is primitive, then the period of \( y \) equals \( m \).

Here the closed walk is called \textit{primitive} if it is not entirely consists of a smaller walk traced several times. Note that Lemma 1 admits a converse statement. Namely, for any periodic point \( x \in [0,1] \) of \( f \) we can consider its \textit{linearization} \( L_x(f) : [0,1] \to [0,1] \) which is a “connect-the-dots” map with respect to the orbit \( \text{orb}_f(x) \). Then each \( m \)-periodic point of \( L_x(f) \) corresponds to some primitive closed walk of length \( m \) in \( G_{L_x(f)}(x) = G_f(x) \).

Full proof of Sharkovsky’s theorem using periodic graphs can be found in [2]. Graph-theoretic properties of periodic graphs were studied in [6–8]. These are include calculation of the number of non-isomorphic periodic graphs with given number of vertices [6] and obtaining graph-theoretic criteria for periodic graphs [7] and their induced subgraphs [8].

Similar approach can be used for dynamics of continuous maps on finite topological trees (see [1] for the Sharkovsky-type result in this case). The defined for combinatorial trees and their vertex maps. Thus, periodic graphs appear as a particular case of Markov graphs where underlying trees are paths and maps are cyclic permutations. M-graphs then defined as Markov graphs up to isomorphism.

In [3] maps on trees were characterized for several classes of M-graphs including complete digraphs, complete bipartite digraphs, disjoint unions of cycles and digraphs in which each arc is a loop. It is also shown [4] that M-graphs satisfy Seymour’s Second Neighbourhood Conjecture as well as Caccetta–Häggkvist Conjecture. Various transformations including
deletion and addition of vertices, doubling and reverse doubling of vertices
and taking disjoint unions of M-graphs are studied in [5]. Also, it is proved
that there exist exactly 11 pairwise non-isomorphic M-graphs which are
tournaments as well as 86 pairwise non-isomorphic 3-vertex M-graphs
(again, see [5]).

In this paper we obtain several sufficient conditions for the disjoint
union of M-graphs to be an M-graph and prove that each weak component
of M-graph is an M-graph itself.

1. Definitions and preliminary results

In what follows map is just a function. For any given map \( \sigma \) by Im \( \sigma \)
and fix \( \sigma \) we denote its image and the set of its fixed points, respectively.

A graph \( G \) is a pair \( (V, E) \), where \( V = V(G) \) is the set of its vertices
and \( E = E(G) \) the set of its edges. By \( E_G(u) \) we denote the set of all edges incident to the vertex \( u \) in \( G \). A vertex \( u \) is called isolated if
\( |E_G(u)| = 0 \). Similarly, \( u \) is a leaf vertex provided \( |E_G(u)| = 1 \). The
unique edge incident to a leaf vertex is called a leaf edge. The set of all leaf
vertices in \( G \) is denoted by \( L(G) \). For the set of vertices \( A \subset V(G) \) we put
\( E(A) = \{uv \in E(G) : u, v \in A\} \) and \( \partial_AG = \{u \in A : E_G(u) - E(A) \neq \emptyset\} \).
By \( G[A] \) and \( G[E'] \) we denote the subgraphs of \( G \) induced by \( A \subset V(G) \)
and \( E' \subset E(G) \), respectively.

A graph \( G \) is called connected if for every pair of its vertices \( u, v \in V(G) \)
there exists a path joining them. The minimum number of edges in such
a path is called the distance \( d_G(u, v) \) between \( u \) and \( v \) in \( G \). The set of
vertices \( A \subset V(G) \) is connected if the induced subgraph \( G[A] \) is connected.
Similarly, \( E' \subset E(G) \) is connected if so is \( G[E'] \).

