Generalized classes of suborbital graphs for the congruence subgroups of the modular group

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Abstract. Let $\Gamma$ be the modular group. We extend a non-trivial $\Gamma$-invariant equivalence relation on $\hat{Q}$ to a general relation by replacing the group $\Gamma_0(n)$ by $\Gamma_K(n)$, and determine the suborbital graph $F_{u,n}$, an extended concept of the graph $F_{u,n}$. We investigate several properties of the graph, such as, connectivity, forest conditions, and the relation between circuits of the graph and elliptic elements of the group $\Gamma_K(n)$. We also provide the discussion on suborbital graphs for conjugate subgroups of $\Gamma$.

Introduction

Let $G$ be a permutation group acting transitively on a nonempty set $X$. Then the action of $G$ can be extended naturally on $X \times X$ by

$$g(v, w) = (g(v), g(w)),$$

where $g \in G$ and $v, w \in X$. The orbit $G(v, w)$ is called a suborbital of $G$ containing $(v, w)$. A suborbital graph $\mathcal{G}(v, w)$ for $G$ on the set $X$ is a graph whose vertex set is the set $X$ and the family of directed edges is

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the suborbital $G(v, w)$. Hence, there exists a directed edge from $v_1$ to $v_2$, denoted by $v_1 \rightarrow v_2$, if $(v_1, v_2) \in G(v, w)$.

The concept of suborbital graphs was first introduced by Sims [14]. Then Jones, Singerman, and Wicks [8] used this idea to construct the suborbital graphs $G_{u,n}$ for the modular group $\Gamma$ acting on the extended set of rational numbers $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. To examine the properties of $G_{u,n}$, they applied the fact that the action of $\Gamma$ on $\hat{\mathbb{Q}}$ is imprimitive, i.e., there is a $\Gamma$-invariant equivalence relation other than the two trivial relations which form the partitions $\{\hat{\mathbb{Q}}\}$ and $\{\{v\} : v \in \hat{\mathbb{Q}}\}$. They used the congruence subgroup $\Gamma_0(n)$ to induce the nontrivial $\Gamma$-invariant equivalence relation on $\hat{\mathbb{Q}}$, and studied the subgraphs $F_{u,n}$ of the graphs $G_{u,n}$ restricted on the block $[\infty]$, the equivalence class containing $\infty$. Note that the graph $G_{u,n}$ is the union of $m$ copies of $F_{u,n}$, where $m$ is the index of $\Gamma_0(n)$ in $\Gamma$. Moreover, if $F_{u,n}$ contains edges, it is actually a suborbital graph for $\Gamma_0(n)$ on the block $[\infty]$.

There are several studies related to the graphs for the modular group, see [1, 4, 5, 7, 11, 13], and other papers about suborbital graphs for other groups, see [2, 3, 6, 9, 10, 12]. In [11], the authors used the different $\Gamma$-invariant equivalence relation obtained from another congruence subgroup $\Gamma_1(n)$ of $\Gamma$, and investigated the connectivity of subgraphs of $G_{u,n}$ on the block containing $\infty$.

Inspired by the results in [8, 11], we introduce a $\Gamma$-invariant equivalence relation using the congruence subgroup $\Gamma_K(n)$ where $K$ is a subgroup of the group of unit $\mathbb{Z}_n^*$. This group is a generalization of $\Gamma_0(n)$ and $\Gamma_1(n)$, so it provides a generalized $\Gamma$-invariant equivalence relation of those induced from $\Gamma_0(n)$ and $\Gamma_1(n)$. We denote by $F^K_{u,n}$ the subgraph of $G_{u,n}$ on the block $[\infty]_K$ with respect to the group $\Gamma_K(n)$, and demonstrate various properties of $F^K_{u,n}$, such as, connectivity, forest conditions, including the relation between circuits of the graph and elliptic elements of the group $\Gamma_K(n)$. In the final section we provide a discussion of the relation of suborbital graphs for congruence subgroups. We show that the suborbital graphs for the group $\Gamma_0(n)$ studied in [7] is isomorphic to some graph $F_{u,n}$. The result is also extended to the case of $\Gamma_K(n)$ and $\Gamma^K(n)$, a generalization of $\Gamma_0(n)$. Moreover, we discuss suborbital graphs for $\Gamma_K(n)$ on $[\infty]_K$ which are more general than $F^K_{u,n}$.

This work can be restricted to the case of $\Gamma_1(n)$ by replacing the group $K$ by the trivial subgroup $\{1\}$ of $\mathbb{Z}_n^*$. This case was studied in [11] already; however, the results in there are different from ours because of
the definition of $\Gamma_1(n)$. The differences will be explained in another our publication.

1. Preliminaries

Let $\Gamma$ be a set of all linear fractional (Möbius) transformations on the upper half-plane $\mathbb{H}^2 = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ of the form

$$z \mapsto \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{Z}$, and $ad - bc = 1$. With the composition of functions, $\Gamma$ forms a group which is called the modular group. The group $\Gamma$ is isomorphic to $\text{PSL}(2, \mathbb{Z})$, the quotient group of the unimodular group $\text{SL}(2, \mathbb{Z})$ by its centre $\{\pm I\}$. Thus, every element of $\Gamma$ of the form (1) can be referred to as the pair of matrices

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}).$$

For convenience, we may leave the sign of matrices representing elements of the group $\Gamma$ and identify them with their negative sign.

Let $n$ be any natural number. One can show that

$$\Lambda(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) : a \equiv 1 \pmod{n}, \text{ and } b \equiv c \equiv 0 \pmod{n} \right\}$$

is a subgroup of $\text{SL}(2, \mathbb{Z})$. The image of $\Lambda(n)$ in $\Gamma = \text{PSL}(2, \mathbb{Z})$ under the quotient mapping is called the principal congruence subgroup of level $n$ and denoted by $\Gamma(n)$. We can see easily that

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv \pm 1 \pmod{n}, \text{ and } b \equiv c \equiv 0 \pmod{n} \right\}.$$

A subgroup of $\Gamma$ containing $\Gamma(n)$, for some $n$, is called a congruence subgroup of $\Gamma$. There are two well-known congruence subgroups of the modular group, that is,

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{n} \right\},$$

and

$$\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv \pm 1 \pmod{n}, \text{ and } c \equiv 0 \pmod{n} \right\}.$$
These two groups are mainly used in [8] and [11], respectively.

