On the saturations of submodules

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Abstract. Let $R \subseteq S$ be a ring extension, and let $A$ be an $R$-submodule of $S$. The saturation of $A$ (in $S$) by $\tau$ is set $A_{[\tau]} = \{x \in S : tx \in A \text{ for some } t \in \tau\}$, where $\tau$ is a multiplicative subset of $R$. We study properties of saturations of $R$-submodules of $S$. We use this notion of saturation to characterize star operations $\star$ on ring extensions $R \subseteq S$ satisfying the relation $(A \cap B)^\star = A^\star \cap B^\star$ whenever $A$ and $B$ are two $R$-submodules of $S$ such that $AS = BS = S$.

1. Introduction

Throughout this paper, all rings considered are commutative with identity. Let $R \subseteq S$ be a ring extension, and let $A$ be an $R$-submodule of $S$. The saturation of $A$ (in $S$) by $\tau$ is set

$$A_{[\tau]} = \{x \in S : tx \in A \text{ for some } t \in \tau\},$$

where $\tau$ is a multiplicative subset of $R$ [7, Definition 10, p. 18]. If $p$ is a prime ideal of $R$, then the saturation of $A$ with respect to $\tau = R \setminus p$ is denoted $A_{[p]}$. If $A$ is an $R$-submodule of $S$ and $M$ is the set of all maximal ideals of $R$, then $A = \bigcap_{p \in M}(R_{[p]} A_{[p]})$ and $R = \bigcap_{p \in M} R_{[p]}$ [7, Remark 5.5, p. 50]. In their book [7], Knebusch and Zhang use this

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notion of saturation to define and study Prüfer extension; see [7, pages 46-73]. In her paper [8], McNair studies properties of saturations of the form
\[ I_\tau = \{ x \in S : tx \in I \text{ for some } t \in \tau \}, \]
whenever \( I \) is an ideal of \( R \), and \( \tau \) is a multiplicative subset of \( R \). In particular, McNair proved that if \( R \subseteq S \) is a ring extension, and \( I \) is a radical ideal of \( R \), then \( I[p] \cap R = p \) and \( I[p] = p(p) \) for any minimal prime ideal \( p \) of \( I \) [8, Proposition 2.1]. Furthermore, if \( R \subseteq S \subseteq T \) and \( p \) is a prime ideal of \( S \) such that \( S[p] = R[p \cap R] \) (saturation in \( T \)), then \( S[p] = R[p \cap R] \) [8, Lemma 2.7]. In the present paper, we do not limit our study to only ideals of \( R \) but to any \( R \)-submodule of \( S \).

In Section 2, we study properties of saturations, and we show that the saturation is distributive with respect to the finite intersection (and finite sum). Note that when \( R \) is a domain with quotient field \( S \), the notion of saturation coincides with the notion of localization, and \( AR[p] = A[p] \) for any \( R \)-submodule \( A \) of \( S \). In Example 2.5, we construct a ring extension \( R \subseteq S \) and an \( R \)-submodule \( A \) of \( S \) such that \( AR[p] \not\subseteq A[p] \). In Proposition 2.7, Proposition 2.9 and Proposition 2.11, we investigate conditions under which the equality \( AR[p] = A[p] \) is satisfied for a ring extension \( R \subseteq S \) and an \( R \)-submodule \( A \) of \( S \).

In Section 3, we use the notion of saturation to study properties of star operations on ring extensions. In Theorem 3.10, which is a generalization of [2, Theorem 4] due to D.D Anderson, we characterize star operations \( \star \) on a ring extension \( R \subseteq S \) satisfying the relation \((A \cap B)\star = A\star \cap B\star\), where \( A \) and \( B \) are \( S \)-regular \( R \)-submodules of \( S \).

2. Some properties of saturations of submodules

In this section, we consider a ring extension \( R \subseteq S \) and we study properties of the saturations of \( R \)-submodules of \( S \). Let \( A \) be an \( R \)-submodule of \( S \). The **saturation** of \( A \) (in \( S \)) by \( \tau \) is set \( A_{[\tau]} = \{ x \in S : tx \in A \text{ for some } t \in \tau \} \), where \( \tau \) is a multiplicative subset of \( R \) [7, Definition 10, p. 18]. If \( p \) is a prime ideal of \( R \) with respect to \( \tau = R \setminus p \) is denoted \( A_{[p]} \). For each \( R \)-submodule \( A \) of \( S \), the set \([R:SA]\) denotes the set of all \( x \in S \) such that \( xA \subseteq R \), and \((R:A)\) denotes the set of all \( x \in R \) such that \( xA \subseteq R \) [7, Definition 2, p. 85]. We also use the notations \([R:A]\) and \((R:A)\) when the context is clear. The \( R \)-submodule \( A \) of \( S \) is called \( S \)-invertible, if there exists an \( R \)-submodule \( B \) of \( S \) such that \( AB = R \) [7, Definition 3, p. 90]. In this case, we write \( B = A^{-1} \), and \( A^{-1} = [R:SA] = \{ x \in S : xA \subseteq R \} \) [7, Remarks 1.10, p. 90].
Proposition 2.1. Let $R \subseteq S$ be a ring extension, and let $\tau$ be a multiplicative subset of $R$. Let $A$ and $B$ be two $R$-submodules of $S$. Then

1. $AR_\tau \subseteq A_\tau$.
2. $(A \cap B)_\tau = A_\tau \cap B_\tau$.
3. $(A + B)_\tau = (A_\tau + B_\tau)_\tau$.
4. If $B$ is finitely generated, then $(A_\tau : B_\tau) = (A : B)_\tau$.

Proof. (1) Let $x \in AR_\tau$. Then $x = \sum_{i=1}^\ell a_iu_i$ with $a_i \in A$ and $u_i \in R_\tau$ for $1 \leq i \leq \ell$. Let $t_i \in \tau$ such that $t_iu_i \in R$ for each $1 \leq i \leq \ell$, and let $t = \prod_{i=1}^\ell t_i$. Then $tx = \sum_{i=1}^\ell a_i(tu_i) \in A$. It follows that $x \in A_\tau$. This shows that $AR_\tau \subseteq A_\tau$.

