Quasi-valuation maps based on positive implicative ideals in BCK-algebras

Young Bae Jun, Kyoung Ja Lee and Seok Zun Song

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Abstract. The notion of PI-quasi-valuation maps of a BCK-algebra is introduced, and related properties are investigated. The relationship between an I-quasi-valuation map and a PI-quasi-valuation map is examined. Conditions for an I-quasi-valuation map to be a PI-quasi-valuation map are provided, and conditions for a real-valued function on a BCK-algebra to be a quasi-valuation map based on a positive implicative ideal are founded. The extension property for a PI-quasi-valuation map is established.

1. Introduction

Logic appears in a ‘sacred’ form (resp., a ‘profane’) which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science is twofold; as a tool for applications in both areas, and a technique for laying the foundations. Non-classical logic including many-valued logic, fuzzy logic, etc., takes the advantage of the classical logic to handle information with various facets of uncertainty (see [11] for generalized theory of uncertainty), such as fuzziness, randomness, and so on. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Among all kinds of uncertainties, incomparability is an important one which can
be encountered in our life. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki (see [2–5]) and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Neggers and Kim [10] introduced the notion of $d$-algebras which is another useful generalization of BCK-algebras, and then they investigated several relations between $d$-algebras and BCK-algebras as well as some other interesting relations between $d$-algebras and oriented digraphs. In [9], Neggers et al. discussed the ideal theory in $d$-algebras. Neggers et al. [8] introduced the concept of $d$-fuzzy function which generalizes the concept of fuzzy subalgebra to a much larger class of functions in a natural way. In addition they discussed a method of fuzzification of a wide class of algebraic systems onto $[0, 1]$ along with some consequences. In [6], Jun et al. introduced the notion of quasi-valuation maps based on a subalgebra and an ideal in BCK/BCI-algebras, and then they investigated several properties. They provided relations between a quasi-valuation map based on a subalgebra and a quasi-valuation map based on an ideal. In a BCI-algebra, they gave a condition for a quasi-valuation map based on an ideal to be a quasi-valuation map based on a subalgebra, and found conditions for a real-valued function on a BCK/BCI-algebra to be a quasi-valuation map based on an ideal. Using the notion of a quasi-valuation map based on an ideal, they constructed (pseudo) metric spaces, and showed that the binary operation $*$ in BCK-algebras is uniformly continuous. In this paper, we introduce the notion of PI-quasi-valuation maps of a BCK-algebra, and investigate related properties. We discuss the relationship between an I-quasi-valuation map and a PI-quasi-valuation map. We provide conditions for an I-quasi-valuation map to be a PI-quasi-valuation map, and find conditions for a real-valued function on a BCK-algebra to be a quasi-valuation map based on a positive implicative ideal. We finally establish an extension property for a PI-quasi-valuation map.

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a $BCI$-algebra if it satisfies the following axioms:

(I) $\forall x, y, z \in X \ ((z * y) * (x * z) * (x * y) = 0),$

(II) $\forall x, y \in X \ ((x * (x * y)) * y = 0),$

(III) $\forall x \in X \ (x * x = 0),$

(IV) $\forall x, y \in X \ (x * y = 0, y * x = 0 \Rightarrow x = y).$

If a BCI-algebra $X$ satisfies the following identity:
(V) \((\forall x \in X) \ (0 \ast x = 0)\),

then \(X\) is called a \(BCK\)-algebra. Any \(BCK/BCI\)-algebra \(X\) satisfies the following conditions:

(a1) \((\forall x \in X) \ (x \ast 0 = x)\),

(a2) \((\forall x, y, z \in X) \ (x \ast y = 0 \Rightarrow (x \ast z) \ast (y \ast z) = 0, \ (z \ast y) \ast (z \ast x) = 0)\),

(a3) \((\forall x, y, z \in X) \ ((x \ast y) \ast z = (x \ast z) \ast y)\),

(a4) \((\forall x, y, z \in X) \ (((x \ast z) \ast (y \ast z)) \ast (x \ast y) = 0)\).

We can define a partial ordering \(\leq\) by \(x \leq y\) if and only if \(x \ast y = 0\). A subset \(A\) of a \(BCK/BCI\)-algebra \(X\) is called an ideal of \(X\) if it satisfies the following conditions:

(b1) \(0 \in A\),

(b2) \((\forall x, y \in X) \ (x \ast y \in A, \ y \in A \Rightarrow x \in A)\).

