On \( n \)-stars in colorings and orientations of graphs

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Abstract. An \( n \)-star \( S \) in a graph \( G \) is the union of geodesic intervals \( I_1, \ldots, I_k \) with common end \( O \) such that the subgraphs \( I_1 \setminus \{O\}, \ldots, I_k \setminus \{O\} \) are pairwise disjoint and \( l(I_1) + \ldots + l(I_k) = n \).

If the edges of \( G \) are oriented, \( S \) is directed if each ray \( I_i \) is directed.

For natural number \( n, r \), we construct a graph \( G \) of \( \text{diam}(G) = n \) such that, for any \( r \)-coloring and orientation of \( E(G) \), there exists a directed \( n \)-star with monochrome rays of pairwise distinct colors.

Let \( G \) be a finite connected graph (with the set of vertices \( V(G) \) and the set of edges \( E(G) \)) endowed with the path metric \( d \) (\( d(u,v) \) is the length of a shortest path between \( u \) and \( v \)). A path \( v_0v_1 \ldots v_n \) is called a geodesic interval if \( d(v_0,v_n) = n \).

For a natural number \( n \), we say that a subgraph \( S \) of \( G \) is an \( n \)-star centered at the vertex \( O \) if \( S \) is the union of geodesic intervals \( I_1, \ldots, I_k \) with common end \( O \) such that the subgraphs \( I_1 \setminus \{O\}, \ldots, I_k \setminus \{O\} \) are pairwise disjoint and \( l(I_1) + \ldots + l(I_k) = n, l(I_i) > 0, i \in \{1, \ldots, k\} \), where \( l(I) \) is the length of \( I \). Each \( I_i \) is called a ray of \( S \). We say that \( S \) is isometrically embedded (in \( G \)) if for any two vertices \( u, v \) of \( S, d(u,v) \) is equal to the distance between \( u \) and \( v \) inside \( S \).

If each edge from \( E(G) \) is oriented, we say that \( S \) is directed if each edge \( v_i v_{i+1} \) in its ray \( v_0 \ldots v_t \) is oriented as \( v_i \rightarrow v_{i+1} \).

We recall that \( \text{diam}(G) \) is the length of a longest geodesic interval in \( G \), and introduce directed and chromatic diameters by

\[
\text{diam}(G) := \text{maximal } p \text{ such that, in each orientation of } E(G), \text{ there is a directed geodesic interval of length } p;
\]

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\( r\)-diam\((G) := \text{maximal } p \text{ such that, in each } r\text{-coloring of } E(G), \text{ there is a monochrome interval of length } p. \)

**Theorem 1.** For any natural numbers \( n, r \), there exists a graph \( G \) of \( \text{diam}(G) = n \) such that, for any \( r\)-coloring and orientation of \( E(G) \), there is a directed isometrically embedded \( n\)-star \( S \) with monochrome rays of pairwise distinct colors. In particular, \( \overrightarrow{\text{diam}}(G) = n \) and \( r\)-diam\((G) \geq n/r. \)

**Proof.** We fix \( r \) and proceed by induction on \( n \). For \( n = 1 \), we take \( G = K_2 \), the complete graph with two vertices.

We assume that a graph \( G \) satisfies the conclusion for some \( n \). We put \( m = r\lfloor |E(G)|/2 \rfloor + 1 \) and consider the Cartesian product \( H = G \times K_m \), \( V(H) = V(G) \times V(K_m) \) and \( (a, c)(b, d) \in E(H) \) if and only if either \( a = b \) or \( c = d \) and \( ab \in E(G) \).

Now we take arbitrary orientation \( \mathcal{O} \) of \( E(H) \) and coloring \( \chi : E(H) \to \{1, \ldots, r\} \). By the choice of \( m \) there are \( c, d \in V(K_m) \) such that the restrictions of \( \chi \) and \( \mathcal{O} \) onto \( G \times \{c\}, G \times \{d\} \) coincide.

By the inductive assumption, there is an \( n\)-star \( S \) in \( G \) centered at \( O \) such that the \( n\)-star \( S \times \{c\} \) is directed and has monochrome rays of pairwise distinct colors. We suppose that the edge \( (O, c)(O, d) \) is directed from \( (O, c) \) to \( (O, d) \) (the opposite case is analogous), look at the color \( i = \chi((O, c)(O, d)) \) and replace the ray of color \( i \) in \( S \times \{c\} \) with the ray \( (O, c)(O, d)I, \) where \( I \) is the ray of color \( i \) in \( S \times \{d\} \). After that, we get the desired \((n + 1)\)-star in \( H \).

By the construction, \( G \) is the Cartesian product of \( n \) complete graphs. Analyzing the proof with vertex-colorings in place of edge-colorings, we get

**Theorem 2.** For any natural numbers \( n, r \), there exists a graph \( G \) of \( \text{diam}(G) = n \) such that, for any \( r\)-coloring of \( V(G) \) and orientation of \( E(G) \), there is a directed monochrome geodesic interval of length \( n - 1 \).

**Comments.** The story began when Taras Banakh transferred me the following question of Krzysztof Pszczoła: can every graph be oriented so that each directed path \( v_0v_1v_2v_3 \) has the shortcut \( \overrightarrow{v_0v_2} \) or \( \overrightarrow{v_1v_3} \). In the case \( r = 1 \), Theorem 1 gives maximally negative answer to this question (perhaps, motivated by comparability graphs ). We mention only three somehow related results from *Ramsey Graph Theory*. For every acyclic directed graph \( G \), there exists \([1]\) a graph \( H \) such that, for every orientation of \( E(H) \), there is an induced copy of \( G \). In the case \( G \) is a tree, see \([3]\) for bounds on \( |V(H)| \) and \( |E(H)| \). For every graph \( G \) and
a natural number $r$, there exists a graph $H$ such that, for every $r$-coloring of $E(H)$, there is a monochrome isometric copy of $G$. This statement can be extracted from Theorem 3.1 in [2].

Applying above Ramsey-isometric fact and Theorem 2, we conclude:

For any natural numbers $m, r$, there exists a graph $H$ such that, for every $r$-coloring of $E(H)$, every $r$-coloring of $V(H)$ and any orientation of $E(H)$, there is a directed, edge-monochrome, vertex-monochrome geodesic interval of length $m$.

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References


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