(2) Let $x \in (A \cap B)_\tau$. Then there exists $t \in \tau$ such that $tx \in A \cap B$. It follows that $tx \in A$ and $tx \in B$. Therefore, $x \in A_\tau$ and $x \in B_\tau$. Thus $x \in A_\tau \cap B_\tau$. This shows that $(A \cap B)_\tau \subseteq A_\tau \cap B_\tau$. On the other hand, if $x$ is an element of $A_\tau \cap B_\tau$, then there exist $t_1, t_2 \in \tau$ such that $t_1x \in A$ and $t_2x \in B$. It follows that $t_1t_2x \in A \cap B$. So $x \in (A \cap B)_\tau$, since $t_1t_2 \in \tau$. This shows that $A_\tau \cap B_\tau \subseteq (A \cap B)_\tau$. Therefore, $(A \cap B)_\tau = A_\tau \cap B_\tau$.

(3) By the definition, the containment $A \subseteq A_\tau$ is always true. Thus $(A + B)_\tau \subseteq (A_\tau + B_\tau)_\tau$, since $A + B \subseteq A_\tau + B_\tau$. Now let $x \in (A_\tau + B_\tau)_\tau$. Then there exists $t \in \tau$ such that $tx = x_1 + x_2$ for some $x_1 \in A_\tau$ and $x_2 \in B_\tau$. Let $t_1, t_2 \in \tau$ such that $t_1x_1 \in A$ and $t_2x_2 \in B$. Then $tt_1t_2x \in A + B$. It follows that $x \in (A + B)_\tau$, since $tt_1t_2 \in \tau$. This shows that $(A_\tau + B_\tau)_\tau \subseteq (A + B)_\tau$. Therefore, $(A + B)_\tau = (A_\tau + B_\tau)_\tau$.

(4) Let $u \in (A : B)_\tau$. Then $tu \in (A : B)$ for some $t \in \tau$. It follows that $tuB \subseteq A$. Let $x \in B_\tau$. Then there exists $t' \in \tau$ such that $t'x \in B$. Thus $tt'xu = (tu)(t'x) \in A$. Hence $xu \in A_\tau$. It follows that $u \in (A_\tau : B_\tau)$, since $x$ was chosen arbitrarily in $B_\tau$. This shows that $(A : B)_\tau \subseteq (A_\tau : B_\tau)$. It remains to show that $(A_\tau : B_\tau) \subseteq (A : B)_\tau$. Let $y_1, \ldots, y_n \in S$ such that $B = (y_1, \ldots, y_n)R$.

Step 1. If $n = 1$, then $B$ is generated by the element $y_1$ of $S$. Let $x \in (A_\tau : B_\tau)$. Then $xB_\tau \subseteq A_\tau$. It follows that $xy_1 \in A_\tau$, since $y_1 \in B \subseteq B_\tau$. So there exists $t \in \tau$ such that $txy_1 = a$ for some $a \in A$. Let $b \in B$. Then $b = ry_1$ for some $r \in R$. Therefore, $t(xb) = r(txy_1) = ra \in A$. It follows that $tx \in (A : B)$, since $b$ was chosen arbitrarily in $B$. Hence $x \in (A : B)_\tau$. This shows that $(A_\tau : B_\tau) \subseteq (A : B)_\tau$.

Step 2. More generally, for $n \geq 1$, we have $B = B_1 + \cdots + B_n$, where each $B_i$ is the $R$-submodule of $S$ generated by $y_i$, $1 \leq i \leq n$. So

$$(A : B)_\tau = (A : B_1 + \cdots + B_n)_\tau = \bigcap_{i=1}^n (A : B_i)_\tau.$$
But by (2), we have $(\bigcap_{i=1}^{n}(A : B_i))_{[\tau]} = \bigcap_{i=1}^{n}(A : B_i)_{[\tau]}$. Furthermore, by Step 1, we have $\bigcap_{i=1}^{n}(A : B_i)_{[\tau]} = \bigcap_{i=1}^{n}(A_{[\tau]} : B_{i[\tau]})$. So

$$(A : B)_{[\tau]} = \bigcap_{i=1}^{n}(A_{[\tau]} : B_{i[\tau]}) = (A_{[\tau]} : B_{1[\tau]} + \cdots + B_{n[\tau]}).$$

But by (3), we have

$$B_{1[\tau]} + \cdots + B_{n[\tau]} \subseteq (B_{1[\tau]} + \cdots + B_{n[\tau]})_{[\tau]} = (B_{1} + \cdots + B_{n})_{[\tau]} = B_{[\tau]}.$$

It follows that $(A_{[\tau]} : B_{[\tau]}) \subseteq (A_{[\tau]} : B_{1[\tau]} + \cdots + B_{n[\tau]}) = (A : B)_{[\tau]}$. □

**Lemma 2.2.** Let $R \subseteq S$ be a ring extension, and suppose that $p$ is a prime ideal of $R$. Then $p = pR_{[p]} \cap R = p_{[p]} \cap R$.

**Proof.** The relation $p \subseteq pR_{[p]} \cap R \subseteq p_{[p]} \cap R$ is a direct consequence of Proposition 2.1(1). Now we show that $p_{[p]} \cap R \subseteq p$. Let $x \in p_{[p]} \cap R$. Then there exists $t \in R \setminus p$ such that $tx \in p$. It follows that $x \in p$, since $p$ is a prime ideal of $R$ and $t \notin p$. This shows that $p_{[p]} \cap R \subseteq p$. Thus $p = pR_{[p]} \cap R = p_{[p]} \cap R$. □

**Remark 2.3.** In [8, Proposition 2.1], McNair showed that if $R \subseteq S$ is a ring extension and $I$ is a radical ideal of $R$, then $I_{[p]} \cap R = p$ for any minimal prime ideal $p$ of $I$. So by taking $I = p$, we obtain Lemma 2.2.

For the rest of the article, if $R$ is a ring, we denote by $\text{Spec}(R)$ the set of all prime ideals of $R$. The intersection of all prime ideal of $R$ be denoted by $\text{Nil}(R)$. A ring extension is said to be **tight** if for every $x \in S \setminus R$, there exists an $S$-invertible ideal $I$ of $R$ such that $xI \subseteq R$ [7, Definition 1, p. 94].