The eccentricity of a vertex \( u \) in a connected graph \( G \) is the value
\( ecc_G(u) = \max_{v \in V(G)} d_G(u, v) \). For the pair of vertices \( u, v \in V(G) \) in a
connected graph \( G \) we put \([u, v]_G = \{w \in V(G) : d_G(u, w) + d_G(w, v) = d_G(u, v)\} \). The set \( A \subset V(G) \) is convex
provided \([u, v]_G \subset A \) for all \( u, v \in A \). The convex hull Conv\(_G(A)\) of \( A \) is defined as the smallest
convex set containing \( A \).

Put \( d_G(u, A) = \min_{v \in A} d_G(u, v) \) and \( d_G(A, B) = \min_{u \in B} d_G(u, A) \)
for all vertex sets \( \{u\}, A, B \subset V(G) \) in a connected graph \( G \). The set
\( A \subset V(G) \) is called Chebyshev if for every vertex \( u \in V(G) \) there exists
a unique \( v \in A \) with \( d_G(u, v) = d_G(u, A) \). The corresponding map \( pr_A : V(G) \to V(G) \), where \( pr_A(u) = v \) is called the projection on a Chebyshev
set \( A \).
A tree is a connected acyclic graph. It should be noted that in a tree each connected set of vertices is Chebyshev.

A directed graph or digraph \( \Gamma \) is a pair \((V, A)\), where \( V = V(\Gamma) \) is the set of its vertices and \( A = A(\Gamma) \subseteq V \times V \) is the set of its arcs. If \((u, v) \in A(\Gamma)\), then we write \( u \rightarrow v \) in \( \Gamma \). The arc of the form \( u \rightarrow u \) is called a loop. For the vertex \( u \in V(\Gamma) \) we put \( N_\Gamma^+(u) = \{ v \in V(\Gamma) : u \rightarrow v \text{ in } \Gamma \} \) and \( N_\Gamma^-(u) = \{ v \in V(\Gamma) : v \rightarrow u \text{ in } \Gamma \} \). The cardinalities \( d_\Gamma^+(u) = |N_\Gamma^+(u)| \) and \( d_\Gamma^-(u) = |N_\Gamma^-(u)| \) are called the outdegree and the indegree of \( u \), respectively.

A digraph \( \Gamma \) is called complete provided \( A(\Gamma) = V(\Gamma) \times V(\Gamma) \). Similarly, \( \Gamma \) is empty if \( A(\Gamma) = \emptyset \). By \( K_n \) and \( \overline{K}_n \) we denote the complete and the empty digraph with \( n \) vertices, respectively.

A digraph is called weakly connected if its underlying graph (which is obtained by “forgetting” orientation of the edges and ignoring loops) is connected. Weak component of a digraph is its maximal weakly connected subgraph. By \( \Gamma_1 \sqcup \Gamma_2 \) we denote the disjoint union of digraphs \( \Gamma_1 \) and \( \Gamma_2 \).

A pair \((X, u_0)\) consisting of a tree \( X \) and its distinguished vertex \( u_0 \in V(X) \) is called a rooted tree. The digraph \( \Gamma \) which is obtained from the rooted tree \((X, u_0)\) by orienting the edges of \( X \) towards \( u_0 \) is called an in-tree. The vertex \( u_0 \) is the center of an in-tree \( \Gamma \). It is easy to see that for an in-tree its center is the unique vertex with zero outdegree.

For every map \( f : X \rightarrow X \) one can define its functional graph as a digraph with the vertex set \( X \) and the arc set \( \{(x, y) : f(x) = y\} \). A digraph is called functional if it is isomorphic to a functional graph for some map. It is easy to see that \( \Gamma \) is functional digraph if and only if \( d_\Gamma^+(v) = 1 \) for all \( v \in V(\Gamma) \). Similarly, \( \Gamma \) is called partial functional if \( d_\Gamma^+(v) \leq 1 \) for all \( v \in V(\Gamma) \). Each partial functional digraph \( \Gamma \) corresponds to some partial map of the form \( f : V(\Gamma) \rightarrow V(\Gamma) \).