A subset \(A\) of a \(BCK\)-algebra \(X\) is called a positive implicative ideal of \(X\) if it satisfies (b1) and

(b3) \((\forall x, y, z \in X) \ ((x \ast y) \ast z \in A, \ y \ast z \in A \Rightarrow x \ast z \in A)\).

Proposition 2.1. [7] For a subset \(A\) of a \(BCK\)-algebra \(X\), the following are equivalent:

(1) \(A\) is a positive implicative ideal of \(X\).

(2) \(A\) is an ideal, and for any \(x, y \in X\), \((x \ast y) \ast y \in A\) implies \(x \ast y \in A\).

We refer the reader to the books [1,7] for further information regarding \(BCK/BCI\)-algebras.

3. Quasi-valuation maps based on a positive implicative ideal

Definition 3.1 ([6]). Let \(X\) be a \(BCK/BCI\)-algebra. By a quasi-valuation map of \(X\) based on a subalgebra (briefly \(S\)-quasi-valuation map of \(X\)), we mean a mapping \(f : X \to \mathbb{R}\) which satisfies the following condition:

\[
(\forall x, y \in X) \ (f(x \ast y) \geq f(x) + f(y)). \tag{3.1}
\]

Proposition 3.2 ([6]). For any \(S\)-quasi-valuation map \(f\) of a \(BCK\)-algebra \(X\), we have

(c1) \((\forall x \in X) \ (f(x) \leq 0)\).

For any real-valued function \(f\) on a \(BCK/BCI\)-algebra \(X\), we consider the following conditions:

(c2) \(f(0) = 0\).

(c3) \(f(x) \geq f(x \ast y) + f(y)\) for all \(x, y \in X\).
(c4) \( f(x \ast y) \geq f(((x \ast y) \ast y) \ast z) + f(z) \) for all \( x, y, z \in X \).

(c5) \( f(x \ast z) \geq f((x \ast y) \ast z) + f(y \ast z) \) for all \( x, y, z \in X \).

(c6) \( f(x \ast y) \geq f((x \ast y) \ast y) \) for all \( x, y \in X \).

(c7) \( f((x \ast z) \ast (y \ast z)) \geq f((x \ast y) \ast z) \) for all \( x, y, z \in X \).

**Definition 3.3** ([6]). Let \( X \) be a BCK/BCI-algebra. By a **quasi-valuation map** of \( X \) based on an ideal (briefly I-quasi-valuation map of \( X \)), we mean a mapping \( f : X \rightarrow \mathbb{R} \) which satisfies the conditions (c2) and (c3).

**Definition 3.4.** Let \( X \) be a BCK-algebra. By a **quasi-valuation map** on \( X \) based on a positive implicative ideal (briefly PI-quasi-valuation map of \( X \)), we mean a mapping \( f : X \rightarrow \mathbb{R} \) which satisfies the conditions (c2) and (c5).

**Example 3.5.** Let \( X = \{0, a, b\} \) be a BCK-algebra with the \( \ast \)-operation given by Table 1.

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Let \( f \) be a real-valued function on \( X \) defined by

\[
\begin{pmatrix}
0 & a & b \\
0 & 0 & -2
\end{pmatrix}
\]

Then \( f \) is a PI-quasi-valuation map of \( X \).

**Example 3.6.** Let \( X = \{0, a, b, c\} \) be a BCK-algebra with the \( \ast \)-operation given by Table 2.

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</table>
Let $f$ be a real-valued function on $X$ defined by
\[
  f = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & -7 \end{pmatrix}.
\]
Then $f$ is a PI-quasi-valuation map of $X$.

**Theorem 3.7.** Let $X$ be a BCK-algebra. Every PI-quasi-valuation map of $X$ is an I-quasi-valuation map of $X$.

**Proof.** Let $f : X \to \mathbb{R}$ be a PI-quasi-valuation map on a BCK-algebra $X$. If we take $z = 0$ in (c5) and use (a1), then we have the condition (c3). Hence $f$ is an I-quasi-valuation map of $X$. \hfill \Box

The converse of Theorem 3.7 may not be true as shown by the following example.

**Example 3.8.** Let $X = \{0, a, b, c\}$ be a BCK-algebra with the $*$-operation given by Table 2 and let $g$ be a real-valued function on $X$ defined by
\[
  g = \begin{pmatrix} 0 & a & b & c \\ 0 & -2 & -3 & 0 \end{pmatrix}.
\]
Then $g$ is an I-quasi-valuation map of $X$, but not a PI-quasi-valuation map of $X$ since $g(b * a) = -2 < 0 = g((b * a) * a) + g(a * a)$.