**Proposition 2.4.** Let $R \subseteq S$ be a tight ring extension. If $\text{Nil}(R) = 0$ then,

$$\bigcup_{p \in \text{Spec}(R)} R_{[p]} = S.$$ 

**Proof.** Let $x \in S$, then there exists an $S$-invertible ideal $I$ of $R$ such that $xI \subseteq R$. Furthermore, we have IS = S. Thus $I \neq 0$. It follows that there exists $p \in \text{Spec}(R)$ such that $I \not\subseteq p$. Let $t \in I \setminus p$. Then $tx \in R$. Hence $x \in R_{[p]}$. This shows that $S \subseteq \bigcup_{p \in \text{Spec}(R)} R_{[p]}$. The containment $\bigcup_{p \in \text{Spec}(R)} R_{[p]} \subseteq S$ is clear. Thus $S = \bigcup_{p \in \text{Spec}(R)} R_{[p]}$. □

In the next example, we construct a ring extension $R \subseteq S$, a prime ideal $p$ of $R$ and an $R$-submodule $A$ of $S$ such that $A_{[p]} \neq AR_{[p]}$. 
Example 2.5. Let $D$ be an integral domain, and let $X$, $Y$ be two indeterminates over $D$. Let $R = D[X^2, Y]$ and $S = D[X, Y]$. Let $p$ be the ideal of $R$ generated by $Y$, and let $A$ be the $R$-submodule of $S$ generated by $X^2Y$. Then $A[p] \neq AR[p]$.

Proof. Let $t = X^4 \in R$. Then $t \notin p$ and $tY = YX^4 = (X^2Y)X^2 \in A$. This shows that $Y \in A[p]$. Now we show that $Y \notin AR[p]$. By contradiction, suppose that $Y \in AR[p]$. Then $Y = \sum_{i=1}^\ell X^2YU_iV_i$ with $U_i \in R$ and $V_i \in R[p]$ for $1 \leq i \leq \ell$. It follows that $1 = \sum_{i=1}^\ell X^2U_iV_i$. This last equality is not possible, since $X$ is not a unit in $S$. Thus $Y \notin AR[p]$. Therefore, $A[p] \neq AR[p]$.

For the rest of this section, we investigate conditions under which $A[p] = AR[p]$, when $A$ is an $R$-submodule of $S$ and $p$ is a prime ideal of $R$. First we recall some definitions. Let $T$ be a ring, and let $\Gamma$ be an additive totally ordered abelian group. Let $\Gamma \cup \{\infty\} = \Gamma \cup \{\infty\}$, where $\infty + g = g + \infty = \infty$ for all $g \in \Gamma \cup \infty$, and $g < \infty$ for all $g \in \Gamma$. A valuation on $T$ with values in $\Gamma$ is a map $v : T \to \Gamma \cup \infty$ such that:

1. $v(xy) = v(x) + v(y)$ for all $x, y \in T$.
2. $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in T$.
3. $v(1) = 0$ and $v(0) = \infty$.

If $v(T) = \Gamma \cup \infty$, then $v$ is called a Manis valuation on $S$ [7, Definition 4, p. 12]. In this case, $V = \{x \in T : v(x) \geq 0\}$ is called a Manis subring of $T$, the extension $V \subseteq T$ is called a Manis extension, and $(V, p)$ is called a Manis pair in $T$, where $p = \{x \in T : v(x) > 0\}$ [7, Definition 1, p. 22].

Let $R \subseteq S$ be a ring extension, and let $A$ be an $R$-submodule of $S$. The $R$-submodule $A$ is said to be $S$-regular if $AS = S$ [7, Definition 1, p. 84]. The ring $S$ is called a Prüfer extension of $R$ if $(R[p], p[p])$ is a Manis pair in $S$ for every maximal ideal $p$ of $R$. In this case, we say that $R$ is Prüfer subring of $S$. The ring extension $R \subseteq S$ is called Bézout extension if $R \subseteq S$ is a Prüfer extension and every $S$-invertible ideal of $R$ is a principal ideal. More on Manis valuations, Prüfer extensions and Bézout extensions can be found in [7].

Lemma 2.6 ([7, Theorem 1.13, p. 91]). If $R \subseteq S$ is a Prüfer extension, then every finitely generated $S$-regular $R$-submodule of $S$ is $S$-invertible.

Proof. Let $A$ be an $S$-invertible $R$-submodule of $S$. By Proposition 2.1(1), we have $AR_{[p]} \subseteq A_{[p]}$. Thus it suffices to show that $A_{[p]} \subseteq AR_{[p]}$. Let $x \in A_{[p]}$. Then there exists $t \in R \setminus p$ such that $tx \in A$. Since $AA^{-1} = R$, there exist $a_1, \ldots, a_\ell \in A$ and $b_1, \ldots, b_\ell \in A^{-1}$ such that $a_1b_1 + \cdots + a_\ell b_\ell = 1$. Moreover, for $1 \leq i \leq \ell$, we have $txb_i \in R$, since $tx \in A$ and $b_i \in A^{-1}$. Thus $xb_i \in R_{[p]}$ for $1 \leq i \leq \ell$. It follows that $x = \sum_{i=0}^\ell a_i(xb_i) \in AR_{[p]}$. \hfill $\Box$

Remark 2.8. Let $R \subseteq S$ be a Prüfer extension, and let $A$ be an $S$-regular finitely generated $R$-submodule of $S$. Then by Lemma 2.6, $A$ is $S$-invertible. It follows from Proposition 2.7 that $AR_{[p]} = A_{[p]}$ for any prime ideal $p$ of $R$.

We recall from [3, Definition 2, p.13] that and $R$-module $M$ is flat, if for every injective homomorphism $v : M_1 \to M_2$ of $R$-modules, the $R$-module homomorphism $1_M \otimes v : M \otimes_R M_1 \to M \otimes_R M_2$ is also injective.

Theorem 2.9. Let $R \subseteq S$ be a ring extension, and suppose that $R = \bigcap_{i=1}^n V_i$. If the extension $V_i \subseteq S$ is Prüfer for each $1 \leq i \leq n$, then $AR_{[p]} = A_{[p]}$ for any $S$-regular flat $R$-submodule $A$ of $S$ and any prime ideal $p$ of $R$.

Proof. Let $A$ be an $S$-regular flat $R$-submodule of $S$. Since $AS = S$, there exist $u_1, \ldots, u_\ell \in A$ and $s_1, \ldots, s_\ell \in S$ such that $u_1s_1 + \cdots + u_\ell s_\ell = 1$. Let $A_i$ be the $V_i$-submodule of $S$ generated by $u_1, \ldots, u_\ell$. Then it is clear that $A_i S = S$. Hence $A_i$ is an $S$-regular finitely generated $V_i$-submodule of $S$.