We now introduce some classes of congruence subgroups of the modular group. Let $K$ be a subgroup of a group of units $\mathbb{Z}_n^*$, and $\overline{a}_n$ denote a congruence class containing an integer $a$ modulo $n$. Without the confusion, we may leave the subscript $n$ and use $\overline{a}$ instead. One can prove easily that

$$\Lambda_K(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) : \overline{a} \in K, \text{ and } c \equiv 0 \mod n \right\}$$

is a subgroup of $\text{SL}(2, \mathbb{Z})$ containing the group $\Lambda(n)$, so the image in $\Gamma$ of this group is certainly a congruence subgroup of $\Gamma$. We let $\Gamma_K(n)$ denote the congruence subgroup of $\Gamma$ obtained in this way. Obviously,

$$\Gamma_K(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \overline{a} \in -K \cup K, \text{ and } c \equiv 0 \mod n \right\},$$

where $-K = \{-\overline{a} : \overline{a} \in K\}$. In the case that $K$ is a trivial subgroup of $\mathbb{Z}_n^*$, $\{1\}$ or $\mathbb{Z}_n^*$, $\Gamma_K(n)$ is such $\Gamma_1(n)$ and $\Gamma_0(n)$, respectively.

We see that every coefficient of a transformation in the modular group is an integer. Then the action (1) can be extended to act on $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. In [8], the authors represented every element in $\hat{\mathbb{Q}}$ as reduced fractions

$$\frac{x}{y} = \frac{-x}{-y},$$

where $x, y \in \mathbb{Z}$ and $\gcd(x, y) = 1$. In the case of $\infty$, it is represented by the fractions $\frac{1}{0} = \frac{-1}{0}$. Now the action (1) of $\Gamma$ on $\hat{\mathbb{Q}}$ can be rewritten as follows,

$$\frac{x}{y} \mapsto \frac{ax + by}{cx + dy}.$$ 

Certainly, $\frac{ax+by}{cx+dy}$ is a reduced fraction. The action of $\Gamma$ on $\hat{\mathbb{Q}}$ is absolutely independent from the non-uniqueness of the representations of fractions. Note that the action of $\Gamma$ on the set $\hat{\mathbb{Q}}$ is transitive, that is, for every $v, w \in \hat{\mathbb{Q}}$ there exists a transformation $\gamma \in \Gamma$ such that $\gamma(v) = w$, equivalently, for every $v \in \hat{\mathbb{Q}}$ there exists $\gamma \in \Gamma$ such that $\gamma(\infty) = v$. This means that we can represent every element in $\hat{\mathbb{Q}}$ by $\gamma(\infty)$, where $\gamma \in \Gamma$.

We see that $\Gamma_\infty < \Gamma_K(n) \leq \Gamma$ where $\Gamma_\infty$ is the stabilizer of $\infty$, the set of all translations $z \to z + b$ with $b \in \mathbb{Z}$. The second inequality is strict if $n > 1$. Then a nontrivial $\Gamma$-invariant equivalence relation on $\hat{\mathbb{Q}}$, see also [8, page 319] for a general definition, related to the group $\Gamma_K(n)$ is given by

$$\gamma(\infty) \sim \gamma'(\infty) \text{ if and only if } \gamma' \in \gamma\Gamma_K(n),$$

where $\gamma\Gamma_K(n)$ is a left coset of $\Gamma_K(n)$ in $\Gamma$. An equivalence class is called a block and denoted by $[v]_K$, the block containing an element $v$ of $\hat{\mathbb{Q}}$. 

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From the relation obtained above, we see that the block \([\gamma(\infty)]_K\) is the set
\[
\{\gamma \Gamma_K(n)\}(\infty) = \{\gamma \gamma_K(\infty) : \gamma_K \in \Gamma_K(n)\}.
\]
In particular, the block \([\infty]_K\) is the \(\Gamma_K(n)\)-orbit,
\[
\{\Gamma_K(n)\}(\infty) = \{\gamma_K(\infty) : \gamma_K \in \Gamma_K(n)\}.
\]
Therefore, \(\Gamma_K(n)\) acts transitively on the block
\[
[\infty]_K = \left\{ \frac{x}{y} \in \hat{\mathbb{Q}} : x \in K \cup K, y \equiv 0 \mod n \right\}.
\]

**Proposition 1.** Let \(n, m\) be positive integers, \(K\) and \(K'\) be subgroups of \(\mathbb{Z}_n^*\) and \(\mathbb{Z}_m^*\), respectively. Then the following statements are equivalent,

1) \(\Gamma_K(n) \leq \Gamma_{K'}(m)\),
2) \([\infty]_K \subseteq [\infty]_{K'}\),
3) \(m \mid n\) and \(\{k \in \mathbb{Z} : \bar{k}_n \in -K \cup K\} \subseteq \{k \in \mathbb{Z} : \bar{k}_m \in -K' \cup K'\}\).

**Proof.** 1) \(\Rightarrow\) 2) It is obvious from the fact that if \(H \leq G\), the orbit \(H(x)\) is always contained in the orbit \(G(x)\).

2) \(\Rightarrow\) 3) Suppose that \([\infty]_K \subseteq [\infty]_{K'}\), and \(a \in \{k \in \mathbb{Z} : \bar{k}_n \in -K \cup K\}\). Then \(\frac{a}{n} \in [\infty]_K \subseteq [\infty]_{K'}\). This implies that \(m \mid n\) and \(\bar{a}_m \in -K' \cup K'\), that is, \(a \in \{k \in \mathbb{Z} : \bar{k}_m \in -K' \cup K'\}\).

3) \(\Rightarrow\) 1) Suppose that the conditions hold, and \((\begin{array}{ll} a & b \\ c & d \end{array})\) belongs to \(\Gamma_K(n)\). Then \(\bar{a}_n \in -K \cup K\) and \(c \equiv 0 \mod n\). Since \(m \mid n\), we have \(c \equiv 0 \mod m\). The remaining condition implies that \(\bar{a}_m \in -K' \cup K'\). Hence, \((\begin{array}{ll} a & b \\ c & d \end{array})\) \(\in \Gamma_{K'}(m)\). \(\square\)