**Definition 1.** Let \( X \) be a tree and \( \sigma : V(X) \rightarrow V(X) \) be some map. The Markov graph \( \Gamma = \Gamma(X, \sigma) \) has the vertex set \( V(\Gamma) = E(X) \) and there is an arc \( e_1 \rightarrow e_2 \) in \( \Gamma \) if \( u_2, v_2 \in [\sigma(u_1), \sigma(v_1)]_X \) for \( e_i = u_iv_i \in V(\Gamma), i = 1, 2 \). In other words, \( N_\Gamma^+(uv) = E([\sigma(u), \sigma(v)]_X) \) for all edges \( uv \in E(X) \).

**Example 1.** Consider the tree \( X \) with \( V(X) = \{1, \ldots, 7\} \), \( E(X) = \{12, 23, 34, 45, 26, 37\} \) and its map \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 3 & 6 & 2 & 4 & 2 \end{pmatrix} \) which are shown in Figure 1. Then the corresponding Markov graph \( \Gamma(X, \sigma) \) is shown in Figure 2.
A digraph $\Gamma$ is called an M-graph if there exists a pair $(X, \sigma)$ such that $\Gamma \simeq \Gamma(X, \sigma)$. Each such a pair is called the realization of $\Gamma$.

**Lemma 2.** [3] Let $X$ be a tree and $\sigma : V(X) \to V(X)$ be a map. Then for every pair of vertices $u, v \in V(X)$ and an edge $xy \in E([\sigma(u), \sigma(v)]_X)$ there exists an edge $wz \in E([u, v]_X)$ with $wz \to xy$ in $\Gamma(X, \sigma)$. In particular,

$$[\sigma(u), \sigma(v)]_X \subset \bigcup_{wz \in E([u, v]_X)} [\sigma(w), \sigma(z)]_X.$$ 

**Lemma 3.** [5] Let $X$ be a tree, $A \subset V(X)$ be some connected set of vertices, $\sigma : V(X) \to V(X)$ be a map and $\Gamma = \Gamma(X, \sigma)$. Then $\Gamma(X[A], \text{pr}_A \circ \sigma) = \Gamma[E(A)]$.

**Proposition 1.** Let $X$ be a tree and $\sigma : V(X) \to V(X)$ be some map. Put $E(\sigma) = \{e \in E(X) : d^-_\Gamma(e) \geq 1\}$. Then $E(\sigma) = E(\text{Conv}_X(\text{Im} \sigma))$. In particular, $X[E(\sigma)]$ is the connected subgraph of $X$.

**Proof.** Let $V_1 = V(E(\sigma))$ and $V_2 = \text{Conv}_X(\text{Im} \sigma)$. If $u \in V_1$, then there exists an edge $e = uv \in E(\sigma)$. By definition, $d^-_\Gamma(e) \geq 1$. This means
that there is an edge $e' = u'v' \in E(X)$ with $e' \to e$ in $\Gamma(X, \sigma)$, i.e. $u, v \in [\sigma(u'), \sigma(v')]_X$. Therefore, $u \in V_2$.

Conversely, suppose $u \in V_2$. Then there exists a pair of vertices $x, y \in V(X)$ such that $u \in [\sigma(x), \sigma(y)]_X$. At first, suppose that $\sigma(x) \neq \sigma(y)$. Then we can fix an edge $e = uv \in E([\sigma(x), \sigma(y)]_X)$. From Lemma 2 it follows that there is an edge $e' \in E([x, y]_X)$ with $e' \to e$ in $\Gamma$. Thus $d^-_\Gamma(e) \geq 1$ and $u \in V_1$. Otherwise, let $\sigma(x) = \sigma(y)$. Then $u \in \operatorname{Im} \sigma$. If $\sigma$ is a constant map, then $E(\sigma) = E(\operatorname{Conv}_X(\operatorname{Im} \sigma)) = \emptyset$. Thus, suppose that $\sigma$ is non-constant. This means that there exists a vertex $v \in \operatorname{Im} \sigma \setminus \{u\}$. Let $\sigma(z) = v$. Since $u \neq v$, we can fix an edge $e = uw \in E([u, v]_X)$. Again, by Lemma 2, $d^-_\Gamma(e) \geq 1$ which implies $u \in V_1$. \hfill \Box

2. Main results

From Lemma 3 it strictly follows that each nontrivial M-graph $\Gamma$ contains a vertex $v \in V(\Gamma)$ such that $\Gamma - \{v\}$ is also an M-graph. In [5] it was proved that any digraph obtained from an M-graph by deletion of a vertex with zero outdegree (in particular, an isolated vertex) is an M-graph itself. We generalize this result using the following theorem.