Let $x \in A_{[p]}$. Then there exists $t \in R \setminus p$ such that $tx \in A$. Let $B_i = txV_i + A_i$. Then $B_i$ is also an $S$-regular finitely generated $V_i$-submodule of $S$. It follows from Lemma 2.6 that $B_i$ is an $S$-invertible $V_i$-submodule of $S$. Therefore, there exists a $V_i$-submodule $C_i$ of $S$ such that $B_iC_i = V_i$. Let $c_{i0}, c_{i1}, \ldots, c_{i\ell} \in C_i$ such that $c_{i0}tx + c_{i1}u_1 + \cdots + c_{i\ell}u_\ell = 1$. Then $x = (tx)(xc_{i0}) + u_1(xc_{i1}) + \cdots + u_\ell(xc_{i\ell})$. Now notice that $t(c_{ik}x) = (tx)c_{ik} \in B_iC_i = V_i$ for $0 \leq k \leq \ell$ and $1 \leq i \leq \ell$. It follows that $c_{ik}x \in V_{i[p]}$ for $0 \leq k \leq \ell$ and $1 \leq i \leq \ell$. Therefore, for $1 \leq i \leq \ell$, we have $x \in AV_{i[p]}$, since $tx, u_1, \ldots, u_\ell \in A$. This shows that $x \in \bigcap_{i=1}^n AV_{i[p]}$. Since $A$ is a flat $R$-module, we have $\bigcap_{i=1}^n AV_{i[p]} = A(\bigcap_{i=1}^n V_{i[p]})$ [3, Proposition 6, p. 18]. Furthermore, by Proposition 2.1(2), we have $\bigcap_{i=1}^n V_{i[p]} = (\bigcap_{i=1}^n V_i)[p] = R_{[p]}$. It follows that $x \in AR_{[p]}$. This shows that $A_{[p]} \subseteq AR_{[p]}$, and hence $A_{[p]} = AR_{[p]}$. \hfill $\Box$

Remark 2.10. Let $R \subseteq S$ be a Prüfer extension, and let $A$ be an $S$-regular $R$-submodule of $S$. 


A star operation ⋆

Proof. Let by saturations. For more on star operation of ring extension, see [6, pages 139 - 164].

is called star operation on \( J \) (in [6, Definition 1, p. 139]. Let \( R \) recall the definition of a star operation as given by Knebusch and Kaiser of star operation to investigate properties of ring extensions. First we

Proposition 2.11. If \( R \subseteq S \) is a Prüfer extension, then for any \( S \)-regular \( R \)-submodule of \( S \), we have \( AR[p] = A[p] \) for each prime ideal \( p \) of \( R \).

Proof. Let \( A \) be an \( S \)-regular \( R \)-submodule of \( S \), and let \( x \in A[p] \). Then there exists \( t \in R \setminus p \) such that \( tx \in A \). Since \( AS = S \), there exist \( a_1, \ldots, a_\ell \in A \) and \( s_1, \ldots, s_\ell \in S \) such that \( a_1s_1 + \cdots + a_\ell s_\ell = 1 \). Let \( A' \) be the \( R \)-submodule of \( S \) generated by \( a_1, \ldots, a_\ell \). Then it is clear that \( A' \) is an \( S \)-regular finitely generated \( R \)-submodule of \( S \). Let \( B = txR + A' \). Then \( B \) is also an \( S \)-regular finitely generated \( R \)-submodule of \( S \). It follows from Lemma 2.6 that \( B \) is \( S \)-invertible. Hence, there exist \( b_0, b_1, \ldots, b_\ell \in B^{-1} \) such that \( b_0tx + b_1a_1 + \cdots + b_\ell a_\ell = 1 \). But for \( 0 \leq i \leq \ell \), we have \( t(b_i x) = b_i(tx) \in R \), since \( tx \in B \) and \( b_i \in B^{-1} \). It follows that \( b_i x \in R[p] \) for each \( 0 \leq i \leq \ell \). Therefore, \( x = (tx)(xb_0) + a_1(xb_1) + \cdots + a_\ell(xb_\ell) \in AR[p] \). This shows that \( A[p] \subseteq AR[p] \). Hence \( A[p] = AR[p] \). \( \square \)

3. Star operations induced by saturations of submodules

In this section, we use the notion of saturation along with the notion of star operation to investigate properties of ring extensions. First we recall the definition of a star operation as given by Knebusch and Kaiser in [6, Definition 1, p. 139]. Let \( R \subseteq S \) be a ring extension. A map \( \star : \mathcal{J}(R, S) \rightarrow \mathcal{J}(R, S) \), where \( \mathcal{J}(R, S) \) is the set of all \( R \)-submodules of \( S \), is called star operation on \( R \subseteq S \) if the following conditions are satisfied for all \( A, B \in \mathcal{J}(R, S) \).

\begin{enumerate}
\item \( A \subseteq A^\star \).
\item If \( A \subseteq B \), then \( A^\star \subseteq B^\star \).
\item \( (A^\star)^\star = A^\star \).
\item \( AB^\star \subseteq (AB)^\star \).
\end{enumerate}

A star operation \( \star \) on a ring extension \( R \subseteq S \) is said to be strict if \( R^\star = R \). For more on star operation of ring extension, see [6, pages 139 - 164].

In the next two results, we give examples of star operations induced by saturations.
Proposition 3.1. Let $R \subseteq S$ be a ring extension, and let $\ast : \mathcal{J}(R, S) \to \mathcal{J}(R, S)$ be the map defined by $A^* = \bigcap_{p \in \mathcal{P}} A_p$, where $\mathcal{P}$ is a set of prime ideals of $R$. Then

(1) The map $\ast$ is a star operation on the extension $R \subseteq S$.

(2) $A_p = (A^*)_p$ for each $A \in \mathcal{J}(R, S)$ and each $p \in \mathcal{P}$.

(3) If $R = \bigcap_{p \in \mathcal{P}} R_p$, then $p^* = p$ for each $p \in \mathcal{P}$.

Proof. (1) Let $A$, $B$ and $J$ be elements of $\mathcal{J}(R, S)$. From the definition of $\ast$, it is clear that $A \subseteq A^*$, and $A^* \subseteq B^*$ if $A \subseteq B$. Furthermore, $J A^* = J \bigcap_{p \in \mathcal{P}} A_p \subseteq J A_p$ for each $p \in \mathcal{P}$. It follows that $J A^* \subseteq \bigcap_{p \in \mathcal{P}} J A_p = (J A)^*$.