### 2. The graph \(\mathcal{G}^{K}_{u,n}\)

In this section we determine the graph \(\mathcal{G}^{K}_{u,n}\) and describe general properties of the graph, for example, edge conditions and isomorphism results. Let \(\mathcal{G}(v, w)\) be a suborbital graph for \(\Gamma\) on \(\hat{\mathbb{Q}}\). The directed graph \(\mathcal{G}(v, w)\) and its arrow reversed graph \(\mathcal{G}(w, v)\) are called *paired* suborbital graphs. In the case \(\mathcal{G}(v, w) = \mathcal{G}(w, v)\), the graphs become undirected, we will call them *self-paired*. Since \(\Gamma\) acts transitively on the set \(\hat{\mathbb{Q}}\), there is a transformation \(\gamma \in \Gamma\) mapping \(v\) to \(\infty\). Hence, a suborbital graphs \(\mathcal{G}(v, w)\) and \(\mathcal{G}(\infty, \gamma(w))\) are identical. If \(\gamma(w) = \frac{u}{n}\) where \(u, n \in \mathbb{Z}\), \(n \geq 0\) and \(\gcd(u, n) = 1\), the graph will be simply denoted by \(\mathcal{G}_{u,n}\). This is a notation traditionally used in [8] and the related research. In the case \(\frac{u}{n} = \infty\), the graph just contains loop at every vertex, and every vertex is not adjacent
to others. This is a trivial case of the suborbital graphs, and need not
be studied. We will consider only the nontrivial case, that is, \( \frac{u}{n} \neq \infty \). In
this case we let all edges be complete geodesics in the upper half-plane \( \mathbb{H}^2 \) joining between two vertices. We denote by \( \mathcal{F}_{u,n}^K \) the subgraph of \( \mathcal{G}_{u,n} \)
whose vertex set is the block \( [\infty]_K \). For the case \( \Gamma_K(n) = \Gamma_0(n) \), the graph
and the block will be simply denoted by \( \mathcal{F}_{u,n} \) and \( [\infty]_0 \), respectively. The
remark below demonstrates a trivial result immediate from Proposition 1
and the definition of \( \mathcal{F}_{u,n}^K \).

**Remark 1.** If \( K \leq K' \), then \( \mathcal{F}_{u,n}^K \) is a subgraph of \( \mathcal{F}_{u,n}^{K'} \). In particular,
\( \mathcal{F}_{u,n}^K \) is a subgraph of \( \mathcal{F}_{u,n} \).

For \( n = 1 \), we obtain \( \Gamma_K(n) = \Gamma \) and \( [\infty]_K = \hat{Q} \). Hence \( \mathcal{G}_{1,1} = \mathcal{F}_{1,1} = \mathcal{F}_{1,1}^K \). We call this graph the Farey graph. The following are some basic
results of suborbital graphs for \( \Gamma \) which were obtained in \([8]\).

**Lemma 1.** \( \Gamma \) acts on vertices and edges of \( \mathcal{G}_{u,n} \) transitively.

**Lemma 2.** \( \mathcal{G}_{u,n} = \mathcal{G}_{u',n'} \) if and only if \( n = n' \) and \( u \equiv u' \mod n \).

**Lemma 3.** \( \mathcal{G}_{u,n} \) is self-paired if and only if \( u^2 \equiv -1 \mod n \).

**Lemma 4.** No edges of \( \mathcal{F}_{1,1} \) cross in \( \mathbb{H}^2 \).

**Proposition 2.** There is an edge \( \frac{r}{s} \rightarrow \frac{x}{y} \) in \( \mathcal{G}_{u,n} \) if and only if one of the
following conditions holds,

1) \( x \equiv ur \mod n, y \equiv us \mod n \) and \( ry - sx = n \),
2) \( x \equiv -ur \mod n, y \equiv -us \mod n \) and \( ry - sx = -n \).

Next we state the first result for the graph \( \mathcal{F}_{u,n}^K \), the edge conditions.
Let us consider the fractions \( \frac{r}{s} \) and \( \frac{x}{y} \) in the previous proposition. If they
are in the block \( [\infty]_K \), then \( s \equiv r \equiv 0 \mod n \), so \( s \equiv \pm ur \mod n \). We
now have the following proposition immediately.

**Proposition 3.** There is an edge \( \frac{r}{s} \rightarrow \frac{x}{y} \) in \( \mathcal{F}_{u,n}^K \) if and only if it satisfies
one of the following conditions,

1) \( x \equiv ur \mod n \) and \( ry - sx = n \),
2) \( x \equiv -ur \mod n \) and \( ry - sx = -n \).

Suppose that \( v \rightarrow w \) is an edge of \( \mathcal{F}_{u,n}^K \). Then there exists a transformation \( \gamma \in \Gamma \) such that \( \gamma(\infty \rightarrow \frac{u}{n}) = v \rightarrow w \). Since \( v \) are in \( [\infty]_K \), we can see easily that \( \gamma \in \Gamma_K(n) \), and so \( \frac{u}{n} = \gamma^{-1}(w) \in (\Gamma_K(n))(\infty) = [\infty]_K \).
This means that if \( \mathcal{F}_{u,n}^K \) contains edges, it also contains the vertex \( \frac{u}{n} \). The
Converse is also true that if $\frac{u}{n}$ is a vertex of $\mathcal{F}_{u,n}^K$, the graph contains edges including the edge $\infty \rightarrow \frac{u}{n}$. We conclude this fact in the following corollary.

**Corollary 1.** $\mathcal{F}_{u,n}^K$ contains edges if and only if $\frac{u}{n} \in [\infty]_K$, i.e., $\overline{u} \in -K \cup K$.

We have known that $\Gamma^K(n)$ acts transitively on the vertex set of $\mathcal{F}_{u,n}^K$, the block $[\infty]_K$. We will show that it also acts transitively on edges of $\mathcal{F}_{u,n}^K$. We may suppose that the graph contains edges. Then Corollary 1 implies that $\frac{u}{n} \in [\infty]_K$. Thus, $\mathcal{F}_{u,n}^K$ is really a suborbital graph for $\Gamma^K(n)$ on the block $[\infty]_K$. We now obtain a trivial consequences coming from [8, Proposition 3.1] that $\Gamma^K(n)$ acts transitively on edges of $\mathcal{F}_{u,n}^K$.

**Corollary 2.** $\Gamma^K(n)$ acts on vertices and edges of $\mathcal{F}_{u,n}^K$ transitively.

The next corollary provides the sufficient and necessary conditions for $\mathcal{F}_{u,n}^K$ to be a self-paired suborbital graph for $\Gamma^K(n)$.

**Corollary 3.** $\mathcal{F}_{u,n}^K$ is self-paired if and only if $\overline{u} \in K$ and $u^2 \equiv -1 \mod n$.

**Proof.** Suppose that $\mathcal{F}_{u,n}^K$ is self-paired. By using Lemma 1, $\mathcal{G}_{u,n}$ is self-paired. Then Lemma 3 implies that $u^2 \equiv -1 \mod n$, and so, $\overline{u} \in K$ if and only if $-\overline{u} \in K$. Since $\mathcal{F}_{u,n}^K$ contains edges, Corollary 1 implies that $\overline{u} \in -K \cup K$. If $\overline{u} \in -K$, we have $-\overline{u} \in K$, and so, $\overline{u} \in K$. For the converse, Lemma 3 implies that $\mathcal{G}_{u,n}$ is self-paired. By Corollary 1, $\mathcal{F}_{u,n}^K$ contains edges, so it is a self-paired suborbital graph on $[\infty]_K$. □

Next we verify the isomorphism results for the graph $\mathcal{F}_{u,n}^K$. The first one shows that the reflection of $\mathcal{F}_{u,n}^K$ across the imaginary axis is another suborbital graph $\mathcal{F}_{-u,n}^K$. For the second one, let us consider the case of the graph $\mathcal{F}_{u,n}$ first. Suppose that $n$ is a multiple of a positive integer $m$. [8, Lemma 5.3 (ii)] shows that $\mathcal{F}_{u,n}$ is an isomorphic subgraph of $\mathcal{F}_{u,m}$. We know by Remark 1 that $\mathcal{F}_{u,n}^K$ is a subgraph of $\mathcal{F}_{u,n}$. Hence, $\mathcal{F}_{u,n}^K$ becomes an isomorphic subgraph of $\mathcal{F}_{u,m}$. Certainly, the graph $\mathcal{F}_{u,m}$ may not be smallest, so we can find the smaller graph $\mathcal{F}_{u,m}'$ containing $\mathcal{F}_{u,n}^K$ as an isomorphic subgraph.