**Theorem 2.** Let $X$ be a tree and $\sigma : V(X) \to V(X)$ be some map. Suppose that we have a collection $A_i \subset V(X)$, $1 \leq i \leq m$ of pairwise disjoint connected sets such that for every $1 \leq i \leq m$ either $|\sigma(\partial_X A_i)| = 1$ or there exists $1 \leq j \leq m$ with $\sigma(\partial_X A_i) \subset A_j$. Then $\Gamma(X, \sigma) - \bigcup_{i=1}^m E(A_i)$ is an M-graph.

**Proof.** Consider the set of indices $I_1 = \{1 \leq i \leq m : |\sigma(\partial_X A_i)| = 1\}$ and the corresponding map $g : I_1 \to V(X)$, where $\sigma(\partial_X A_i) = \{g(i)\}$ for all $i \in I_1$. Similarly, the set of indices $I_2 = \{1, \ldots, m\} - I_1$ defines the map $f : I_2 \to \{1, \ldots, m\}$, where $\sigma(\partial_X A_i) \subset A_{f(i)}$ for all $i \in I_2$.

Take a graph $X - \bigcup_{i=1}^m A_i$ and add to it $m$ new vertices $z_i$ for each $1 \leq i \leq m$ with new edges $z_iy_i$ for all $y_i \in \partial_X(V(X) - A_i)$ to obtain a new graph $X'$. It is easy to see that $X'$ is a tree (one can think of $X'$ as of tree which is obtained from $X$ by “contracting” sets $A_i$ into points). Put

$$
\sigma'(x) = \begin{cases} 
z_i & \text{if } \sigma(x) \in A_i, 
g(i) & \text{if } x = z_i \text{ and } i \in I_1, 
z_{f(i)} & \text{if } x = z_i \text{ and } i \in I_2, 
\sigma(x) & \text{otherwise},
\end{cases}
$$

for all $x \in V(X')$. Then $\Gamma(X, \sigma) - \bigcup_{i=1}^m E(A_i) \simeq \Gamma(X', \sigma')$. \hfill \Box
Corollary 1. Let $\Gamma$ be an M-graph and $v \in V(\Gamma)$ be its vertex with $N^+_{\Gamma}(v) \subset \{v\}$. Then $\Gamma - \{v\}$ is an M-graph. Moreover, if $N^+_{\Gamma}(v) = \{v\}$, then there exists a realization $(X, \sigma)$ of $\Gamma - \{v\}$ such that $\text{fix} \sigma \neq \emptyset$.

Proof. Fix some realization $(X, \sigma)$ of $\Gamma$. Let the edge $e = ux \in E(X)$ corresponds to the vertex $v \in V(\Gamma)$. If $N^+_{\Gamma}(v) = \emptyset$, then $\sigma(u) = \sigma(x)$. In this case for the connected set of vertices $A = \{u, x\}$ we have $\left|\sigma(\partial_X A)\right| = 1$. By Theorem 2, $\Gamma - \{v\}$ is an M-graph.

Otherwise, let $N^+_{\Gamma}(v) = \{v\}$. Then $\sigma(u) = u$ and $\sigma(x) = x$, or $\sigma(u) = x$ and $\sigma(x) = u$. In both cases $\sigma(\partial_X A) \subset A$. Again, by Theorem 2, $\Gamma - \{v\}$ is an M-graph. Moreover, with the notation of Theorem 2, $\sigma'(z_1) = z_1$ (here $A = A_1$). Therefore, in this case $\text{fix} \sigma' \neq \emptyset$.