Now we show that $(A^*)^* = A^*$. Let $x \in (A^*)^*$. Then for each $p_0 \in \mathcal{P}$, we have $x \in (A^*)_p = (A^*)_{p_0}$. So there exists $t_0 \in R \setminus p_0$ such that $t_0 x \in A^* = \bigcap_{p \in \mathcal{P}} A_p$. In particular $t_0 x \in A_{p_0}$. Therefore, $t_0 s_0 x \in A$ for some $s_0 \in R \setminus p_0$. It follows that $x \in A_{[p_0]}$, since $t_0 s_0 \in R \setminus p_0$. Since $p_0$ was chosen arbitrarily in $\mathcal{P}$, we have $x \in \bigcap_{p \in \mathcal{P}} A_p = A^*$. Thus, $(A^*)^* \subseteq A^*$. The containment $A^* \subseteq (A^*)^*$ is clear. Hence $(A^*)^* = A^*$.

(2) Let $A \in \mathcal{J}(R, S)$, and let $p \in \mathcal{P}$. By part (1), we have $A \subseteq A^*$. So $A_{[p]} \subseteq (A^*)_{[p]}$. It remains to show that $(A^*)_{[p]} \subseteq A_{[p]}$. Notice that for each $p \in \mathcal{P}$, we have $A^* \subseteq \bigcap_{q \in \mathcal{P}} A_{[q]} \subseteq A_{[p]}$. Thus $(A^*)_{[p]} \subseteq (A_{[p]})_{[p]} = A_{[p]}$.

(3) Let $p_0 \in \mathcal{P}$. Then by part (1), we have $p_0 \subseteq p_0^*$. Now we show the containment $p_0^* \subseteq p_0$. Let $x \in p_0^*$. Then $x \in \bigcap_{p \in \mathcal{P}} p_0 \subseteq \bigcap_{p \in \mathcal{P}} R_p = R$. Furthermore, there exists $t_0 \in R \setminus p_0$ such that $t_0 x \in p_0$. It follows that $x \in p_0$, since $p_0$ is prime and $t_0 \notin p$. This shows that $p_0^* \subseteq p_0$. Hence $p = p^*$.

Proposition 3.2. Let $R \subseteq S$ be a ring extension, and let $\ast : \mathcal{J}(R, S) \to \mathcal{J}(R, S)$ be the map defined by $A^* = \bigcap_{p \in \mathcal{P}} A R_p$, where $\mathcal{P}$ is a set of prime ideals of $R$. Then

(1) The map $\ast$ is a star operation on the extension $R \subseteq S$.

(2) $A_p = (A^*)_p$ for each $A \in \mathcal{J}(R, S)$ and each $p \in \mathcal{P}$.

(3) If $R = \bigcap_{p \in \mathcal{P}} R_p$, then $p^* = p$ for each $p \in \mathcal{P}$.

Proof. (1) Let $A$, $B$ and $J$ be elements of $\mathcal{J}(R, S)$. From the definition of $\ast$, it is clear that $A \subseteq A^*$. Also, if $A \subseteq B$, then $A R_p \subseteq B R_p$ for each prime ideal $p$ of $\mathcal{P}$. Hence $A^* = \bigcap_{p \in \mathcal{P}} A R_p \subseteq \bigcap_{p \in \mathcal{P}} B R_p = B^*$. Furthermore, $J A^* = J \bigcap_{p \in \mathcal{P}} A R_p \subseteq J A R_p$ for each $p \in \mathcal{P}$. It follows that $J A^* \subseteq \bigcap_{p \in \mathcal{P}} J A R_p = (J A)^*$. Notice that if $q \in \mathcal{P}$, then $(A^*)_q = \bigcap_{p \in \mathcal{P}} A^* R_p \subseteq A^* R_{[q]} = (\bigcap_{p \in \mathcal{P}} A R_p) R_{[q]} \subseteq (A R_{[q]}) R_{[q]} \subseteq A R_{[q]}$. It follows that $(A^*)_q \subseteq A R_{[q]}$ for each $q \in \mathcal{P}$. Thus $(A^*)_q = \bigcap_{q \in \mathcal{P}} A R_q = A^*$. 

L. Paudel, S. Tchamna

117
(2) Let \( A \in \mathcal{J}(R, S) \), and let \( p \in \mathcal{P} \). By part (1), we have \( A \subseteq A^* \). It follows that \( A[p] \subseteq (A^*)[p] \). It remains to show that \( (A^*)[p] \subseteq A[p] \). We have \( (A^*)[p] = (\bigcap_{q \in \mathcal{P}} AR[q])[p] \subseteq (AR[p])[p] \). By Proposition 2.1(1), we have \( AR[p] \subseteq A[p] \). Thus \( (A^*)[p] \subseteq (A[p])[p] = A[p] \). This shows that \( (A^*)[p] = A[p] \).

(3) Let \( p_0 \in \mathcal{P} \). Then by part (1), we have \( p_0 \subseteq p_0^* \). Now we show the containment \( p_0^* \subseteq p_0 \). We have \( p_0^* = \bigcap_{p \in \mathcal{P}} p_0 R[p] \). But by Proposition 2.1(1), we have \( p_0 R[p] \subseteq p_0[p] \) for each \( p \in \mathcal{P} \). Thus \( p_0^* \subseteq \bigcap_{p \in \mathcal{P}} p_0[p] \). Furthermore, by Proposition 3.1(3), we have \( \bigcap_{p \in \mathcal{P}} p_0[p] = p_0 \). Therefore, \( p_0^* = p_0 \).

Let \( R \subseteq S \) be a ring extension, and let \( X \) be an indeterminate over \( S \). For each element \( f \) of \( S[X] \), and each subring \( L \) of \( S \) containing \( R \), we denote by \( c_L(f) \) the \( L \)-submodule of \( S \) generated by the coefficients of \( f \). Let \( T = \{ g \in S[X] : c_S(g) = S \} \). Then \( T \) is a multiplicative subset of \( S[X] \) [5, ]. We denote by \( S(X) \) the quotient ring \( (S[X])_T \). Hence \( S(X) \) is the set of all elements of the form \( \frac{f}{g} \) with \( f, g \in S[X] \) such that \( c_S(g) = S \) (for properties of the ring \( S(X) \), see [5, p. 410]). A subring \( B \) of \( S(X) \) is said to be a Kronecker subring of \( S(X) \) if \( X \in B \), and for every \( f = a_0 + a_1 X + \cdots + a_n X^n \in S[X] \) with \( c_S(f) = S \) the elements \( \frac{a_0}{f}, \frac{a_1}{f}, \ldots, \frac{a_n}{f} \) are contained in \( B \) [6, Definition 4, p. 127].