Let $K' = \{\overline{k}_m : \overline{k}_n \in K\}$. It is not difficult to see that $K'$ is closed under the multiplication modulo $m$, so $K' \leq \mathbb{Z}_m^*$. We use $K'$ to obtain the general version of [8, Lemma 5.3 (ii)]. We are now ready to prove the proposition.
Proposition 4.
1) \( F_{u,n}^K \) is isomorphic to \( F_{-u,n}^K \) by the mapping \( v \mapsto -v \).

2) If \( m \mid n \), then \( F_{u,n}^K \) is an isomorphic subgraph of \( F_{u,m}^{K''} \) by the mapping \( v \mapsto \frac{um}{m} \), where \( K'' \) is a supergroup of \( K' = \{ k_m : \bar{k}_n \in K \} \). In particular, \( F_{u,n}^K \) is an isomorphic subgraph of \( F_{u,m}^{K'} \).

Proof. 1) We can see easily that \( \frac{r}{s} \in [\infty]_K \) if and only if \( -\frac{r}{s} \in [\infty]_K \). Clearly, the mapping is bijective. We need to check that the mapping is edge-preserving so that it is an isomorphism. Let \( \frac{r}{s} \rightarrow \frac{x}{y} \) be an edge in \( F_{u,n}^K \). Then by Proposition 3, \( x \equiv \pm ur \mod n \) and \( ry - sx = \pm n \). This implies that \( -x \equiv \mp (-u)(-r) \mod n \) and \( (-r)y - s(-x) = \mp n \). Therefore, \( \frac{-r}{s} \rightarrow \frac{-x}{y} \) is an edge of \( F_{-u,n}^K \).

2) We will prove only the particular case, the general case will be obtained directly after applying Remark 1 which implies that \( F_{u,m}^{K'} \) is a subgraph of \( F_{u,m}^{K''} \). Let \( m \mid n \), and \( v = \frac{r}{sm} \) be a vertex of \( F_{u,n}^K \), \( s \in \mathbb{Z} \). We then have \( v \rightarrow \frac{nu}{m} = \frac{r}{sm} \). Since \( \gcd(r, sn) = 1 \) and \( m \mid n \), \( \gcd(r, sm) = 1 \). Since \( \bar{r}_n \in -K \cup K \), we have \( \bar{r}_m \in -K' \cup K' \). Thus, \( \frac{r}{sm} \) is a vertex of \( F_{u,n}^{K'} \). The injective property is obvious, so we prove only the edge-preserving property. Suppose that \( \frac{r}{sm} \rightarrow \frac{x}{ym} \) be an edge of \( F_{u,n}^K \). Proposition 3 implies that \( x \equiv \pm ur \mod n \), and \( r(yn) - (sn)x = \pm n \). Since \( m \mid n \), \( x \equiv \pm ur \mod m \). We see that \( ry - sx = \pm 1 \), so \( r(ym) - (sm)x = \pm m \). Therefore, there is an edge \( \frac{r}{sm} \rightarrow \frac{x}{ym} \) in \( F_{u,n}^{K''} \).

Corollary 4. No edges of \( F_{u,n}^K \) cross in \( \mathbb{H}^2 \)

Proof. By using the second result of Proposition 4 with \( m = 1 \), \( F_{u,n}^K \) becomes an isomorphic subgraph of \( F_{1,1} \). Lemma 4 said that there are no edges of \( F_{1,1} \) crossing in \( \mathbb{H}^2 \). Then so does \( F_{u,n}^K \).  

3. Connectivity of graphs

In this section we investigate connectivity of the graph \( F_{u,n}^K \). The goal of this section is to show the following theorem.

Theorem 1. The graph \( F_{u,n}^K \) is connected if and only if \( n \leq 4 \).

To prove this theorem we consider each case of \( n \). Proposition 5 and Proposition 7 will result the conclusion when \( n \leq 4 \) and \( n \geq 5 \), respectively. Now let us consider the graph \( F_{u,n}^K \). We have known from Remark 1 that \( F_{u,n}^K \) is a subgraph of \( F_{u,n} \). The connectivity of \( F_{u,n} \) was already concluded.
in [8, Theorem 5.10]. However, results for the subgraph does not depend on its supergraph in general. Thus, it is worth examining the connectivity of $\mathcal{F}_{u,n}^K$. One can verify that $\mathcal{F}_{u,n}^K = \mathcal{F}_{u,n}$ if and only if $-K \cup K = \mathbb{Z}_n^*$, that is, $\Gamma_K(n) = \Gamma_0(n)$. Then we prove only the case $-K \cup K \subset \mathbb{Z}_n^*$.

The cases $n \leq 4$ or $n = 6$ need not be proved since $\mathcal{F}_{u,n}^K = \mathcal{F}_{u,n}$ for every subgroup $K$ of $\mathbb{Z}_n^*$. For completeness, we conclude them again in the proposition below using the notation $\mathcal{F}_{u,n}^K$, and then prove the remaining cases.

**Proposition 5.** $\mathcal{F}_{u,6}^K$ is not connected, but $\mathcal{F}_{u,n}^K$ is connected for every $n \leq 4$.

**Lemma 5.** Let $\frac{j}{k}$ be a reduced fraction where $k \mid n$. Then there are not adjacent vertices $v$ and $w$ of $\mathcal{F}_{u,n}^K$ such that $v < \frac{j}{k} < w$.

**Proof.** We assume by contrary that $v$ and $w$ are adjacent vertices of $\mathcal{F}_{u,n}^K$. By using Proposition 4 with $m = 1$, the vertices $nv$ and $nw$ are adjacent in $\mathcal{F}_{1,1}$. Then the edge joining these two vertices crosses an edge $\frac{nj}{k} \to \infty$ of $\mathcal{F}_{1,1}$ in $\mathbb{H}^2$ that provides a contradiction to Lemma 4. Thus, $v$ and $w$ cannot be adjacent.