Example 2. Consider the tree $X$ with $V(X) = \{1, \ldots, 7\}$, $E(X) = \{12, 23, 34, 16, 25, 67\}$ and its map $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 5 & 4 & 5 & 2 & 3 \end{pmatrix}$. Then $d^-_{\Gamma(X, \sigma)}(16) = 0$, however $\Gamma(X, \sigma) - \{16\}$ is not an M-graph (see Figure 3).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{Markov graph $\Gamma(X, \sigma)$ for which $\Gamma(X, \sigma) - \{16\}$ is not an M-graph.}
\end{figure}

Denote by $V_0(\Gamma) = \{v \in V(\Gamma) : d^-_{\Gamma}(v) = 0\}$ the set of vertices with zero indegree in $\Gamma$.

Proposition 2. For every M-graph $\Gamma$ and $0 \leq k \leq |V_0(\Gamma)|$ there exists $V' \subset V_0(\Gamma)$ with $|V'| = k$ such that $\Gamma - V'$ is an M-graph. In particular, $\Gamma - V_0(\Gamma)$ is an M-graph.

Proof. Fix a realization $(X, \sigma)$ of $\Gamma$. Let the edge set $E' \subset E(X)$ corresponds to $V_0(\Gamma)$. By Proposition 1, the set $E(X) - E' = E(\sigma)$ is connected. Since $X$ is a connected graph, for any $0 \leq k \leq |V_0(\Gamma)|$ there exists a connected set of edges $E'' \subset E(X)$ with $E(X) - E' \subset E''$. 

and $|E'| = |E(X)| - k$. Let $V' \subset V(\Gamma)$ corresponds to $E(X) - E''$. Then $|V'| = k$ and by Lemma 3, $\Gamma - V' \simeq \Gamma(X, \sigma) - (E(X) - E'') = \Gamma(X, \sigma)[E''] = \Gamma(X[E''], pr_{V(E')} \circ \sigma)$ is an M-graph.

Note that any digraph obtained from an M-graph by addition of an isolated vertex is also an M-graph. Using this fact one can conclude that $\Gamma$ is an M-graph if and only if so is $\Gamma \sqcup \overline{K}_1$. However, not every disjoint union of two M-graphs is an M-graph itself.

**Example 3.** Suppose that $\Gamma$ is obtained from the complete digraph with two vertices $K_2$ by deletion of a loop. Then $\Gamma$ is an M-graph, but $\Gamma \sqcup K_1$ is not (see Figure 4).

**Figure 4.** Disjoint union of two M-graphs which is not an M-graph.

**Remark 1.** [5] If we have a pair of trees $X_i, i = 1, 2$ and a pair of their maps $\sigma_i : V(X_i) \to V(X_i)$ with fix $\sigma_i \neq \emptyset, i = 1, 2$, then the disjoint union $\Gamma(X_1, \sigma_1) \sqcup \Gamma(X_2, \sigma_2)$ is an M-graph. Indeed, “gluing” realizations $(X_1, \sigma_1)$ and $(X_2, \sigma_2)$ together along some pair of fixed vertices we obtain the realization of $\Gamma(X_1, \sigma_1) \sqcup \Gamma(X_2, \sigma_2)$.

As a corollary of the construction in Remark 1 one can obtain a sufficient condition for the disjoint union of two M-graphs to be an M-graph.

**Corollary 2.** [5] Let $\Gamma_1$ and $\Gamma_2$ be a pair of M-graphs with even numbers of loops in each. Then $\Gamma_1 \sqcup \Gamma_2$ is an M-graph. In particular, any disjoint union of two M-graphs without loops is an M-graph itself.

It turns out that for any given M-graph we can provide a graph-theoretic criterion for the existence of its realization $(X, \sigma)$ with fix $\sigma \neq \emptyset$.

