Cross-cap singularities counted with sign

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Communicated by I. Protasov

Abstract. A method for computing the algebraic number of cross-cap singularities for mapping from $m$-dimensional compact manifold with boundary $M \subset \mathbb{R}^m$ into $\mathbb{R}^{2m-1}$, $m$ is odd, is presented. As an application, the intersection number of an immersion $g: S^{m-1}(r) \rightarrow \mathbb{R}^{2m-2}$ is described as the algebraic number of cross-caps of a mapping naturally associated with $g$.

Introduction

Mappings from the $m$-dimensional, smooth, orientable manifold $M$ into $\mathbb{R}^{2m-1}$ are natural object of study. In [9], Whitney described typical mappings from $M$ into $\mathbb{R}^{2m-1}$. Those mappings have only isolated critical points, called cross-caps (or Whitney umbrellas).

According to [1, Theorem 4.6], [11, Lemma 2], a mapping $M \rightarrow \mathbb{R}^{2m-1}$ has a cross-cap at $p \in M$, if and only if in the local coordinate system near $p$ this mapping has the form

$$(x_1, \ldots, x_m) \mapsto (x_1^2, x_2, \ldots, x_m, x_1 x_2, \ldots, x_1 x_m).$$

In [11], for $m$ odd, Whitney presented a method to associate a sign with a cross-cap. Put $\zeta(f)$ to be an algebraic sum of cross-caps of $f : M \rightarrow \mathbb{R}^{2m-1}$, where $M$ is $m$-dimensional compact orientable manifold. Then according to Whitney, [11, Theorem 3], $\zeta(f) = 0$, if $M$ is closed. If $M$

2010 MSC: 14P25, 57R45, 57R42, 12Y05.

Key words and phrases: cross-cap, immersion, Stiefel manifold, intersection number, signature.
has a boundary, then following Whitney, [11, Theorem 4], for a homotopy $f_t : M \to \mathbb{R}^{2m-1}$ regular in some open neighbourhood of $\partial M$, if the only singular points of $f_0$ and $f_1$ are cross-caps, then $\zeta(f_0) = \zeta(f_1)$. Moreover arbitrarily close to any mapping $h : M \to \mathbb{R}^{2m-1}$, there is mapping regular near boundary, with only cross-caps as singular points (see [11]). In the case where $m$ even, it is impossible to associate sign with cross-cap in the same way as in the odd case, but if $m$ is even, it is enough to consider number of cross-caps mod 2, to get similar results (see [11]).

In [6], the authors studied a mapping $\alpha$ from a compact and oriented $(n-k)$-manifold $M$ into the Stiefel manifold $\widetilde{V}_k(\mathbb{R}^n)$, for $n-k$ even. They constructed a mapping $\tilde{\alpha} : S^{k-1} \times M \to \mathbb{R}^n \setminus \{0\}$ associated with $\alpha$, and defined $\Lambda(\alpha)$ as half of topological degree of $\tilde{\alpha}$. In case $M = S^{m-k}$, they showed that $\Lambda(\alpha)$ corresponds with the class of $\alpha$ in $\pi_{n-k} \mathbb{V}_k(\mathbb{R}^n) \cong \mathbb{Z}$. According to [6], in the case where $M \subset \mathbb{R}^{n-k+1}$ is an algebraic hypersurface and $\alpha$ is polynomial, with some additional assumptions concerning $M$ and $\alpha$, $\Lambda(\alpha)$ can be presented as a sum of signatures of two quadratic forms defined on $\mathbb{R}[x_1, \ldots, x_{n-k+1}]$. And so, easily computed.

In this paper we prove that in the case where $m$ is odd, for $f : (M, \partial M) \to \mathbb{R}^{2m-1}$, where $M \subset \mathbb{R}^m$, $\zeta(f)$ can be expressed as $\Lambda(\alpha)$, for some $\alpha$ associated with $f$. And so, with some additional assumptions concerning $M$ and $f$, $\zeta(f)$ can be easily computed for polynomial mapping $f$. Moreover we present a method that can be used to check effectively that $f$ has only cross-caps as singular points. In case when $m$ is even, the effective method to compute number of cross-caps modulo 2 is presented in [5].

Take a smooth map $g : \mathbb{R}^m \to \mathbb{R}^{2m-2}$, let us assume that $g|_{S^{m-1}}$ is an immersion. In [10], Whitney introduced the intersection number $I(g|_{S^{m-1}})$ of immersion $g|_{S^{m-1}}$. In this paper we show that $I(g|_{S^{m-1}})$, can be presented as an algebraic sum of cross-caps of the mapping $(\omega, g)|_{B^m}$, where $\omega$ is sum of squares of coordinates.