**Lemma 6.** Let $a, b, k \in \mathbb{Z}$, and $b \neq 0 \neq k$. Then $\frac{1+2abk}{4b^2k}$ is a reduced fraction.

**Proof.** Let $p = \gcd(1+2abk, 4b^2k)$ and $q = \gcd(p, 2bk)$. Then $q \mid 1+2abk$ and $q \mid 2bk$. Thus $q \mid 1$, so $q = 1$. Hence, $p = \gcd(p, 4b^2k) = 1$.

**Proposition 6.** If $n \geq 5$, the graph $\mathcal{F}_{u,n}^K$ with $u \equiv \pm 1 \mod n$ is not connected.

**Proof.** The case $n = 6$ is concluded in Proposition 5. Then we suppose that $n \neq 6$. By using Lemma 2 and Lemma 4, we can consider only the case $u = 1$. We see that the block $[\infty]_K$ always contains all fractions $\frac{r}{s}$ with $r \equiv \pm 1 \mod n$ and $s \equiv 0 \mod n$. If the block $[\infty]_K$ contains another fraction $\frac{x}{y}$ with $x \neq \pm 1 \mod n$, by using Proposition 3, $\frac{x}{y}$ is never joined to $\frac{r}{s}$. This provides disconnectedness of the graph. Next we suppose that the block $[\infty]_K$ contains only fractions $\frac{r}{s}$ where $r \equiv \pm 1 \mod n$ and $s \equiv 0 \mod n$. Since $n \geq 5$ and $n \neq 6$, there are at least two proper fractions $\frac{z}{n}$ and $\frac{z'}{n}$ such that $\frac{1}{n} < \frac{z}{n} < \frac{z'}{n} < \frac{n-1}{n}$. We will show that the interval $(\frac{z}{n}, \frac{z'}{n})$ contains some vertices of $\mathcal{F}_{u,n}^K$. Certainly, every vertex of the graph in this interval is not adjacent to $\infty$. By using Lemma 6 with $a = z + z'$ and $b = n$, we obtain that $\frac{1+2(z+z')nk}{4n^2k}$ is a reduced fraction. Obviously, it is contained in $[\infty]_K$ for every $k \in \mathbb{N}$. If we consider this
fraction as an infinite sequence over the index $k$, the sequence converges to the fraction $\frac{z + z'}{2n}$, the middle value of the open interval $(\frac{z}{n}, \frac{z'}{n})$. Thus, the interval contains vertices of $F_{u,n}^K$. We now replace $\frac{k}{n}$ in Lemma 5 by $\frac{z}{n}$ and $\frac{z'}{n}$. Hence, vertices of $F_{u,n}^K$ in the interval $(\frac{z}{n}, \frac{z'}{n})$ is separated from others outside the interval providing disconnectedness of the graph.

Lemma 7. If $u \not\equiv \pm 1 \pmod{n}$, then there are not adjacent vertices $v$ and $w$ of $F_{u,n}^K$ such that $v < \frac{1}{2} < w$.

Proof. The case that $n$ is even follows from Lemma 5. We then suppose that $n$ is odd. Assume that $v$ is adjacent to $w$. Then Lemma 4 implies that $nv$ and $nw$ are adjacent vertices in $F_{1,1}$. By using [8, Lemma 4.1], $nv$ and $nw$ are adjacent term in some $F_m$, the Farey sequence of order $m$. Since $nv < \frac{n}{2} < nw$, we obtain $m = 1$. Then $nv = \frac{n-1}{2}$ and $nw = \frac{n+1}{2}$, so $v = \frac{(n-1)/2}{n}$ and $w = \frac{(n+1)/2}{n}$. If $v \to w$ is an edge in $F_{u,n}^K$, Proposition 3 implies that $\frac{n-1}{2} \equiv -u(n-1)/2 \pmod{n}$. Then $1 \equiv -u(-1) \equiv u \pmod{n}$ which contradicts to the assumption. For the case that $w \to v$ is an edge of $F_{u,n}^K$, we will obtain $u \equiv -1 \pmod{n}$. This also provides a contradiction. Therefore, $v$ and $w$ are not adjacent in $F_{u,n}^K$.

Proposition 7. $F_{u,n}^K$ is not connected for every $n \geq 5$.

Proof. In this proposition we prove the remaining cases. Here, we can assume that $-K \cup K \subset \mathbb{Z}_n^*$ and $u \not\equiv \pm 1 \pmod{n}$. Since $-K \cup K \subset \mathbb{Z}_n^*$, there exists $\frac{t}{n} \in (0,1)$ such that $\frac{t}{n} \not\in [\infty]_K$. By using Proposition 3, one can compute that there are at most two vertices of $F_{u,n}^K$ in the interval $(0,1)$ adjacent to $\infty$. Hence, there is at least one interval $(\frac{r}{s}, \frac{x}{y})$, where $\frac{r}{s}, \frac{x}{y} \in \{0, \frac{1}{2}, \frac{t}{n}, 1\}$, not containing these two vertices. We now put $a = ry + sx, b = sy$, and apply Lemma 6 with the same step used in Proposition 6. We finally obtain at least one vertex of $F_{u,n}^K$ contained in $(\frac{r}{s}, \frac{x}{y})$. Certainly, every vertex of $F_{u,n}^K$ in $(\frac{r}{s}, \frac{x}{y})$ is not adjacent to $\infty$. By applying Lemma 5, some cases may require Lemma 7, the vertices in $(\frac{r}{s}, \frac{x}{y})$ is not adjacent to other vertices this interval. Thus $F_{u,n}^K$ is not connected.

4. Circuits of graphs

This section discusses circuits of the graph $F_{u,n}^K$. A circuit of $F_{u,n}^K$ is a sequence of $m \geq 3$ different vertices $v_1, v_2, \ldots, v_m \in F_{u,n}^K$ such that $v_1 \to v_2 \to \cdots \to v_m \to v_1$ and some arrows may be reversed. If $m = 3$,
we call it a \textit{triangle}. A \textit{directed triangle} is a triangle whose arrows are in the same direction. Otherwise, called an \textit{anti-directed triangle}. The next two statements, Proposition 8 and Remark 2, provide sufficient and necessary conditions for the graph \( \mathcal{F}_{u,n}^K \) to contain triangles.

\textbf{Proposition 8.} \( \mathcal{F}_{u,n}^K \) contains directed triangles if and only if \( \frac{u}{n} \in [\infty]_K \) and \( u^2 \pm u + 1 \equiv 0 \mod n \).

