On recurrence in $G$-spaces

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To the memory of Vitaly Sushchansky

Abstract. We introduce and analyze the following general concept of recurrence. Let $G$ be a group and let $X$ be a $G$-space with the action $G \times X \to X$, $(g, x) \mapsto gx$. For a family $\mathcal{F}$ of subsets of $X$ and $A \in \mathcal{F}$, we denote $\Delta_{\mathcal{F}}(A) = \{ g \in G : gB \subseteq A \text{ for some } B \in \mathcal{F}, B \subseteq A \}$, and say that a subset $R$ of $G$ is $\mathcal{F}$-recurrent if $R \cap \Delta_{\mathcal{F}}(A) \neq \emptyset$ for each $A \in \mathcal{F}$.

Let $G$ be a group with the identity $e$ and let $X$ be a $G$-space, a set with the action $G \times X \to X$, $(g, x) \mapsto gx$. If $X = G$ and $gx$ is the product of $g$ and $x$ then $X$ is called a left regular $G$-space.

Given a $G$-space $X$, a family $\mathcal{F}$ of subsets of $X$ and $A \in \mathcal{F}$, we denote

$$\Delta_\mathcal{F}(A) = \{ g \in G : gB \subseteq A \text{ for some } B \in \mathcal{F}, B \subseteq A \}.$$

Clearly, $e \in \Delta_\mathcal{F}(A)$ and if $\mathcal{F}$ is upward directed ($A \in \mathcal{F}, A \subseteq C$ imply $C \in \mathcal{F}$) and if $\mathcal{F}$ is $G$-invariant ($A \in \mathcal{F}$, $g \in G$ imply $gA \in \mathcal{F}$) then

$$\Delta_\mathcal{F}(A) = \{ g \in G : gA \cap A \in \mathcal{F} \}, \quad \Delta_\mathcal{F}(A) = (\Delta_\mathcal{F}(A))^{-1}.$$

If $X$ is a left regular $G$-space and $\emptyset \notin \mathcal{F}$ then $\Delta_\mathcal{F}(A) \subseteq AA^{-1}$.

For a $G$-space $X$ and a family $\mathcal{F}$ of subsets of $X$, we say that a subset $R$ of $G$ is $\mathcal{F}$-recurrent if $\Delta_\mathcal{F}(A) \cap R \neq \emptyset$ for every $A \in \mathcal{F}$. We denote by $\mathcal{R}_\mathcal{F}$ the filter on $G$ with the base $\cap \{ \Delta_\mathcal{F}(A) : A \in \mathcal{F} \}$, where $\mathcal{F}'$ is a finite subfamily of $\mathcal{F}$, and note that, for an ultrafilter $p$ on $G$, $\mathcal{R}_\mathcal{F} \in p$ if and only if each member of $p$ is $\mathcal{F}$-recurrent.

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The notion of an $\mathcal{F}$-recurrent subset is well-known in the case in which $G$ is an amenable group, $X$ is a left regular $G$-space and $\mathcal{F} = \{ A \subseteq X : \mu(A) > 0 \text{ for some left invariant Banach measure } \mu \text{ on } X \}$. See [1] and [2] for historical background.

Now we endow $G$ with the discrete topology and identify the Stone-Čech compactification $\beta G$ of $G$ with the set of all ultrafilters on $G$. Then the family $\{ \overline{A} : A \subseteq G \}$, where $\overline{A} = \{ p \in \beta G : A \in p \}$, forms a base for the topology of $\beta G$. Given a filter $\varphi$ on $G$, we denote $\varphi = \cap \{ A : A \subseteq \varphi \}$.

We use the standard extension [3] of the multiplication on $G$ to the semigroup multiplication on $\beta G$. We take two ultrafilters $p, q \in \beta G$, choose $P \in p$ and, for each $x \in P$, pick $Q_x \in q$. Then $\cup_{x \in P} xQ_x \in pq$ and the family of these subsets forms a base of the ultrafilter $pq$.

We recall [4] that a filter $\varphi$ on a group $G$ is left topological if $\varphi$ is a base at the identity $e$ for some (uniquely at defined) left translation invariant (each left shift $x \mapsto gx$ is continuous) topology on $G$. If $\varphi$ is left topological then $\varphi$ is a subgroup of $\beta G$ [4]. If $G = X$ and a filter $\varphi$ is left topological then $\varphi = R\varphi$.