Take $f : (\mathbb{R}^m, 0) \to \mathbb{R}^{2m-1}$ with cross-cap at 0. In [3], Ikegami and Saeki defined the sign of a cross-cap singularity for mapping $f$ as the intersection number of immersion $f|_S : S = f^{-1}(S^{2m-2}(\epsilon)) \to S^{2m-2}(\epsilon)$, for $\epsilon$ small enough. It is easy to see that this definition complies with Whitney definition from [11]. In [3], the authors showed that for generic map (in sense of [3]) $g : (\mathbb{R}^m, 0) \to \mathbb{R}^{2m-1}$, the number of cross-caps appearing in a $C^\infty$ stable perturbation of $g$, counted with signs, is an invariant of the topological $A_\pm$-equivalence class of $g$, and is equal to the intersection number of $g|_S : S = g^{-1}(S^{2m-2}(\epsilon)) \to S^{2m-2}(\epsilon)$. Using our methods, this number can be easily computed for polynomial mappings.
We use notation $S^n(r)$, $B^n(r)$, $\overline{B^n(r)}$ for sphere, open ball, closed ball (resp.) centred at the origin of radios $r$ and dimension $n$. If we omit symbol $r$, we assume that $r = 1$.

1. Cross-cap singularities

Let $M, N$ be smooth manifolds. Take a smooth mapping $f : M \to N$.

**Lemma 1.** Let $W$ be a submanifold of $N$. Take $p \in M$ such that $f(p) \in W$. Let us assume that there is a neighbourhood $U$ of $f(p)$ in $N$ and a smooth mapping $\phi : U \to \mathbb{R}^s$ such that $\text{rank } D\phi(f(p)) = k = \text{codim } W$ and $W \cap U = \phi^{-1}(0)$. Then $f \cap W$ at $p$ if and only if $\text{rank } D(\phi \circ f)(p) = k$.

**Proof.** Of course $\text{Ker } D\phi(f(p)) = T_{f(p)} W$, and so we get $\dim T_{f(p)} N = \dim \text{Ker } D\phi(f(p)) + k$. Then:

\[
\begin{align*}
f \cap W \text{ at } p & \iff T_{f(p)} N = T_{f(p)} W + Df(p)T_p M \\
& \iff T_{f(p)} N = \text{Ker } D\phi(f(p)) + Df(p)T_p M.
\end{align*}
\]

The above equality holds if and only if there exist vectors $v_1, \ldots, v_k$ in $Df(p)T_p M$, such that any nontrivial combination of $v_1, \ldots, v_k$ is outside the $\text{Ker } D\phi(f(p))$ and so $\text{rank } D\phi(f(p))[v_1 \ldots v_k] = k$. We get that $f \cap W$ at $p$ if and only if $\text{rank } D(\phi \circ f)(p) = k$. \hfill \Box

By $j^1 f$ we mean the canonical mapping associated with $f$, from $M$ into the spaces of 1-jets $J^1(M, N)$. We say that $f : M \to N$ is 1-generic, if $j^1 f \cap S_r$, for $r \geq 0$, where $S_r = \{ \sigma \in J^1(M, N) \mid \text{corank } \sigma = r \}$. Put $S_r(f) = \{ x \in M \mid \text{corank } Df(p) = r \} = (j^1 f)^{-1}(S_r)$.

Let us assume that $M$ and $N$ are manifolds of dimension $m$ and $2m - 1$ respectively. In this case (see [1]) $\text{codim } S_r = r^2 + r(m - 1)$, and so $\text{codim } S_1 = m$ and $\text{codim } S_r > m$, for $r \geq 2$. So $f$ is 1-generic if and only if $f \cap S_1$ and $S_r(f) = \emptyset$ for $r \geq 2$. The typical singularity for mapping $f : M \to N$ is a cross-cap singularity. Following [9], [11], [1] we present equivalent definitions of a cross-cap.