\textit{Proof.} Let \( \mathcal{F}_{u,n}^K \) contains directed triangles. Then so does \( \mathcal{F}_{u,n} \) since \( \mathcal{F}_{u,n}^K \) is a subgraph of \( \mathcal{F}_{u,n} \). By [8, Theorem 5.11] we have \( u^2 \pm u + 1 \equiv 0 \mod n \). Since \( \mathcal{F}_{u,n}^K \) contains edges, Corollary 1 implies that \( \frac{u}{n} \in [\infty]_K \). For the converse implication, we suppose that the conditions hold. Then \( u \in -K \cup K \). Since \( u^2 \pm u + 1 \equiv 0 \mod n \), \( u \pm 1 = \mp u^2 \in -K \cup K \). We now obtain \( \frac{u\pm 1}{n} \in [\infty]_K \). By Proposition 3, one can easily check that the graph \( \mathcal{F}_{u,n}^K \) contains the directed triangle of the form \( \infty \rightarrow \frac{u}{n} \rightarrow \frac{u\pm 1}{n} \rightarrow \infty \). \( \square \)

It is not difficult to see that \( \mathcal{F}_{u,1} = \mathcal{F}_{1,1} \) is a self-paired graph containing directed triangles. Then, it contains anti-directed triangles. \[8, \text{Theorem 5.11 (ii)}\] said that \( \mathcal{F}_{u,n} \) contains no anti-directed triangles if \( n \geq 1 \). Since \( \mathcal{F}_{u,n}^K \) is a subgraph of \( \mathcal{F}_{u,n} \) and they are identical if \( n = 1 \), it is worth to remark that,

\textbf{Remark 2.} \( \mathcal{F}_{u,n}^K \) contains anti-directed triangles if and only if \( n = 1 \).

The next proposition was proved in [1, Theorem 10] for the case of \( \mathcal{F}_{u,n} \) that the graph is a \textit{forest}, a graph contains no circuits, if and only if it contains no triangles. The general case can be proved by using this fact together with Proposition 8.

\textbf{Theorem 2.} \( \mathcal{F}_{u,n}^K \) is a forest if and only if it contains no triangles, i.e., \( \frac{u}{n} \notin [\infty]_K \) or \( u^2 \pm u + 1 \not\equiv 0 \mod n \).

\textit{Proof.} The forward implication is clear by the definition of a forest. For the converse we assume the contrary that \( \mathcal{F}_{u,n}^K \) contains circuits. Then so does \( \mathcal{F}_{u,n} \). By the proof of [1, Theorem 10], \( \mathcal{F}_{u,n} \) contains triangles. Thus, we have \( u^2 \pm u + 1 \equiv 0 \mod n \). Since there is an edge in \( \mathcal{F}_{u,n}^K \), Corollary 1 implies that \( \frac{u}{n} \in [\infty]_K \). By Proposition 8, \( \mathcal{F}_{u,n}^K \) contains triangles. \( \square \)

We know from Theorem 1 that \( \mathcal{F}_{u,n} \) is connected if and only if \( n \leq 4 \). Combine with Theorem 2, we obtain the following corollary.

\textbf{Corollary 5.} \( \mathcal{F}_{u,n}^K \) is a tree if and only if \( n = 2, 4 \).
In Section 2 we have proved that $\Gamma_K(n)$ acts transitively on vertices and edges of $F_{u,n}$, see Corollary 2. This situation also occurs for directed triangles. The proof can be done by using the transitivity of the action of $\Gamma_K(n)$ on edges of $F_{u,n}$.

**Proposition 9.** $\Gamma_K(n)$ acts on directed triangles of $F_{u,n}$ transitively.

**Proof.** By Proposition 8, we see that if $F_{u,n}$ contains triangles, it always contains the triangle $\frac{1}{0} \rightarrow \frac{u}{n} \rightarrow \frac{u+1}{n} \rightarrow \frac{1}{0}$. Suppose that $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_1$ is an arbitrary directed triangle in $F_{u,n}$. It is sufficient to show that there is a transformation $\gamma \in \Gamma_K(n)$ such that $\gamma(\infty \rightarrow \frac{u}{n} \rightarrow \frac{u+1}{n} \rightarrow \infty) = v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_1$. Since $v_1 \rightarrow v_2$ is an edge of the graph $F_{u,n}$, Corollary 2 implies that there is an element $\gamma \in \Gamma_K(n)$ such that $\gamma(\infty \rightarrow \frac{u}{n}) = v_1 \rightarrow v_2$. One can verify that $\gamma$ is unique. Next we prove $\gamma(\frac{u+1}{n}) = v_3$. Since $v_3 \rightarrow v_1$ and $v_2 \rightarrow v_3$ are edges of $F_{u,n}$, we obtain that $\gamma^{-1}(v_3 \rightarrow v_1) = \gamma^{-1}(v_3) \rightarrow \infty$ and $\gamma^{-1}(v_2 \rightarrow v_3) = \frac{u}{n} \rightarrow \gamma^{-1}(v_3)$ are edges of $F_{u,n}$. First, we apply edge conditions, Proposition 3, to the first identity and obtain $\gamma^{-1}(v_3) = \frac{x}{n}$ for some $x \in \mathbb{Z}$. Next we replace $\gamma^{-1}(v_3)$ in the second identity by $\frac{x}{n}$ and apply Proposition 3 again. Then $un - xn = \pm n$, and so $x = u \pm 1$. Thus, $\gamma(\frac{u+1}{n}) = v_3$. The proof is now complete.

In the proof of the previous proposition, the triangle $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_1$ is arbitrary. If we replace $v_1, v_2$ and $v_3$ by $\frac{u}{n}, \frac{u+1}{n}$ and $\infty$, respectively, then there is a unique transformation $\gamma_1 \in \Gamma_K(n)$ rotating the triangle $\infty \rightarrow \frac{u}{n} \rightarrow \frac{u+1}{n} \rightarrow \infty$ in such a way that $\gamma_1(\infty \rightarrow \frac{u}{n} \rightarrow \frac{u+1}{n} \rightarrow \infty) = \frac{u}{n} \rightarrow \frac{u+1}{n} \rightarrow \infty \rightarrow \frac{u}{n}$. One can show easily that

$$
\gamma_1 = \begin{pmatrix}
    u & -(u^2 \pm u + 1)/n \\
    n & -(u \pm 1)
\end{pmatrix}.
$$

Therefore, $\gamma_1$ and $\gamma$ in the proof above induce a unique transformation $\gamma_1 \gamma^{-1}$ rotating another given directed triangle in $F_{u,n}$. We provide the lemma below after concluding this result with more precisely.

**Lemma 8.** Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an elliptic element of the modular group $\Gamma$, that is $|a + d| < 2$. if $|a + d| = 0$, then $\gamma$ has order 2, otherwise, $\gamma$ has order 3.