**Proposition 1.** For every $G$-space $X$ and any family $\mathcal{F}$ of subsets of $X$, the filter $R\mathcal{F}$ is left topological.

**Proof.** By [4], a filter $\varphi$ on a group $G$ is left topological if and only if, for every $\Phi \in \varphi$, there is $H \in \varphi, H \subseteq \Phi$ such that, for every $x \in H$, $xH_x \subseteq \Phi$ for some $H_x \in \varphi$.

We take an arbitrary $A \in \mathcal{F}$, put $\Phi = \triangle \mathcal{F}(A)$ and, for each $g \in \triangle \mathcal{F}(A)$, choose $B_g \in \mathcal{F}$ such that $gB_g \in A$. Then $g\triangle \mathcal{F}(B_g) \subseteq \triangle \mathcal{F}(A)$ so put $H = \Phi$.

To conclude the proof, let $A_1, \ldots, A_n \in \mathcal{F}$. We denote

$$\Phi_1 = \triangle \mathcal{F}(A_1), \ldots, \Phi_n = \triangle \mathcal{F}(A_n), \quad \Phi = \Phi_1 \cap \ldots \cap \Phi_n.$$ 

We use the above paragraph, to choose $H_1, \ldots, H_n$ corresponding to $\Phi_1, \ldots, \Phi_n$ and put $H = H_1 \cap \ldots \cap H_n$. 

Let $X$ be a $G$-space and let $\mathcal{F}$ be a family of subsets of $X$. We say that a family $\mathcal{F}'$ of subsets of $X$ is $\mathcal{F}$-disjoint if $A \cap B \notin \mathcal{F}$ for any distinct $A, B \in \mathcal{F}'$.

A family $\mathcal{F}'$ of subsets of $X$ is called $\mathcal{F}$-packing large if, for each $A \in \mathcal{F}'$, any $\mathcal{F}$-disjoint family of subsets of $X$ of the form $gA, g \in G$ is finite.
We say that a subset $S$ of a group $G$ is a $\Delta_\omega$-set if $e \in A$ and every infinite subset $Y$ of $G$ contains two distinct elements $x, y$ such that $x^{-1}y \in S$ and $y^{-1}x \in S$.

**Proposition 2.** Let $X$ be a $G$-space and let $\mathcal{F}$ be a $G$-invariant upward directed family of subsets of $X$. Then $\mathcal{F}$ is $\mathcal{F}$-packing large if and only if, for each $A \in \mathcal{F}$, the subset $\Delta_\mathcal{F}(A)$ of $G$ is a $\Delta_\omega$-set.

**Proof.** We assume that $\mathcal{F}$ is $\mathcal{F}$-packing large and take an arbitrary infinite subset $Y$ of $G$. Then we choose distinct $g, h \in Y$ such that $gA \cap hA \in \mathcal{F}$, so $g^{-1}h \in \Delta_\mathcal{F}(A)$, $hg \in \Delta_\mathcal{F}(A)$ and $\Delta_\mathcal{F}(A)$ is a $\Delta_\omega$-set.

Now we suppose that $\Delta_\mathcal{F}(A)$ is a $\Delta_\omega$-set and take an arbitrary infinite subset $Y$ of $G$. Then there are distinct $g, h \in Y$ such that $g^{-1}h \in \Delta_\mathcal{F}(A)$ so $g^{-1}hA \cap A \in \mathcal{F}$ and $gA \cap hA \in \mathcal{F}$. It follows that the family $\{gA : g \in Y\}$ is not $\mathcal{F}$-disjoint. \hfill $\square$

**Proposition 3.** For every infinite group $G$, the following statements hold

(i) a subset $A \subseteq G$ is a $\Delta_\omega$-set if and only if $e \in A$ and every infinite subset $Y$ of $G$ contains an infinite subset $Z$ such that $x^{-1}y \in A$, $y^{-1}x \in A$ for any distinct $x, y \in Z$;

(ii) the family $\varphi$ of all $\Delta_\omega$-sets of $G$ is a filter;

(iii) if $A \in \varphi$ then $G = FA$ for some finite subset $F$ of $G$.