**Definition 1.** A point $p$ is a cross-cap of a mapping $f : M \to N$ if the following equivalent conditions are fulfilled:

1) $p \in S_1(f)$ and $j^1 f \cap S_1$ at $p$;
2) there are coordinate systems near $p$ and $f(p)$, such that

\[
\frac{\partial f}{\partial x_1}(p) = 0 \tag{1}
\]
and vectors
\[
\frac{\partial^2 f}{\partial x_1^2}(p), \frac{\partial f}{\partial x_2}(p), \ldots, \frac{\partial f}{\partial x_m}(p), \frac{\partial^2 f}{\partial x_1 \partial x_2}(p), \ldots, \frac{\partial^2 f}{\partial x_1 \partial x_m}(p) \tag{2}
\]
are linearly independent;
3) there are coordinate systems near \( p \) and \( f(p) \) such that the mapping \( f \) has the form
\[
(x_1, \ldots, x_m) \mapsto (x_1^2, x_2, \ldots, x_m, x_1 x_2, \ldots, x_1 x_m).
\]
According to [9, Section 2], if \( p \) is a cross-cap singularity and (1) holds, then vectors (2) are linearly independent.

Take \( f = (f_1, \ldots, f_{2m-1}) : \mathbb{R}^m \to \mathbb{R}^{2m-1} \). Put \( \mu : \mathbb{R}^m \to \mathbb{R}^s \) such that \( \mu(x) \) is given by all the \( m \)-minors of \( Df(x) \). Of course \( s = \binom{2m-1}{m} \).

**Lemma 2.** A point \( p \in \mathbb{R}^m \) is a cross-cap singularity of \( f \) if and only if \( \text{rank} \ Df(p) = m-1 \) and \( \text{rank} \ D\mu(p) = m \).

**Proof.** A point \( p \) is a cross-cap singularity if and only if \( p \in S_1(f) \) and \( j^1 f \cap S_1 \) at \( p \). Note that \( p \in S_1(f) \) if and only if \( \text{rank} \ Df(p) = m-1 \).

Of course \( J^1(\mathbb{R}^m, \mathbb{R}^{2m-1}) \cong \mathbb{R}^m \times \mathbb{R}^{2m-1} \times M(2m-1, m) \), where \( M(2m-1, m) \) is a space of real matrices of dimension \( (2m-1) \times m \). Take an open neighbourhood \( U \) of \( j^1 f(p) \) in \( J^1(\mathbb{R}^m, \mathbb{R}^{2m-1}) \), and a mapping \( \phi : U \to \mathbb{R}^s \), where \( \phi(x, y, [a_{ij}]) \) is given by all \( m \)-minors of \( [a_{ij}] \). We may assume that
\[
\text{det} \frac{\partial (f_1, \ldots, f_{m-1})}{\partial (x_1, \ldots, x_{m-1})}(p) \neq 0.
\]
Put \( A = [a_{ij}]_{1 \leq i, j \leq m-1} \) the submatrix of \( [a_{ij}] \), then for \( U \) small enough, \( \text{det} A \neq 0 \). Let \( M_i \) be the determinant of submatrix of \( [a_{ij}] \) composed of first \( m-1 \) rows and row number \( (m+i-1) \), for \( i = 1, \ldots, m \). Then
\[
M_i = (-1)^{2m-1+i} \text{det} A \cdot a_{m+i-1,m} + b_i,
\]
for \( i = 1, \ldots, m \) and \( b_i \) does not depend on \( a_{mm}, \ldots, a_{2m-1,m} \), and so
\[
\text{rank} \ \frac{\partial (M_1, \ldots, M_m)}{\partial (a_{m,m}, \ldots, a_{2m-1,m})} = m.
\]
We get that
\[
\text{rank} \ D\phi(j^1 f(p)) \geq m.
\]
Let us recall that \( \text{codim} S_1 = m \). We can choose \( U \) small enough such that

\[
\phi^{-1}(0) = U \cap S_1. 
\]

So we get that rank \( D\phi(j^1f(p)) = \text{codim} S_1 = m \). Of course \( \phi \circ j^1f = \mu \) in the small neighbourhood of \( p \). According to Lemma 1, \( j^1f \cap S_1 \) at \( p \) if and only if rank \( D\mu(p) = m \).

\[ \square \]

2. **Algebraic sum of cross-cap singularities**

First we want to recall some well-known facts concerning the topological degree. Let \((N, \partial N)\) be \( n \)-dimensional compact oriented manifold with boundary. For smooth mapping \( f : N \to \mathbb{R}^n \) such that \( f|_{\partial N} : \partial N \to \mathbb{R}^n \setminus \{0\} \), by \( \deg f|_{\partial N} \) or \( \deg(f, N, 0) \) we denote the topological degree of mapping \( f|f| : \partial N \to S^{n-1} \). Note that if \( f^{-1}(0) \) is a finite set then

\[
\deg f|_{\partial N} = \sum_{p \in f^{-1}(0)} \deg_p f, 
\]

where \( \deg_p f \) stands for the local topological degree of \( f \) at \( p \) (see [8]).