**Proof.** Suppose that $|a + d| = 0$. Then $\gamma = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, so $-a^2 - bc = 1$. We see that

$$
\gamma^2 = \begin{pmatrix} a^2 + bc & ab - ab \\ ac - ac & a^2 + bc \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
$$
becomes the identity transformation. Hence, \( \gamma \) has order 2.

Next suppose that \(|a + d| = 1\). Then \( d = -(a \pm 1)\). We will prove only the case \( d = -a - 1\). The other case can be proved similarly. Now we have \( \gamma = \left( \begin{array}{cc} a & b \\ c & -a - 1 \end{array} \right) \) and \(-a^2 - a - bc = 1\). Consider

\[
\gamma^2 = \left( \begin{array}{cc} a^2 + bc & ab - ab - b \\ ac - ac - c & a^2 + 2a + 1 + bc \end{array} \right) = \left( \begin{array}{cc} -a - 1 & -b \\ -c & a \end{array} \right).
\]

We see that \( \gamma^2 \) is the inverse transformation of \( \gamma \). Then \( \gamma \) has order 3.

**Remark 3.** Elements of \( \Gamma \) which are conjugate to elliptic elements are elliptic, so \( \gamma_1 \gamma^{-1} \) is elliptic.

**Corollary 6.** There is a unique elliptic element \( \gamma \) of order 3 in \( \Gamma_K(n) \) rotating a triangle \( v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_1 \) of \( \mathcal{F}^K_{u,n} \) in such a way that \( \gamma(v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_1) = v_2 \rightarrow v_3 \rightarrow v_1 \rightarrow v_2 \).

The above corollary and the two consequences below are all about relations between elliptic elements in the group \( \Gamma_K(n) \) and its suborbital graph \( \mathcal{F}^K_{u,n} \). All of them were proved already in [1] for the version of \( \Gamma_0(n) \) and \( \mathcal{F}_{u,n} \). The proofs of the two results below follow from the former.

**Theorem 3.** \( \Gamma_K(n) \) contains an elliptic element of order 3 if and only if there exists \( u \in -K \cup K \) such that \( \mathcal{F}^K_{u,n} \) contains a triangle.

**Proof.** Suppose that

\[
\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)
\]

is an elliptic element of order 3 contained in \( \Gamma_K(n) \). Then Lemma 8 implies that \(|a + d| = 1\). Since \( \gamma \in \Gamma_K(n) \), \( ad \equiv 1 \mod n \). Thus \( a^2 \pm a + 1 \equiv 0 \mod n \). Certainly, \( \bar{a} \in -K \cup K \). If we choose \( u \equiv a \mod n \), Theorem 8 implies that \( \mathcal{F}^K_{u,n} \) contains a triangle. Conversely, suppose that the conditions hold. Again, by using Theorem 8, we obtain \( u^2 \pm u + 1 \equiv 0 \mod n \). Now let \( \gamma \) be the transformation

\[
\gamma_1 = \left( \begin{array}{cc} u & -(u^2 \pm u + 1)/n \\ n & -(u \pm 1) \end{array} \right)
\]

defined before Lemma 8. It is certainly an elliptic element of order 3 in \( \Gamma_K(n) \).

**Theorem 4.** \( \Gamma_K(n) \) contains an elliptic element of order 2 if and only if there exists \( \bar{u} \in K \) such that \( \mathcal{F}^K_{u,n} \) is self-paired.
Proof. Suppose that 
\[ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]
is an elliptic element of order 2 contained in \( \Gamma_K(n) \). Then Lemma 8 implies that \( a + d = 0 \). Since \( \gamma \in \Gamma_K(n) \), we have \( ad \equiv 1 \mod{n} \). Then \( a^2 \equiv -1 \mod{n} \).

Certainly, \( a \in -K \cup K \). If \( a \in K \), we choose \( u \equiv a \mod{n} \). If \( a \in -K \), we choose \( u \equiv -a \mod{n} \). Thus, we have \( u \in K \) and \( u^2 \equiv -1 \mod{n} \).

Now apply Corollary 3, we obtain that \( F_{u,n}^K \) is self-paired. Conversely, suppose that the conditions hold. Again, by using Corollary 3, we obtain \( u^2 \equiv -1 \mod{n} \), that is, \( u^2 + 1 \equiv 0 \mod{n} \). Hence by computation, the transformation
\[
\begin{pmatrix} u & -(u^2 + 1)/n \\ n & -u \end{pmatrix}
\]
belongs to \( \Gamma_K(n) \). Lemma 8 implies that it is an elliptic element of order 2.

\[ \square \]

5. Graphs for conjugate subgroups of \( \Gamma \)

This section is inspired by \([5, 7]\) which studied suborbital graphs for the groups \( \Gamma_0(n) \) and \( \Gamma^0(n) \), respectively. As subgroups of the modular group \( \Gamma \), they are considered to act on \( \hat{\mathbb{Q}} \), and their specific suborbital graphs were determined on their orbits whom they act transitively. We extend the topic to the case of \( \Gamma_K(n) \) and \( \Gamma^K(n) \). The discussion shows that we can study only the suborbital graph \( F_{u,n}^K \) to conclude some general properties of a suborbital graph for \( \Gamma^K(n) \) through a graph isomorphism.

We start with the spacial case of the groups \( \Gamma_0(n) \) and \( \Gamma^0(n) \). The group \( \Gamma^0(n) \) is another congruence subgroup of \( \Gamma \) determined by,
\[
\Gamma^0(n) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma : b \equiv 0 \mod{n} \right\}.
\]
It is conjugate to the group \( \Gamma_0(n) \). More precisely, \( \Gamma^0(n) = \gamma \Gamma_0(n) \gamma^{-1} \) where \( \gamma = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \in \Gamma \). In \([5]\), the authors determined the suborbital graphs of \( \Gamma_0(n) \) on its orbit containing \( \infty \), \( (\Gamma_0(n))(\infty) = \{ \frac{x}{y} \in \hat{\mathbb{Q}} : y \equiv 0 \mod{n} \} \).

They assumed and studied for the case that \( n \) is a prime number \( p \). Suborbital graphs whom they studied are, in fact, the graph \( F_{u,p}^K \) on the block \([\infty]_0 \). Likewise, in \([7]\), the authors studied suborbital graphs for \( \Gamma^0(p) \). In this case the graphs were determined on the orbit of \( 0 \), \( (\Gamma^0(p))(0) = \{ \frac{x}{y} \in \hat{\mathbb{Q}} : x \equiv 0 \mod{p} \} \). We shall roughly denote it by
\( \bar{F}_{p,u} \), the suborbital graph for \( \Gamma^0(p) \) whose edges from the suborbital \( (\Gamma^0(p))(0, \frac{p}{u}) \). What is the relation between the graphs \( F_{u,p} \) and \( \bar{F}_{p,u} \)?