**Proof.** (i) We assume that $A$ is a $\Delta_\omega$-set and define a coloring $\chi$ of $[Y]^2$, $\chi : [Y]^2 \to \{0, 1\}$ by the rule: $\chi(\{x, y\}) = 1$ if and only if $x^{-1}y \in A$, $y^{-1}x \in A$. By the Ramsey theorem, there is an infinite subset $Z$ of $Y$ such that $\chi$ is monochrome on $[Z]^2$. Since $A$ is a $\Delta_\omega$-set $\chi(\{x, y\}) = 1$ for all $\{x, y\} \in [Z]^2$.

(ii) follows from (i).

(iii) We assume the contrary and choose an injective sequence $(x_n)_{n \in \omega}$ in $G$ such that $x_{n+1} \notin x_iA$ for each $i \in \{0, \ldots, n\}$, and denote $Y = \{x_n : n \in \omega\}$. Then $x_{m}^{-1}x_n \in A$ for every $m, n, m < n$, so $A$ is not a $\Delta_\omega$-set. \hfill $\square$

**Proposition 4.** Let $G$ be a infinite group and let $\varphi$ denotes the filter of all $\Delta_\omega$-sets of $G$. Then $\varphi$ is the smallest closed subset of $\beta G$ containing all ultrafilters on $G$ of the form $q^{-1}q$, $q \in \beta G$, $g^{-1} = \{A^{-1} : A \in q\}$.

**Proof.** We denote by $Q$ the smallest closed subset of $\beta G$ containing all $q^{-1}q$, $q \in \beta G$. It follows directly from the definition of the multiplication in $\beta G$ that $p \in Q$ if and only if either $p$ is principal and $p = e$ or, for each $P \in p$, there is an injective sequence $(x_n)_{n \in \omega}$ in $G$ such that $x_{m}^{-1}x_n \in P$ for all $m < n$. 

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Applying Proposition 3(i), we conclude that $q^{-1}q \in \overline{\varphi}$ for each $q \in \beta G$ so $Q \subseteq \overline{\varphi}$. On the other hand, if $p \notin \overline{\varphi}$ then there is $P \in p$ such that $G \setminus P$ is a $\Delta_\omega$-set. By above paragraph, $p \notin Q$ so $\overline{\varphi} \subseteq Q$. \hfill \Box

Now let $G$ be an amenable group, $X$ be a left regular $G$-space and $\mathcal{F} = \{A \in X : \mu(A) > 0\}$ for some left invariant Banach measure $\mu$ on $G$. For combinatorial characterization of $\mathcal{F}$ see [6]. Clearly, $\mathcal{F}$ is upward directed $G$-invariant and $\mathcal{F}$-packing large. By Proposition 2, $\overline{\varphi} \subseteq \overline{\mathcal{F}}$. By Proposition 4, $\overline{\mathcal{F}}$ contains all ultrafilters of the form $q^{-1}q$, $q \in \beta G$, so we get Theorem 3.14 from [1].

We suppose that a $G$-space $X$ is endowed with a $G$-invariant probability measure $\mu$ defined on some ring of subsets of $X$. Then the family $\mathcal{F}\{A \subseteq X : \mu(B) > 0\text{ for some } B \subseteq A\}$ is $\mathcal{F}$-packing large.

In particular, we can take a compact group $X$, endow $X$ with the Haar measure, choose an arbitrary subgroup $G$ of $X$ and endow $G$ with the discrete topology.

Another example: let a discrete group $G$ acts on a topological space $X$ so that, for each $g \in G$, the mapping $X \rightarrow X$, $(g, x) \mapsto gx$ is continuous. We take a point $x \in X$, denote by $\mathcal{F}$ the filter of all neighborhoods of $x$, and recall that $x$ is recurrent if, for every $U \in \mathcal{F}$, there exists $g \in G \setminus \{e\}$ such that $gx \in U$. Clearly, $x$ is a recurrent point if and only if $G \setminus \{e\}$ if a set of $\mathcal{F}$-recurrence, so by Proposition 1, $x$ defines some non-discrete left translation invariant topology on $G$.