Let \( M \) be a \( m \)-dimensional manifold and \( m \) be odd. Take a smooth mapping \( f : M \to \mathbb{R}^{2m-1} \) and let \( p \in M \) be a cross-cap of \( f \). According to [11], \( p \) is called positive (negative) if the vectors (2) determine the negative (positive) orientation of \( \mathbb{R}^{2m-1} \). According to [11, Lemma 3], this definition does not depend on choosing the coordinate system on \( M \).

Let us assume, that \( f : \mathbb{R}^m \to \mathbb{R}^{2m-1} \) is a smooth mapping such that 0 is a cross-cap of \( f \). Of course it is an isolated critical point of \( f \). Denote by \( v_i \) the \( i \)th column of \( Df \), for \( i = 1, \ldots, m \). There exists \( r > 0 \) such that \( v_1(x), \ldots, v_m(x) \) are linearly independent for \( x \in \bar{B}^m(r) \setminus \{0\} \). Following [6] we can define

\[
\tilde{\alpha}(\beta, x) = \beta_1 v_1(x) + \ldots + \beta_m v_m(x) 
\]
\[
= Df(x)(\beta) : S^{m-1} \times \bar{B}^m(r) \to \mathbb{R}^{2m-1}.
\]

Then the topological degree of the mapping

\[
\tilde{\alpha}|_{S^{m-1} \times S^{m-1}(r)} : S^{m-1} \times S^{m-1}(r) \to \mathbb{R}^{2m-1} \setminus \{0\}
\]

is well defined. By [6, Proposition 2.4], \( \deg(\tilde{\alpha}|_{S^{m-1} \times S^{m-1}(r)}) \) is even.
Theorem 1. Let $m$ be odd. If 0 is a cross-cap of a mapping $f : \mathbb{R}^m \to \mathbb{R}^{2m-1}$, then it is positive if and only if $\frac{1}{2} \deg(\bar{\alpha}|_{S^{m-1} \times S^{m-1}(r)}) = -1$, and so it is negative if and only if $\frac{1}{2} \deg(\bar{\alpha}|_{S^{m-1} \times S^{m-1}(r)}) = +1$.

Proof. We can find linear coordinate system $\phi : \mathbb{R}^m \to \mathbb{R}^m$, such that $\phi(0) = 0$ and $f \circ \phi$ fulfills condition (1) at 0. Denote by $A$ the matrix of $\phi$. Let $w_1, \ldots, w_m$ denote columns of $D(f \circ \phi)$. Then $w_1(0) = 0$ and since 0 is a cross-cap then vectors

$$\frac{\partial w_1}{\partial x_1}(0), \ w_2(0), \ldots, \ w_m(0), \ \frac{\partial w_1}{\partial x_2}(0), \ldots, \ \frac{\partial w_1}{\partial x_m}(0) \quad (3)$$

are linearly independent. Put $\tilde{\gamma}(\beta, x) = (\beta_1 w_1(x) + \ldots + \beta_m w_m(x)) : S^{m-1} \times \bar{B}^m(r) \to \mathbb{R}^{2m-1}$. We can assume that $r$ is such that $\tilde{\gamma} \neq 0$ on $S^{m-1} \times \bar{B}^m(r) \setminus \{0\}$. Let us see that

$$\tilde{\gamma}(\beta, x) = D(f \circ \phi)(x) \cdot \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} = Df(\phi(x)) \cdot A \cdot \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$$

$$= Df(\phi(x)) \cdot \begin{bmatrix} \phi_1(\beta) \\ \vdots \\ \phi_m(\beta) \end{bmatrix}.$$ 

So $\tilde{\gamma} = \tilde{\alpha} \circ (\phi \times \phi)$. It is easy to see that $\phi \times \phi$ preserve the orientation of $S^{m-1} \times S^{m-1}(r)$. We can assume that $r > 0$ is so small, that $\deg(\bar{\alpha}|_{S^{m-1} \times S^{m-1}(r)}) = \deg(\tilde{\alpha}|_{\phi(S^{m-1}) \times \phi(S^{m-1}(r))})$. So we get that

$$\deg(\tilde{\gamma}|_{S^{m-1} \times S^{m-1}(r)}) = \deg(\tilde{\alpha}|_{\phi(S^{m-1}) \times \phi(S^{m-1}(r))}) \deg(\phi \times \phi) =$$