**Proposition 3.** Let $\Gamma$ be a digraph. Then $\Gamma \sqcup K_1$ is an M-graph if and only if $\Gamma$ is an M-graph and there exists its realization $(X, \sigma)$ with fix $\sigma \neq \emptyset$.

**Proof.** Sufficiency of this condition follows from Remark 1, since for $K_1$ there obviously exists its realization $(X, \sigma)$ with fix $\sigma \neq \emptyset$. Thus, we must prove only the necessity of this condition. To do so fix a realization $(X', \sigma')$ of $\Gamma \sqcup K_1$. Let the vertex $v \in V(\Gamma \sqcup K_1)$ corresponds to a unique vertex from $K_1$. Then $N^+_{\Gamma \sqcup K_1}(v) = \{v\}$ implying that by Corollary 1, $\Gamma$ is an M-graph and there exists its realization $(X, \sigma)$ with fix $\sigma \neq \emptyset$. □
Corollary 3. If $\Gamma \sqcup K_1$ is an M-graph, then there exists its realization $(X, \sigma)$ with $\text{fix } \sigma \neq \emptyset$.

Combining Remark 1 and Proposition 3, we obtain the following result.

Proposition 4. If for a pair of digraphs $\Gamma_1$ and $\Gamma_2$ the digraphs $\Gamma_1 \sqcup K_1$ and $\Gamma_2 \sqcup K_1$ are M-graphs, then $\Gamma_1 \sqcup \Gamma_2$ is an M-graph.

Theorem 3. Let $\Gamma_1$ be an M-graph and $\Gamma_2$ be acyclic partial functional digraph. Then $\Gamma_1 \sqcup \Gamma_2$ is also an M-graph.

Proof. Without loss of generality, we can assume that $\Gamma_2$ is weakly connected. Since $\Gamma_2$ is acyclic and partially functional, $\Gamma_2$ is an in-tree. Let $x_0 \in V(\Gamma_2)$ be its center (thus, $d^+_2(x_0) = 0$). Denote by $X'$ the underlying tree of $\Gamma_2$. For every $0 \leq i \leq \text{ecc}_{X'}(x_0)$ put $a_i = |N^i_{X'}(x_0)|$ for the cardinality of the sphere with radius $i$ centered at $x_0$ in $X'$.

Now fix a realization $(X, \sigma_0)$ of $\Gamma_1$. Since $V(X)$ is finite, $\sigma_0$ has a periodic point $u_0 \in V(X)$ with period $m \geq 1$. Consider the restriction $\sigma = \sigma_{0|\text{orb}_{\sigma_0}(u_0)}$ of $\sigma_0$ to $\text{orb}_{\sigma_0}(u_0)$. Clearly, $\sigma$ is a cyclic permutation of $\text{orb}_{\sigma_0}(u_0)$.

For every $0 \leq i \leq \text{ecc}_{X'}(x_0)$ add $a_i$ new vertices $y^i_1, \ldots, y^i_{a_i}$ to $X$ with the new edges $y^i_j \sigma^{-i \mod m}(u_0)$ for all $1 \leq j \leq a_i$ (of course, $\sigma^0_0(u_0) = u_0$). Denote the obtained tree as $X''$. For all $u \in V(X'')$ put

$$\sigma'(u) = \begin{cases} 
\sigma(u) & \text{if } u \in V(X), \\
 y^i_{j-1} & \text{if } u = y^i_j, i \geq 1 \text{ and } N^i_{\Gamma_2}(x^i_j) = \{x^i_{j-1}\}, \\
 \sigma(u_0) & \text{if } u = y^i_0,
\end{cases}$$

where $N^i_{X'}(x_0) = \{x^i_j : 1 \leq j \leq a_i\}$ (for example, $N^0_{X'}(x_0) = \{x^0_1\} = \{x_0\}$). Then $\Gamma(X'', \sigma') \simeq \Gamma_1 \sqcup \Gamma_2$ (the edges from $E(X)$ correspond to the vertices of $\Gamma_1$ and edges of the form $y^i_j \sigma^{-i \mod m}(u_0)$ correspond to the vertices $x^i_j$).