We see that \( (\Gamma^0(p))(0, \frac{p}{u}) = (\gamma_0(n))(\infty, \infty) \). Then \( \bar{F}_{p,u} \) is actually a subgraph of \( G_{u,p} \) on the block \([0]_0 = [\gamma(\infty)]_0 \). It is certainly isomorphic to the graph \( F_{u,p} \), and so, isomorphic to the graph \( F_{u,p} \) after applying Proposition 4. This fact can be directly extended to the case of \( \Gamma_K(n) \) and \( \Gamma^K(n) \) where \( \Gamma^K(n) \) is a congruence subgroup of \( \Gamma \) defined by

\[
\Gamma^K(n) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma : \bar{a} \in -K \cup K, \text{ and } b \equiv 0 \mod n \right\}.
\]

Certainly, \( \Gamma^0(n) = \gamma \Gamma_0(n) \gamma^{-1} \). We let \( \bar{F}^K_{n,u} \) denote the suborbital graph for \( \Gamma^K(n) \) on the orbit \( (\Gamma^K(n))(0) = [0]_K \) where the suborbital \( (\Gamma^K(p))(0, \frac{n}{u}) \) is the set of edges.

**Proposition 10.** \( \bar{F}^K_{u,n}, \bar{F}^K_{-u,n}, \bar{F}^K_{n,u} \) and \( \bar{F}^K_{-n,-u} \) are isomorphic.

Next we discuss the more general suborbital graph for \( \Gamma_K(n) \) on \([\infty]_K \).

We have known that if \( F^K_{u,n} \) contains edges, it is certainly a suborbital graph for \( \Gamma_K(n) \). However, not all suborbital graphs for \( \Gamma_K(n) \) can be represented by some \( F^K_{u,m} \). We need to introduce some notations before clarifying this claim by a trivial example on \( \Gamma_0(2) \).

**Notation.** We denote by \( F^K_n(\infty, v) \) the suborbital graph for \( \Gamma_K(n) \) on \([\infty]_K \) whose edges from the suborbital \( (\Gamma_K(n))(\infty, v) \), and denote by \( F^K_n(0, v) \) the suborbital graph for \( \Gamma^K(n) \) on \([0]_K \) whose edges from the suborbital \( (\Gamma^K(n))(0, v) \). For the case of \( \Gamma_0(n) \) and \( \Gamma^0(n) \) we will leave the letter \( K \) for the notation of graphs and replace \( K \) by \( 0 \) for the notation of blocks.

Let us consider the block \([\infty]_0 \) of the group \( \Gamma_0(2) \). It certainly contains the fraction \( \frac{1}{4} \). We show that \( F^2(\infty, \frac{1}{4}) \) cannot be written as the graph \( F^K_{u,n} \) for some \( \frac{u}{n} \in \mathbb{Q} \), and some \( K \leq \mathbb{Z}_n^* \). If \( F^2(\infty, \frac{1}{4}) = F^K_{u,n} \), then \( \frac{1}{4} \in [\infty]_K \), and so \( n \mid 4 \). Thus \( n = 1, 2, 4 \). It is obvious that \( n \neq 1 \) and \( n \neq 4 \) because they provide vertex sets which are larger and smaller than \([\infty]_0 \), respectively. For the remaining case, it is clear that \( F^2(\infty, \frac{1}{4}) \neq F^K_{1,2} \) since \( F^K_{1,2} \) does not contain the edge \( \infty \rightarrow \frac{1}{4} \). Surely, the same situation occurs on the graphs for \( \Gamma^K(n) \). However, we see that \( F^K_n(\infty, \frac{u}{m}) \) and \( F^K_n(0, -\frac{w}{u}) \) are subgraphs of \( G_{u,m} \) restricted on the blocks \([\infty]_K \) and \([0]_K \), respectively. Then the following result is still true.

**Proposition 11.** \( F^K_n(\infty, \frac{u}{m}) \) and \( F^K_n(0, -\frac{w}{u}) \) are isomorphic.
We have shown that some suborbital graph for $\Gamma_K(n)$ on the block $[\infty]_K$ can not be written as $F^{K'}_{u,m}$. However, the graph is, in fact, the disjoint union of copies of some graph $F^{K'}_{u,m}$. This is the reason why we can study only the graph which is represented by $F^K_{u,n}$ to obtain the results for this general case.

Let us consider the graph $F^K_n(\infty; \frac{u}{m})$. Certainly, $n \mid m$ and $\bar{u}_n \in -K \cup K$. We may assume that $\bar{u}_n \in K$, and define $K' = \langle \bar{u}_n \rangle$, the cyclic subgroup of $\mathbb{Z}_n^*$ generated by $\bar{u}_n$. One can verify easily that the union of all congruence classes in $K'$ is a subset of the union of those congruence classes in $K$. Thus, Proposition 1 implies that $\Gamma_{K'}(m) \leq \Gamma_K(n)$. We now have $\Gamma_K(n) \leq \Gamma_{K'}(m) \leq \Gamma_K(n)$, where $\Gamma_K(n)$ is the stabilizer subgroup of $\Gamma_K(n)$ fixing $\infty$. Similar to the case of $\Gamma$ and its congruence subgroup, this provides the $\Gamma_K(n)$-invariant equivalence relation on the block $[\infty]_K$ related to $\Gamma_{K'}(m)$, and the partition $\{((\gamma \Gamma_{K'}(m))(\infty) : \gamma \in \Gamma_K(n))\}$ on $[\infty]_K$ is formed. We see that the orbit $(\Gamma_{K'}(m))(\infty)$ is, in fact, the block $[\infty]_{K'}$ and the restriction of the graph $F^K(\infty, \frac{u}{m})$ on $[\infty]_{K'}$ is actually the graph $F^K_{u,m}$. Therefore $F(\infty, \frac{u}{m})$ is the disjoint union of $j$ copies of the graph $F^{K'}_{u,m}$ where $j = |\Gamma_{K'}(m) : \Gamma_K(n)|$, the index of $\Gamma_{K'}(m)$ in $\Gamma_K(n)$. Since $F^{K'}_{u,m}$ and $F^{K'}_{u,m}$ are isomorphic, then $F(\infty, \frac{u}{m})$ and $F(\infty, \frac{-u}{m})$ are isomorphic. After applying this result together with the previous proposition, we now have the following consequences immediately.

**Theorem 5.** $F^K(\infty, \frac{u}{m}), F^K(\infty, -\frac{u}{m}), \bar{F}^K(0, \frac{m}{u}), \bar{F}^K(0, -\frac{m}{u})$ are isomorphic.

**Corollary 7.** $F(\infty, \frac{u}{m}), F(\infty, -\frac{u}{m}), \bar{F}(0, \frac{m}{u}), \bar{F}(0, -\frac{m}{u})$ are isomorphic.

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Generalized classes of suborbital graphs


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