$$= \deg(\bar{\alpha}|_{S^{m-1} \times S^{m-1}(r)}).$$ 

Since $f \circ \phi$ fulfills (1), vectors $w_2, \ldots, w_m$ are independent on $\bar{B}^m(r)$. Let us see that $\tilde{\gamma}(\beta, x) = 0$ on $S^{m-1} \times \bar{B}^m(r)$ if and only if $x = 0$ and $\beta = (\pm 1, 0, \ldots, 0)$. So $\deg(\tilde{\gamma}|_{S^{m-1} \times S^{m-1}(r)})$ is a sum of local topological degrees of $\tilde{\gamma}$ at $(1, 0, \ldots, 0; 0, \ldots, 0)$ and at $(-1, 0, \ldots, 0; 0, \ldots, 0)$.

Near the point $(1, 0, \ldots, 0; 0, \ldots, 0)$ the well-oriented parametrisation of $S^{m-1} \times \bar{B}^m(r)$ is given by

$$(\beta_2, \ldots, \beta_m; x) = (\sqrt{1 - \beta_2^2 - \ldots - \beta_m^2}, \beta_2, \ldots, \beta_m; x).$$ 

And then the derivative matrix of $\tilde{\gamma}$ at $(1, 0, \ldots, 0; 0, \ldots, 0)$ has a form

$$A_1 = \begin{bmatrix} w_2(0) & \ldots & w_m(0) & \frac{\partial w_1}{\partial x_1}(0) & \ldots & \frac{\partial w_1}{\partial x_m}(0) \end{bmatrix}.$$
Near \((-1, 0, \ldots, 0; 0, \ldots, 0)\) the well-oriented parametrisation of \(S^{m-1} \times \bar{B}^m(r)\) is given by

\[
(\beta_2, \ldots, \beta_m; x) = (-\sqrt{1 - \beta_2^2 - \cdots - \beta_m^2}; -\beta_2, \ldots, -\beta_m; x).
\]

And then the derivative matrix of \(\tilde{\gamma}\) at \((-1, 0, \ldots, 0; 0, \ldots, 0)\) has a form

\[
A_2 = \begin{bmatrix} -w_2(0) & \cdots & w_m(0) & -\frac{\partial w_1}{\partial x_1}(0) & \cdots & -\frac{\partial w_1}{\partial x_m}(0) \end{bmatrix}.
\]

Let us recall that \(m\) is odd. System of vectors (3) is independent, so 0 is a regular value of \(\tilde{\gamma}\), and

\[
\frac{1}{2}\deg(\tilde{\gamma}|_{S^{m-1} \times S^{m-1}(r)}) = \frac{1}{2}(\text{sgn det } A_1 + \text{sgn det } A_2) = \text{sgn det } A_1.
\]

Moreover 0 is a positive cross-cap if and only if vectors (3) determine negative orientation of a \(R^{2m-1}\), i.e. if and only if \(\frac{1}{2}\deg(\tilde{\alpha}|_{S^{m-1} \times S^{m-1}(r)}) = -1\).

Let \(U \subset \mathbb{R}^m\) be an open bounded set and \(f : U \rightarrow \mathbb{R}^{2m-1}\) be smooth. We say that \(f\) is generic if only critical points of \(f\) are cross-caps and \(f\) is regular in the neighborhood of \(\partial U\). Let us denote by \(\zeta(f)\) the algebraic sum of cross-caps of \(f\). Then using Theorem 1 we get the following.

**Proposition 1.** Let \(U \subset \mathbb{R}^m\), \((m \text{ is odd})\), be a bounded \(m\)-dimensional manifold such that \(\overline{U}\) is an \(m\)-dimensional manifold with a boundary. For \(f : \overline{U} \subset \mathbb{R}^m \rightarrow \mathbb{R}^{2m-1}\) generic, \(\zeta(f) = -\frac{1}{2}\deg(\tilde{\alpha})\), where \(\tilde{\alpha}(\beta, x) = Df(x)(\beta) : S^{m-1} \times \partial U \rightarrow \mathbb{R}^{2m-1} \setminus \{0\}\).