**Example 3.9.** Let $X = \{0, a, b, c\}$ be a BCK-algebra with the $*$-operation given by Table 3.

\[
\begin{array}{c|cccc}
  * & 0 & a & b & c \\
\hline
  0 & 0 & 0 & 0 & 0 \\
  a & a & 0 & 0 & 0 \\
  b & b & b & 0 & 0 \\
  c & c & c & b & 0 \\
\end{array}
\]

Let $f$ be a real-valued function on $X$ defined by
\[
  f = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & -3 & -4 \end{pmatrix}.
\]
Then $f$ is an I-quasi-valuation map of $X$, but not a PI-quasi-valuation map of $X$ since $f(c * b) = -3 < 0 = f((c * b) * b) + f(b * b)$. 

We give conditions for an I-quasi-valuation map to be a PI-quasi-valuation map. We first consider the following lemma.

**Lemma 3.10.** [6] For any I-quasi-valuation map $f$ of $X$, we have the following assertions:

1. $f$ is order reversing.
2. $f(x * y) + f(y * x) \leq 0$ for all $x, y \in X$.
3. $f(x * y) \geq f(x * z) + f(z * y)$ for all $x, y, z \in X$.

**Theorem 3.11.** Let $f$ be an I-quasi-valuation map of a BCK-algebra $X$. If $f$ satisfies the condition (c6), then $f$ is a PI-quasi-valuation map of $X$.

**Proof.** Let $f$ be an I-quasi-valuation map of $X$ which satisfies the condition (c6). Notice that $((x * z) * z) * (y * z) \leq (x * z) * y = (x * y) * z$ for all $x, y, z \in X$. Since $f$ is order reversing, it follows that

$$f(((x * z) * z) * (y * z)) \geq f((x * y) * z)$$

so from (c6) and (c3) that

$$f(x * z) \geq f((x * z) * z) \geq f(((x * z) * z) * (y * z)) + f(y * z) \geq f((x * y) * z) + f(y * z).$$

Therefore $f$ is a PI-quasi-valuation map of $X$. □

For any function $f : X \to \mathbb{R}$, consider the following set:

$$I_f := \{ x \in X | f(x) = 0 \}.$$

**Lemma 3.12.** [6] Let $X$ be a BCK-algebra. If $f$ is an I-quasi-valuation map of $X$, then the set $I_f$ is an ideal of $X$.

**Lemma 3.13.** [6] In a BCK-algebra, every I-quasi-valuation map is an $S$-quasi-valuation map.

**Lemma 3.14.** Every PI-quasi-valuation map $f$ of a BCK-algebra $X$ satisfies the condition (c6).

**Proof.** Let $f$ be a PI-quasi-valuation map of $X$. Then $f$ is an I-quasi-valuation map of $X$ by Theorem 3.7. If we take $z = y$ in (c5), then $f(x * y) \geq f((x * y) * y) + f(y * y) = f((x * y) * y) + f(0) = f((x * y) * y)$ for all $x, y \in X$. Thus the condition (c6) is valid. □

**Theorem 3.15.** Let $X$ be a BCK-algebra. If $f$ is a PI-quasi-valuation map of $X$, then the set $I_f$ is a positive implicative ideal of $X$. 

Proof. Suppose $f$ is a PI-quasi-valuation map of $X$. Then $f$ is an I-quasi-valuation map of $X$ by Theorem 3.7, and so $I_f$ is an ideal of $X$ by Lemma 3.12. Let $x, y \in X$ be such that $(x \ast y) \ast y \in I_f$. Then $f((x \ast y) \ast y) = 0$ and so $f(x \ast y) \geq f((x \ast y) \ast y) = 0$ by Lemma 3.14. Using Lemma 3.13 and Proposition 3.2, we get $f(x) \leq 0$ for all $x \in X$. Thus $f(x \ast y) = 0$ which means that $x \ast y \in I_f$. Thus, by Proposition 2.1, we conclude that $I_f$ is a positive implicative ideal of $X$.

The following examples show that the converse of Theorem 3.15 may not be true, that is, there exist a BCK-algebra $X$ and a function $f : X \to \mathbb{R}$ such that

1. $f$ is not a PI-quasi-valuation map of $X$,
2. $I_f$ is a positive implicative ideal of $X$.

Example 3.16. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the $\ast$-operation given by Table 4.