For each star operation \( \ast \) on a ring extension \( R \subseteq S \), we denote by \( \text{Kr}(\ast) \) the set of all elements \( \frac{f}{g} \in S(X) \) with \( (c_R(f)H)^{\ast} \subseteq (c_R(g)H)^{\ast} \) for some finitely generated \( S \)-regular \( R \)-submodule \( H \) of \( S \). The set \( \text{Kr}(\ast) \) is a Kronecker subring of \( S(X) \) [6, Theorem 3.10, p. 143], and the extension \( \text{Kr}(\ast) \subseteq S(X) \) is Bézout [6, Theorem 1.5, p. 128].

**Remark 3.3.** Let \( R \subseteq S \) be a ring extension, and let \( \mathcal{P} \) be a set of prime ideals of \( R \). Let \( \ast_1, \ast_2 : \mathcal{J}(R, S) \rightarrow \mathcal{J}(R, S) \) be two maps defined for each \( A \in \mathcal{J}(R, S) \) by \( A^{\ast_1} = \bigcap_{p \in \mathcal{P}} A[p] \) and \( A^{\ast_2} = \bigcap_{p \in \mathcal{P}} AR[p] \) respectively. Then by Proposition 3.1, \( \ast_1 \) is a star operation on \( R \subseteq S \). Also, by Proposition 3.2, the map \( \ast_2 \) is also a star operation on \( R \subseteq S \). Let \( \frac{f}{g} \in \text{Kr}(\ast_2) \). Then there exists a finitely generated \( S \)-regular \( R \)-submodule \( H \) of \( S \) such that \( c_R(f)H \subseteq (c_R(f)H)^{\ast_2} \subseteq (c_R(g)H)^{\ast_2} \). But by Proposition 2.1(1), \( (c_R(g)H)^{\ast_2} \subseteq (c_R(g)H)^{\ast_1} \). It follows that \( (c_R(f)H) \subseteq (c_R(g)H)^{\ast_1} \), and hence \( (c_R(f)H)^{\ast_1} \subseteq (c_R(g)H)^{\ast_1} \). Therefore, \( \frac{f}{g} \in \text{Kr}(\ast_1) \). This shows that \( \text{Kr}(\ast_2) \subseteq \text{Kr}(\ast_1) \).

**Remark 3.4.** Let \( R \subseteq S \) be a ring extension, and let \( \mathcal{M} \) be the set of all maximal ideals of \( R \). Then by [7, Remark 5.5, p. 50], we have \( A^{b_1} = \bigcap_{p \in \mathcal{M}} A[p] = \bigcap_{p \in \mathcal{M}} AR[p] = A^{b_2} \). Hence \( \text{Kr}(b_1) = \text{Kr}(b_2) \).
Proposition 3.5. Let \( R \subseteq S \) be a ring extension, and let \( \mathcal{P} \) be a set of prime ideals of \( R \). Let \( \star_1 \) and \( \star_2 \) be the two star operations on \( R \subseteq S \) defined for each \( R \)-submodule \( A \) of \( S \) by \( A^{\star_1} = \bigcap_{p \in \mathcal{M}} A_{[p]} \) and \( A^{\star_2} = \bigcap_{p \in \mathcal{M}} AR_{[p]} \).

If the extension \( R \subseteq S \) is Prüfer, then \( \text{Kr}(\star_1) = \text{Kr}(\star_2) \).

**Proof.** Suppose that the extension \( R \subseteq S \) is Prüfer. Let \( \frac{f}{g} \in \text{Kr}(\star_1) \). Then there exists a finitely generated \( S \)-regular \( R \)-submodule \( H \) of \( S \) such that \( c_R(f)H \subseteq (c_R(f)H)^{\star_1} \subseteq (c_R(g)H)^{\star_1} \). Furthermore, \((c_R(g)H)S = S\), since \( c_R(S)S = S \) and \( HS = S \). Thus \( c_R(g)H \) is an \( S \)-regular finitely generated \( R \)-submodule of \( S \). Therefore, by Proposition 2.11, we have \((c_R(g)H)^{\star_2} = (c_R(g)H)^{\star_1}\). It follows from the relation \( c_R(f)H \subseteq (c_R(g)H)^{\star_1} \) that \( c_R(f)H \subseteq (c_R(g)H)^{\star_2} \). Therefore,

\[
(c_R(f)H)^{\star_2} \subseteq ((c_R(g)H)^{\star_2})^{\star_2} = (c_R(g)H)^{\star_2}.
\]

Thus \( \frac{f}{g} \in \text{Kr}(\star_2) \). This shows that \( \text{Kr}(\star_1) \subseteq \text{Kr}(\star_2) \). But by Remark 3.3, we have \( \text{Kr}(\star_2) \subseteq \text{Kr}(\star_1) \). Hence \( \text{Kr}(\star_1) = \text{Kr}(\star_2) \).

**Remark 3.6.** Let \( R \subseteq S \) be a ring extension, and let \( A, B \) be two \( R \)-submodules of \( S \). By definition of \( (A : B) \), we have \( (A : B)B \subseteq A \). Furthermore, the containment \( (A : B) \subseteq R \) implies that \( (A : B)B \subseteq B \). Thus \( (A : B)B \subseteq A \cap B \).

**Lemma 3.7.** Let \( R \subseteq S \) be a ring extension, and let \( A \) be an \( R \)-submodule of \( S \). For each \( S \)-invertible \( R \)-submodule \( I \) of \( S \), we have \( A \cap I = (A : I)I \).