**Proposition 2.** Let \(U \subset \mathbb{R}^m\), \((m \text{ is odd})\), be a bounded \(m\)-dimensional manifold such that \(\overline{U}\) is an \(m\)-dimensional manifold with a boundary. Take \(h : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^{2m-1}\) a smooth mapping such that \(h\) is regular in a neighborhood of \(\partial U\). Then for every generic \(f : \overline{U} \subset \mathbb{R}^m \rightarrow \mathbb{R}^{2m-1}\) close enough to \(h\) in \(C^1\)-topology we have, \(\zeta(f) = -\frac{1}{2}\deg(\tilde{\alpha})\), where \(\tilde{\alpha}(\beta, x) = Dh(x)(\beta) : S^{m-1} \times \partial U \rightarrow \mathbb{R}^{2m-1} \setminus \{0\}\).

### 3. Examples

To compute some examples we want first to recall the theory presented in [6].

Take \(\alpha = (\alpha_1, \ldots, \alpha_k) : \mathbb{R}^{n-k+1} \rightarrow M(n, k)\) a polynomial mapping, \(n - k\) even, where \(M(n, k)\) is a space of real matrices of dimension \(n \times k\).
By \([a_{ij}(x)], 1 \leq i \leq n, 1 \leq j \leq k\), we denote the matrix given by \(\alpha (x)\) (i.e. \(\alpha_j(x)\) stands in the \(j\)th column). Then one can define \(\tilde{\alpha}: \mathbb{R}^k \times \mathbb{R}^{n-k+1} \to \mathbb{R}^n\) as

\[
\tilde{\alpha}(\beta, x) = \beta_1 \alpha_1(x) + \ldots + \beta_k \alpha_k(x) = [a_{ij}(x)] \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}.
\]

Let \(I\) be the ideal in \(\mathbb{R}[x_1, \ldots, x_{n-k+1}]\) generated by all \(k \times k\) minors of \([a_{ij}(x)]\), and \(V(I) = \{x \in \mathbb{R}^{n-k+1} | h(x) = 0 \text{ for all } h \in I\}\).

Take

\[
m(x) = \det \begin{bmatrix} a_{12}(x) & \ldots & a_{1k}(x) \\ a_{k-1,2}(x) & \ldots & a_{k-1,k}(x) \end{bmatrix}.
\]

For \(k \leq i \leq n\), we define

\[
\Delta_i(x) = \det \begin{bmatrix} a_{11}(x) & \ldots & a_{1k}(x) \\ a_{k-1,1}(x) & \ldots & a_{k-1,k}(x) \\ a_{i1}(x) & \ldots & a_{ik}(x) \end{bmatrix}.
\]

Put \(A = \mathbb{R}[x_1, \ldots, x_{n-k+1}]/I\). Let us assume that \(\dim A < \infty\), so that \(V(I)\) is finite. For \(h \in A\), we denote by \(T(h)\) the trace of the linear endomorphism \(A \ni a \mapsto h \cdot a \in A\). Then \(T: A \to \mathbb{R}\) is a linear functional.

Let \(u \in \mathbb{R}[x_1, \ldots, x_{n-k+1}]\). Assume that \(\overline{U} = \{x \mid u(x) \geq 0\}\) is bounded and \(\nabla u(x) \neq 0\) at each \(x \in u^{-1}(0) = \partial U\). Then \(\overline{U}\) is a compact manifold with boundary, and \(\dim \overline{U} = n - k + 1\).

Put \(\delta = \partial(\Delta_k, \ldots, \Delta_n)/\partial(x_1, \ldots, x_{n-k+1})\). With \(u\) and \(\delta\) we associate quadratic forms \(\Theta_\delta, \Theta_{u-\delta} : A \to \mathbb{R}\) given by \(\Theta_\delta(a) = T(\delta \cdot a^2)\) and \(\Theta_{u-\delta}(a) = T(u \cdot \delta \cdot a^2)\).

**Theorem 2.** [6, Theorem 3.3] If \(n - k\) is even, \(\alpha = (\alpha_1, \ldots, \alpha_k) : \mathbb{R}^{n-k+1} \to M(n, k)\) is a polynomial mapping such that \(\dim A < \infty\), \(I + \langle m \rangle = \mathbb{R}[x_1, \ldots, x_{n-k+1}]\) and quadratic forms \(\Theta_\delta, \Theta_{u-\delta} : A \to \mathbb{R}\) are non-degenerate, then the restricted mapping \(\alpha|_{\partial U}\) goes into \(\tilde{V}_k(\mathbb{R}^n)\) and

\[
\Lambda(\alpha|_{\partial U}) = \frac{1}{2} \deg(\tilde{\alpha}|_{S^{k-1} \times \partial U}) = \frac{1}{2} (\text{signature } \Theta_\delta + \text{signature } \Theta_{u-\delta}),
\]

where \(\tilde{\alpha}(\beta, x) = \beta_1 \alpha_1(x) + \ldots + \beta_k \alpha_k(x)\).