Note that the acyclicity condition in Theorem 3 is essential as can be seen from the digraph in Example 3.

Example 4. Consider the pair $(X, \sigma)$ from Example 1 and the corresponding Markov graph $\Gamma_1 = \Gamma(X, \sigma)$. Also, let $\Gamma_2$ be the in-tree depicted in Figure 5 (the vertices of $\Gamma_2$ are labeled according to the notation in the proof of Theorem 3). Thus, $x_0$ is the center of $\Gamma_2$, $\text{ecc}_{X'}(x_0) = 2$, $N^1_{X'}(x_0) = \{x^1_1, x^1_2\}$, $N^2_{X'}(x_0) = \{x^2_1, x^2_2\}$ and $a_1 = a_2 = 2$. Put $u_0 = 4.$
Then \( \text{orb}_\sigma(u_0) = \{4, 6\} \) and therefore \( m = 2 \). The corresponding tree \( X'' \) is shown in Figure 6. We also have \( \sigma'(y_1^0) = 6, \sigma'(y_1^1) = \sigma'(y_2^1) = y_1^0 \) and \( \sigma'(y_2^2) = \sigma'(y_2^3) = y_2^3 \).

**Theorem 4.** The disjoint union of any collection of weak components (in particular, each weak component) in an M-graph is an M-graph itself.

**Proof.** It is sufficient to prove that for any M-graph \( \Gamma \) and its weak component \( \Gamma' \) the digraph \( \Gamma - V(\Gamma') \) is an M-graph. To do so fix a realization \( (X, \sigma) \) of \( \Gamma \). Let the set of edges \( E' \subset E(X) \) corresponds to the vertex set of \( \Gamma' \). Consider the components \( X_1, \ldots, X_m \) of the induced subgraph \( X[E'] \) in \( X \) and put \( A_i = V(X_i) \) for all \( 1 \leq i \leq m \).

Suppose that for some \( 1 \leq i \leq m \) there exists a vertex \( x \in A_i \) with \( \sigma(x) \notin \cup_{j=1}^m A_j = V(X[E']) \). Then \( E_X(\sigma(x)) \cap E' = \emptyset \). If for some \( y \in A_i \) we have \( \sigma(x) \neq \sigma(y) \), then \( d_X(\sigma(x), \sigma(y)) \geq 2 \). This means that there is a vertex \( u \in [\sigma(x), \sigma(y)]_X \) such that \( e = u\sigma(x) \in E(X) \). By Lemma 2, there exists an edge \( e' \in E([x, y]_X) \subset E(A_i) \) with \( e' \rightarrow e \) in \( \Gamma(X, \sigma) \). Since
e \notin E', \Gamma' is not a weak component of \Gamma. The obtained contradiction implies that in this case we have \sigma(x) = \sigma(y) for every y \in A_i. In other words, |\sigma(A_i)| = 1.

Now let 1 \leq i \leq m is fixed and for every vertex x \in A_i there exists 1 \leq j_x \leq m with \sigma(x) \in A_{j_x}. We want to prove that in this case j_x = j_y for each pair of vertices x, y \in A_i. To the contrary, suppose j_x \neq j_y for some x, y \in A_i. Then d_X(A_{j_x}, A_{j_y}) \geq 1. This implies the existence of an edge e \in E([\sigma(x), \sigma(y)]_X) - \cup_{k=1}^m E(A_k). Again, from Lemma 2 it follows that there exists e' \in E([x, y]_X) \subset E(A_i) with e' \rightarrow e in \Gamma(X, \sigma) which is a contradiction. Thus, in this case \sigma(A_i) \subset A_j for some 1 \leq j \leq m. Theorem 2 now asserts that \Gamma - V(\Gamma') is an M-graph.