**Proof.** Let \( A \) be an \( R \)-submodule of \( S \), and let \( I \) be an \( S \)-invertible \( R \)-submodule of \( S \). Let \( p \) be a prime ideal of \( R \). By [7, Proposition 2.3, p. 97], there exists \( w_p \in I \) such that the \( R_p \)-module \( I_p \) is generated by \( w_p \). Let \( x \in (A \cap I)_p = A_p \cap I_p \). Then \( x = \frac{ru_p}{t} \) with \( r \in R \) and \( t \in R \setminus p \). Since \( \frac{ru_p}{t} \in A_p \), we have \( \frac{t}{r} \in (A : R I)_p \). Furthermore, \( I \) is a finitely generated \( R \)-submodule of \( S \) [7, Remark 1.10 (a)]. Hence \( (A_p :_{R_p} I_p) = (A :_{R} I)_p \) [1, Corollary 3.15 p.43]. Thus \( \frac{t}{r} \in (A :_{R} I)_p \). It follows that \( x = \frac{ru_p}{t} \in (A :_{R} I)_pI_p = ((A :_{R} I)I)_p \). This shows that \( (A \cap I)_p \subseteq ((A :_{R} I)I)_p \). On the other hand, the containment \( (A :_{R} I)I \subseteq A \cap I \) follows from Remark 3.6. Hence \( ((A :_{R} I)I)_p \subseteq (A \cap I)_p \). Therefore, \( ((A :_{R} I)I)_p = (A \cap I)_p \). It follows from [7, Lemma 1.1, p. 85] that \( A \cap I = (A : I)I \), since \( p \) was chosen arbitrarily in \( R \).

**Remark 3.8.** Let \( R \subseteq S \) be a tight ring extension, and let \( A \) be an \( S \)-regular \( R \)-submodule of \( A \). Then for each element \( x \) of \( A \), there exists an \( S \)-invertible \( R \)-submodule \( E \) of \( S \) such that such that \( x \in E \subseteq A \).
Proof. Let $A$ be an $S$-regular $R$-submodule of $S$. Then $AS = S$. Therefore, there exist $a_1, \ldots, a_n \in A$ and $s_1, \ldots, s_n \in S$ such that $a_1s_1 + \cdots + a_ns_n = 1$.

Let $x \in A$. Since the extension $R \subseteq S$ is tight, there exists $u \in S$ such that $xu \in R$. Let $E$ be the $R$-submodule of $S$ generated by $x, a_1, \ldots, a_n$. Then $E$ is $S$-invertible, and $E^{-1}$ is the $R$-submodule of $S$ generated by $u, s_1, \ldots, s_n$. Furthermore, $x \in E \subseteq A$. \hfill \square

Remark 3.9. Let $R \subseteq S$ be a tight ring extension, and let $B$ be a subring of $S$ containing $R$. Let $x \in B$. If $x \in R$, then $xR \subseteq R$. Thus $R \subseteq (R : x)$, and hence $(R : x)S = S$. If $x \in S \setminus R$, then by the definition of a tight extension, there exists a $S$-invertible ideal $I$ of $R$ such that $xI \subseteq R$. Hence $I \subseteq (R : x)$. Thus $IS \subseteq (R : x)S$. By [7, Remarks 1.10(d), p. 90], we have $IS = S$. Hence $(R : x)S = S$. It follows from [7, Theorem 3.13, p. 37] and [7, Proposition 1.15, p. 91] that if $A$ and $B$ are two $S$-regular $R$-submodules of $S$, then $A \cap B$ and $(A : B)$ are also $S$-regular $R$-submodules of $S$.

The following theorem is inspired by [2, Theorem 4], which is a result due to D.D. Anderson. We consider a tight ring extension $R \subseteq S$ and we characterize star operations $\star$ on $R \subseteq S$ satisfying the condition $(A \cap B)^\star = A^\star \cap B^\star$ where $A$ and $B$ are $S$-regular $R$-submodule of $S$.

Theorem 3.10. Let $R \subseteq S$ be a tight ring extension, and let $\mathcal{P}$ be a set of prime ideals of $R$ such that each $S$-regular ideal of $R$ is contained in some $p \in \mathcal{P}$. Then for each star operation $\star$ on the ring extension $R \subseteq S$ satisfying $R = R^\star$, the following statements are equivalent.

1. $A^\star = \bigcap_{p \in \mathcal{P}} A_p$ for each $S$-regular $R$-submodules $A$ of $R$.
2. (a) $R = \bigcap_{p \in \mathcal{P}} R_p$.
   (b) $(A \cap B)^\star = A^\star \cap B^\star$ for all $S$-regular $R$-submodules $A, B$ of $R$.
3. (a) $R = \bigcap_{p \in \mathcal{P}} R_p$.
   (b) $(A :_R B)^\star = (A^\star :_R B^\star)$ for all $S$-regular $R$-submodule $A, B$ of $R$ with $B$ finitely generated.

Proof. (1) $\Rightarrow$ (2) Since $R$ is an $S$-regular $R$-submodule of $R$, we have $R^\star = \bigcap_{p \in \mathcal{P}} R_p$. It follows that $R = \bigcap_{p \in \mathcal{P}} R_p$, since by hypothesis, we have $R = R^\star$. Let $A$ and $B$ be two $S$-regular $R$-submodules of $S$. Then by Remark 3.9, $A \cap B$ is also an $S$-regular $R$-submodule of $S$. Thus by hypothesis, we have $(A \cap B)^\star = \bigcap_{p \in \mathcal{P}} (A \cap B)_p$. But by Proposition 2.1(2), $(A \cap B)_p = A_p \cap B_p$. Therefore,