Using the theory presented in [6], particularly [6, Theorem 3.3], and computer system \textsc{Singular} ([2]), one can apply the results from Sections 1 and 2 to compute algebraic sum of cross-caps for polynomial mappings.
Example 1. Let us take \( f : \mathbb{R}^3 \to \mathbb{R}^5 \) given by

\[
\begin{align*}
    f(x, y, z) &= (12y^2 + z, 6x^2 + y^2 + 6y, 18xy + 13y^2 + 9x, \\
    &\quad 8x^2z + 10x^2 + 5x^2 + 3xz, x^2 + 4xyz + yz + 4z^2).
\end{align*}
\]

Applying Lemma 2 and using SINGULAR one can check that \( f \) is 1-generic. Moreover, according to Proposition 1 and [6], one can check that

\[
\zeta(f|_{\overline{B}^3(\sqrt{3})}) = 2, \quad \zeta(f|_{\overline{B}^3(10)}) = 1.
\]

We can also check that \( f \) has 11 cross-caps in \( \mathbb{R}^3 \), 6 of them are positive, 5 negative.

Example 2. Take \( f : \mathbb{R}^5 \to \mathbb{R}^9 \) given by

\[
\begin{align*}
    f(s, t, x, y, z) &= (y, z, t, 20x^2 + 17sz + x, 13sy + 13sz + 5t, 25st + 4x^2 + 28z, \\
    &\quad 3x^2 + 19yz + 22s, 11ts^2 + 8t^2z + xz, 27txy + 9sxz + 20st).
\end{align*}
\]

One may check that \( f \) is 1-generic, has 3 cross-caps in \( \mathbb{R}^5 \) and

\[
\zeta(f|_{\overline{B}^3(1/10)}) = 0, \quad \zeta(f|_{\overline{B}^3(2)}) = -1, \quad \zeta(f|_{\overline{B}^3(1000)}) = 1.
\]

4. Intersection number of immersions

Take \( n \)-dimensional, compact, oriented manifold \( N \) and immersion \( g : N \to \mathbb{R}^{2n} \). As in [10] we say that an immersion \( g : N \to \mathbb{R}^{2n} \) has a regular self-intersection at the point \( g(p) = g(q) \) if

\[
Dg(p)T_p N + Dg(q)T_q N = \mathbb{R}^{2n}.
\]

An immersion \( g : N \to \mathbb{R}^{2n} \) is called completely regular if it has only regular self-intersections and no triple points.

Assume that \( n \) is even. Let \( g : N \to \mathbb{R}^{2n} \) be a completely regular immersion having a regular self-intersection at the point \( g(p) = g(q) \). Let \( u_1, \ldots, u_n \in T_p N, v_1, \ldots, v_n \in T_q N \) be sets of well-oriented, independent vectors in respective tangent spaces of \( N \). Then the vectors \( Dg(p)u_1, \ldots, Dg(p)u_n, Dg(q)v_1, \ldots, Dg(q)v_n \) form a basis in \( \mathbb{R}^{2n} \). As in [10] we will say that the self-intersection at the point \( g(p) = g(q) \) is positive or negative according to whether this basis determines the positive or negative orientation of \( \mathbb{R}^{2n} \).

Following [10], the intersection number \( I(g) \) of a completely regular immersion \( g \) is the algebraic sum of its self-intersections. For any immersion
$g : N \to \mathbb{R}^{2n}$ the intersection number $I(g)$ is defined as the intersection number of a completely regular immersion $\tilde{g}$, regularly homotopic to $g$ (homotopy by immersions). For other equivalent description of $I(g)$ see [7], [4].

As in previous Sections we assume that $m$ is odd. Take a smooth map 
\[g = (g_1, \ldots, g_{2m-2}) : \mathbb{R}^m \to \mathbb{R}^{2m-2} \text{.} \]
Denote by $\omega = x_1^2 + \ldots + x_m^2$. Then $S^{m-1}(r) = \{x | \omega(x) = r^2 \}$. According to [4, Lemma 18], $g|_{S^{m-1}(r)}$ is an immersion if and only if
\[
\begin{bmatrix}
2x_1 & \ldots & 2x_m \\
\frac{\partial g_1}{\partial x_1}(x) & \ldots & \frac{\partial g_1}{\partial x_m}(x) \\
\vdots & & \vdots \\
\frac{\partial g_{2m-2}}{\partial x_1}(x) & \ldots & \frac{\partial g_{2m-2}}{\partial x_m}(x)
\end{bmatrix} = m,
\]
for $x \in S^{m-1}(r)$.