\textbf{Corollary 4.} If for a pair of digraphs \Gamma_1 and \Gamma_2 their disjoint union \Gamma_1 \sqcup \Gamma_2 is an M-graph, then both \Gamma_1 and \Gamma_2 are M-graphs.

\textit{Proof.} Clearly, \Gamma_1 and \Gamma_2 are both disjoint unions of weak components in \Gamma_1 \sqcup \Gamma_2.

\textbf{Proposition 5.} Let \Gamma_1 and \Gamma_2 be a pair of nontrivial digraphs having loops at each of their vertices. Then \Gamma_1 \sqcup \Gamma_2 is an M-graph if and only if \Gamma_1 \sqcup K_1 and \Gamma_2 \sqcup K_1 are both M-graphs.

\textit{Proof.} The sufficiency of this condition strictly follows from Proposition 4. To prove its necessity fix a realization (X, \sigma) of \Gamma = \Gamma_1 \sqcup \Gamma_2. Let \Gamma' be a weak component in \Gamma and E' \subset E(X) be the corresponding set of edges in X. We want to prove that E' is connected. To the contrary, suppose that there is a partition E' = E_1 \sqcup E_2 with d_X(V(E_1), V(E_2)) \geq 1.

Since \Gamma' is weakly connected, there is a pair of edges e_1 = u_1v_1 \in E_i, i = 1, 2 with e_1 \rightarrow e_2 or e_2 \rightarrow e_1 in \Gamma'. Without loss of generality, assume e_1 \rightarrow e_2 in \Gamma'. We have e_1, e_2 \in N^-_{\Gamma(X, \sigma)}(e_1) which implies [u_1, u_2]_X \subset [\sigma(u_1), \sigma(v_1)]_X. However, the inequality d_X(V(E_1), V(E_2)) \geq 1 asserts E([u_1, u_2]_X) - E' \neq \emptyset. In other words, there exists an edge e' \notin E' such that e_1 \rightarrow e' in \Gamma. Therefore, \Gamma' is not a weak component in \Gamma. The obtained contradiction proves that E' is connected.

By Lemma 3, \Gamma' \simeq \Gamma(X, \sigma)[E'] = \Gamma(X[E'], pr_{V(E')} \circ \sigma). Furthermore, since \Gamma_1 and \Gamma_2 are nontrivial digraphs, we have \Gamma \neq \Gamma'. This implies \partial_X V(E') \neq \emptyset. Fix a vertex w \in \partial_X V(E') and an edge e \in E_X(w) - E'. Let E'' be the vertex set of the weak component in \Gamma(X, \sigma) which contains e. Similarly, we can prove that E'' is connected. Also, note that w \in \partial_X V(E''). Finally, the proof of Theorem 4 implies that \sigma(\partial_X V(E')) \subset
$\partial_X V(E')$ as well as $\sigma(\partial_X V(E'')) \subset \partial_X V(E'').$ Hence, we can conclude that $\sigma(w) = w.$

Thus, for every weak component $\Gamma'$ of $\Gamma$ there exists its realization $(X', \sigma')$ with $\operatorname{fix} \sigma' \neq \emptyset.$ But $\Gamma_1$ (as well as $\Gamma_2$) is a disjoint union of weak components in $\Gamma.$ Combining this fact with Remark 1 and Proposition 4, we obtain that $\Gamma_1 \sqcup K_1$ as well as $\Gamma_2 \sqcup K_1$ are M-graphs. \hfill \square

**Example 5.** Consider the path $X \simeq P_4$ with $V(X) = \{1, 2, 3, 4\}, \ E(X) = \{12, 23, 34\}$ and its vertex map $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}.$ Then $\Gamma(X, \sigma)$ has a loop at each vertex, but $\Gamma(X, \sigma) \sqcup K_1$ is not an M-graph.

**References**


**Contact Information**

S. Kozerenko

Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, Volodymyrska str., 64, 01033 Kyiv, Ukraine

E-Mail(s): kozerenko@ukr.net

Received by the editors: 12.03.2017
and in final form 02.11.2017.