$$(A \cap B)^\star = \bigcap_{p \in \mathcal{P}} (A \cap B)_p = \bigcap_{p \in \mathcal{P}} (A_p \cap B_p) = \bigcap_{p \in \mathcal{P}} A_p \cap \bigcap_{p \in \mathcal{P}} B_p = A^\star \cap B^\star.$$
(2) $\Rightarrow$ (1) Let $A$ be an $S$-regular $R$-submodule of $S$. Then $AS = S$. It follows that $A^* S = S$, since $A \subseteq A^*$. Let $x \in A^*$. Then by Remark 3.8, there exists an $S$-invertible $R$-submodule $I$ of $S$ such that $x \subseteq I \subseteq A^*$. Since $I \subseteq A^*$, we have $I^* = A^* \cap I^*$. But by hypothesis, we have $A^* \cap I^* \neq (A \cap I)^*$. Also, by Lemma 3.7, we have $A \cap I = (A : I)I$. It follows that $I^* = ((A : I)I)^*$. But by [6, Proposition 4.1(b), p. 146], we have $((A : I)I)^* = (A : I)^*$. This shows that $I^* = (A : I)^*$, and so $I \subseteq (A : I)^*$. By multiplying the last relation by $I^{-1}$, we get $R \subseteq (A : I)^*$. Furthermore, by [7, Remarks 1.10(d)], $I$ is an $S$-regular $R$-submodule of $S$. It follows from Remark 3.9 that $(A : I)$ is an $S$-regular ideal of $R$. If $(A : I)$ is a proper ideal of $R$, then by hypothesis, $(A : I) \subseteq p_0$ for some $p_0 \in \mathcal{P}$ of $R$. In this case $S = (A : I)S \subseteq p_0S$. Hence $p_0S = S$. So $p_0$ is an $S$-regular ideal of $R$. It follows that $p_0^\ast = \bigcap_{p \in \mathcal{P}} p_0[p]$. Therefore, as in the the proof of Proposition 3.1(3), we have $p_0 = p_0^\ast$. Hence $R \subseteq (A : I)^* \subseteq p_0^\ast = p_0$; which is impossible. Thus $R = (A : I)$. Let $p \in \mathcal{P}$ and let $t \in R \setminus p$. Then $t \in (A : I) = R$. It follows that $tI \subseteq A$. In particular, $tx \in A$. Thus $x \in A[p]$. This shows that $A^* \subseteq \bigcap_{p \in \mathcal{P}} A[p]$. For the containment $\bigcap_{[p] \in \mathcal{P}} A[p] \subseteq A^*$, let $A' = \bigcap_{[p] \in \mathcal{P}} A[p]$. Then $A \subseteq A'$. It follows that $A'S = S$. Let $y \in A'$. Then by Remark 3.8, there exists an $S$-invertible $R$-submodule $J$ of $S$ such that $y \in J \subseteq A'$. Let $p \in \mathcal{P}$. Then by Proposition 3.1(2), we have $A'[p] = A[p]$. Thus $J \subseteq A' \subseteq A'[p] = A[p]$. But $J$ is a finitely generated $R$-submodule of $S$ [7, Remarks 1.10(a), p. 90]. So there exist $u_1, \ldots, u_\ell \in S$ such that $J = (u_1, \ldots, u_\ell)R$. For $1 \leq i \leq \ell$, let $t_i \in R \setminus p$ such that $t_i u_i \in A$, and let $t = \prod_{i=1}^{\ell} t_i$. Then $t \in R \setminus p$ and $tI \subseteq A$. It follows that $(A : J) \cap (R \setminus p) \neq \emptyset$. Notice that $J$ is an $S$-regular $R$-submodule of $S$, since each $S$-invertible $R$-submodule of $S$ is $S$-regular[7, Remarks 1.10(d), p. 90]. Therefore, by Remark 3.9, we conclude that $(A : J)$ is an $S$-regular ideal of $R$. It follows from the hypothesis and the relation $(A : J) \cap (R \setminus p) \neq \emptyset$ for each $p \in \mathcal{P}$, that $(A : J)$ is not a proper ideal of $R$. Hence $(A : J) = R$, and thus $(A : J)^* = R^*$. Furthermore, $J^* = (JR)^* = JR^*$ [6, Proposition 4.1, p. 146]. Therefore, $J^* = JR^* = J(A : J)^* = (J(A : J))^* = (J \cap A)^*$. It follows that $y \in J \subseteq J^* \subseteq A^*$. This shows that $A' = \bigcap_{p \in \mathcal{P}} A[p] \subseteq A^*$. Therefore, $A^* = \bigcap_{p \in \mathcal{P}} A[p]$.

(1) $\Rightarrow$ (3) We have $R^* \subseteq \bigcap_{p \in \mathcal{P}} R[p]$, since $R$ is an $S$-regular $R$-submodule of $S$. It follows that $R = \bigcap_{p \in \mathcal{P}} R[p]$, since by hypothesis, we have $R = R^*$. Let $A$ and $B$ be two $S$-regular $R$-submodules of $S$, and suppose that $B$ is finitely generated. Then by Remark 3.9, the ideal $(A : B)$ of $R$ is also $S$-regular. Thus by the hypothesis, we have
(A : B)^* = \bigcap_{p \in P} (A : B)_{[p]}^* . It follows from Proposition 2.1(4) that
(A : B)^* = \bigcap_{p \in P} (A_{[p]} : B_{[p]}) \subseteq (\bigcap_{p \in P} A_{[p]} : \bigcap_{p \in P} B_{[p]}) = (A^* : B^*) .

Now we show that (A^* : B^*) \subseteq (A : B)^* . First notice that the containment (A^* : B^*) \subseteq (A^* : B) is always true. Let d \in (A^* : B^*) . Then for each p \in P , we have d \in (A^* : B) \subseteq (A : B)_{[p]} . But by Proposition 2.1(4), we have (A^* : B)_{[p]} = ((A^*)_{[p]} : B_{[p]} ) . Thus d \in ((A^*)_{[p]} : B_{[p]} ) . Furthermore, A_{[p]} \subseteq (A^*)_{[p]} = (\bigcap_{q \in P} A_{[q]} )_{[p]} \subseteq (A_{[p]})_{[p]} = A_{[p]} . Thus (A^*)_{[p]} = A_{[p]} . Hence d \in (A_{[p]} : B_{[p]} ) . It follows from Proposition 2.1(4) that d \in (A : B)_{[p]} . This shows that d \in \bigcap_{p \in P} (A : B)_{[p]} = (A : B)^* . Hence (A^* : B^*) \subseteq (A : B)^* , since d was chosen arbitrarily in (A^* : B^*) . Thus (A^* : B^*) = (A : B)^* .

(3) \Rightarrow (1) Let A be an S-regular R-submodule of S . Let x \in A^* . The R-submodule A^* is S-regular, since A \subseteq A^* . Therefore, by Remark 3.8, there exits an S-invertible R-submodule I of S such that x \in I \subseteq A^* . The containment I \subseteq A^* implies I^* \subseteq A^* . Thus R = (A^* : I^* ) . Furthermore, I is a finitely generated R-submodule of S [7, Remark 1.10 (a)] . Thus R = (A^* : I^* ) = (A : I)^* . Suppose that (A : I) is a proper ideal of R . Then by hypothesis, (A : I) \subseteq p_0 for some p_0 \in P of R . Thus S = (A : I)S \subseteq p_0 S . Hence p_0 S = S . So p_0 is an S-regular ideal of R . It follows that p_0^* = \bigcap_{p \in P} p_0_{[p]} . Thus, as in the the proof of Proposition 3.1(3), we have p_0 = p_0^* . Hence R \subseteq (A : I)^* \subseteq p_0^* = p_0 ; which is impossible. Thus R = (A : I) . The rest of the proof is identical to the proof of (2) \Rightarrow (1) .

References


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