**Proposition 5.** Let $G$ be an infinite group, $A$ be a $\Delta_\omega$-set of $G$ and let $\tau$ be a left translation invariant topology on $G$ with continuous inversion $x \mapsto x^{-1}$ at the identity $e$. Then the closure $cl_\tau A$ is a neighborhood of $e$ in $\tau$.

**Proof.** On the contrary, we suppose that $cl_\tau A$ is not a neighborhood of $e$, put $U = G \setminus cl_\tau A$. Then $U$ is open and $e \in cl_\tau U$.

We take an arbitrary $x_0 \in U$ and choose an open neighborhood $U_0$ of the identity such that $x_0U_0^{-1} \subseteq U$. Then we take $x_1 \in U_0 \cap U$ and choose an open neighborhood $U_1$ of $e$ such that $U_1 \subseteq U_0$ and $x_1U_1^{-1} \subseteq U$. We take $x_2 \in U_1 \cap U$ and choose an open neighborhood $U_0$ of $e$ such that $U_2 \subseteq U_1$ and $x_2U_2^{-1} \subseteq U$ and so on. After $\omega$ steps, we get a sequence $(x_n)_{n \in \omega}$ in $G$ such that $x_nx_m^{-1} \subseteq U$ for all $n < m$. We denote $Y = \{x_n^{-1} : n \in \omega\}$. Then $(x_n^{-1})^{-1}x_m^{-1} \in A$ for all $n < m$, so $A$ is not a $\Delta_\omega$-set. \hfill \Box

A subset $A$ of an infinite group $G$ is called a $\Delta_{<\omega}$-set if $e \in A$ and there exists a natural number $n$ such that every subset $Y$ of $G$, $|Y| = n$
contains two distinct \( x, y \in Y \) such that \( x^{-1}y \in A \), \( y^{-1}x \in A \). These subsets were introduced in [5] under name thick subsets, but thick subsets are well-known in combinatorics with another meaning [3]: \( A \) is thick if, for every finite subset \( F \) of, there is \( g \in A \) such that \( Fg \subseteq A \). The family \( \psi \) of all \( \triangleleft \omega \)-sets of \( G \) is a filter [5], clearly, \( \psi \subseteq \varphi \). Every infinite group \( G \) has a \( \triangle \omega \)-set but not \( \triangleleft \omega \)-set \( A \): it suffices to choose inductively a sequence \( (X_n)_{n \in \omega} \) of subsets of \( G \), \( |X_n| = n \) such that \( \bigcup_{n \in \omega} X_n^{-1}X_n \) has no infinite subsets of the form \( Y^{-1}Y \) and put

\[
A = \{e\} \cup (G \setminus \bigcup_{n \in \omega} X_n^{-1}X_n),
\]
so \( \psi \subset \varphi \).

By analogy with Propositions 3 and 4, we can prove

**Proposition 6.** Let \( G \) be an infinite group and let \( \psi \) be the filter of all \( \triangleleft \omega \)-subsets of \( G \). Then \( p \in \bar{\psi} \) if and only if either \( p \) is principal and \( p = e \) or, for every \( A \in p \), there exists a sequence \( (X_n)_{n \in \omega} \) of subsets of \( G \), \( |X_n| = n + 1 \), \( X_n = \{x_{n0}, \ldots, x_{nn}\} \) such that \( x_{ni}^{-1}x_{nj} \in A \) for all \( i < j \leq n \).

Let \( A \) be a subset of a group \( G \) such that \( e \in A \), \( A = A^{-1} \). We consider the Cayley graph \( \Gamma_A \) with the set of vertices \( G \) and the set of edges \( \{\{x, y\} : x^{-1}y \in A, x \neq y\} \). We recall that a subset \( S \) of vertices of a graph is **independent** if any two distinct vertices from \( S \) are not incident. Clearly, \( A \) is a \( \triangle \omega \)-set if and only if any independent set in \( \Gamma_A \) is finite, and \( A \) is \( \triangle \omega \)-set if and only if there exists a natural number \( n \) such that any independent set \( S \) is of size \( |S| < n \).

**Problem 1.** Characterize all infinite graphs with only finite independent set of vertices.

**Problem 2.** Given a natural number \( n \), characterize all infinite graphs such that any independent set \( S \) of vertices is of size \( |S| < n \).

In the context of this note, above problems are especially interesting in the case of Cayley graphs of groups.

**References**


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