Take $0 < r_1 < r_2$, such that $g|_{S^{m-1}(r_1)}$ and $g|_{S^{m-1}(r_2)}$ are immersions. Denote by $P = \{x | r_1^2 \leq \omega(x) \leq r_2^2 \}$. Then $P$ is an $m$-dimensional oriented manifold with boundary. Then $(\omega, g) : \mathbb{R}^m \to \mathbb{R}^{2m-1}$ is a regular map in the neighbourhood of $\partial P$. Let us define $\tilde{\alpha} : S^{m-1} \times P \to \mathbb{R}^{2m-1}$ as
\[
\tilde{\alpha}(\beta, x) = \begin{bmatrix}
2x_1 \\
\frac{\partial g_1}{\partial x_1}(x) \\
\vdots \\
\frac{\partial g_{2m-2}}{\partial x_1}(x)
\end{bmatrix} \begin{bmatrix}
\beta_1 \\
\vdots \\
\beta_m
\end{bmatrix}
\]

**Proposition 3.** Let us assume that $g|_{S^{m-1}(r_1)}$ and $g|_{S^{m-1}(r_2)}$ are immersions, then
\[
I(g|_{S^{m-1}(r_2)}) - I(g|_{S^{m-1}(r_1)}) = \zeta(\omega, g)|_P.
\]

**Proof.** Let us recall that $m$ is odd. Then
\[
\deg(\tilde{\alpha}|_{S^{m-1} \times \partial P}) = \deg(\tilde{\alpha}|_{S^{m-1} \times S^{m-1}(r_2)}) - \deg(\tilde{\alpha}|_{S^{m-1} \times S^{m-1}(r_1)}).
\]
According to [6, Theorem 4.2], we get that
\[
\deg(\tilde{\alpha}|_{S^{m-1} \times S^{m-1}(r_1)}) = I(g|_{S^{m-1}(r_1)}),
\]
for $i = 1, 2$. Then applying Proposition 2 we get that $\zeta(\omega, g)|_P = -\frac{1}{2} \deg(\tilde{\alpha}|_{S^{m-1} \times \partial P})$. And so
\[
\zeta(\omega, g)|_P = I(g|_{S^{m-1}(r_2)}) - I(g|_{S^{m-1}(r_1)}). \quad \square
\]
Corollary 1. If $g|_{S^{m-1}(r)}$ is an immersion, then

$$I(g|_{S^{m-1}(r)}) = \zeta((\omega, g)|_{\bar{B}^m(r)}).$$

Remark 1. If the only singular points of $(\omega, g)|_{\bar{B}^m(r)}$ are cross-caps, then the intersection number of an immersion $g|_{S^{m-1}(r)}$ is equal to the algebraic sum of cross-caps of $(\omega, g)|_{\bar{B}^m(r)}$. Also, in generic case, the difference between intersection numbers of immersions $g|_{S^{m-1}(r_1)}$ and $g|_{S^{m-1}(r_2)}$, is equal to the algebraic sum of cross-caps of $(\omega, g)$ appearing in $P$.

Remark 2. If $(\omega, g)$ has finite number of singular points, and all of them are cross-caps, then for any $R > 0$ big enough, $g|_{S^{m-1}(R)}$ is an immersion with the same intersection number equal to the algebraic sum of cross-caps of $(\omega, g)$.

Example 3. Take $g : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$g = (-3y^2 + 5yz - x + 2, -4x^2 + z^2 + 9y - 6z + 5, 4x^2z - 2x^2 + 2xy - y - 3, 3y^2z + xy - 4yz + 4x - 5y - 5),$$

and $\omega = x^2 + y^2 + z^2$. In the same way as in Section 3, one may check that the only singular points of $(\omega, g)$ are cross-caps, moreover $(\omega, g)$ has 8 cross-caps, 5 of them are positive and 3 negative. According to previous results $g|_{S^2(r)}$ is an immersion for all $r > 0$, except at most 8 values of $r$. And if $g|_{S^2(R)}$ is an immersion, then

$$-3 \leq I(g|_{S^2(r)}) \leq 5.$$

Moreover for $R > 0$ big enough $g|_{S^2(R)}$ is an immersion with

$$I(g|_{S^2(R)}) = 2.$$

References


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Received by the editors: 22.09.2015  
and in final form 02.03.2018.