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<th>$\ast$</th>
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Let $g$ be a real-valued function on $X$ defined by

$$g = \begin{pmatrix} 0 & a & b & c & d \\ 0 & 0 & 0 & -8 & 0 \end{pmatrix}.$$

Then $I_g = \{0, a, c\}$ is a positive implicative ideal of $X$. But $g$ is not a PI-quasi-valuation map of $X$ since $g(b \ast c) = g(b) = -8 \not\leq -6 = g((b \ast d) \ast c) + g(d \ast c)$.

Proposition 3.17. Let $X$ be a BCK-algebra. Then every PI-quasi-valuation map $f$ of $X$ satisfies the condition (c7).

Proof. Let $f$ be a PI-quasi-valuation map of $X$. Then $f$ satisfies the condition (c6) (see Lemma 3.14) and $f$ is an I-quasi-valuation map $f$ of $X$ (see Theorem 3.7). It follows from [6, Proposition 3.13] that $f$ satisfies the condition (c7).
Notice that an I-quasi-valuation map $f$ of a BCK-algebra $X$ does not satisfy the condition (c7). In fact, consider a BCK-algebra $X = \{0, a, b, c\}$ in which the $*$-operation is given by the Table 5.

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Let $f$ be a real-valued function on $X$ defined by

$$f = \begin{pmatrix} 0 & a & b & c \\ 0 & -3 & -3 & -8 \end{pmatrix}.$$  

Then $f$ is an I-quasi-valuation map of $X$. Since

$$f((b * a) * (a * a)) = f(a * 0) = f(a) = -3 < 0 = f((b * a) * a),$$

$f$ does not satisfy the condition (c7).

**Theorem 3.18.** Let $X$ be a BCK-algebra. If an I-quasi-valuation map $f$ of $X$ satisfies the condition (c7), then it is a PI-quasi-valuation map of $X$.

**Proof.** Let $f$ be an I-quasi-valuation map of $X$ which satisfies the condition (c7). For any $x, y, z \in X$, we have

$$f(x * z) \geq f((x * z) * (y * z)) + f(y * z) \geq f((x * y) * z) + f(y * z)$$

by (c3) and (c7). Therefore $f$ is a PI-quasi-valuation map of $X$. \qed

**Theorem 3.19.** Let $f$ be a real-valued function on a BCK-algebra $X$. If $f$ satisfies conditions (c2) and (c4), then $f$ is a PI-quasi-valuation map of $X$.

**Proof.** Assume that $f$ satisfies conditions (c2) and (c4). Then

$$f(x) = f(x * 0) \geq f(((x * 0) * 0) * z) + f(z) = f(x * z) + f(z)$$

for all $x, z \in X$. Hence $f$ is an I-quasi-valuation map of $X$. Taking $z = 0$ in (c4) and using (a1) and (c2), we have

$$f(x * y) \geq f(((x * y) * y) * 0) + f(0) = f((x * y) * y)$$
for all $x, y \in X$. It follows from Theorem 3.11 that $f$ is a PI-quasi-valuation map of $X$. 

**Proposition 3.20.** Every PI-quasi-valuation map $f$ of a BCK-algebra $X$ satisfies the following implication for all $x, y, a, b \in X$:

$$(((x \ast y) \ast y) \ast a) \ast b = 0 \Rightarrow f(x \ast y) \geq f(a) + f(b). \quad (3.2)$$

**Proof.** Note that $f$ is an I-quasi-valuation map of $X$ by Theorem 3.7. Assume that $(((x \ast y) \ast y) \ast a) \ast b = 0$ for all $x, y, a, b \in X$. Using [6, Proposition 3.14], we have $f((x \ast y) \ast y) \geq f(a) + f(b)$. It follows from (III), (a1) and (c7) that

$$f(x \ast y) = f((x \ast y) \ast 0) = f((x \ast y) \ast (y \ast y)) \geq f((x \ast y) \ast y) \geq f(a) + f(b).$$

This completes the proof. 

**Lemma 3.21.** [6, Theorem 3.16] If a real-valued function $f$ on $X$ satisfies the conditions (c2) and

$$(\forall x, y, z \in X) \ ( (x \ast y) \ast z = 0 \ \Rightarrow \ f(x) \geq f(y) + f(z)), \quad (3.3)$$

then $f$ is an I-quasi-valuation map of $X$.

**Theorem 3.22.** Let $f$ be a real-valued function on a BCK-algebra $X$. If $f$ satisfies conditions (c2) and (3.2), then $f$ is a PI-quasi-valuation map of $X$.

**Proof.** Let $x, y, z \in X$ be such that $(x \ast y) \ast z = 0$. Then

$$(((x \ast 0) \ast 0) \ast y) \ast z = 0.$$ 

It follows from (a1) and (3.2) that $f(x) = f(x \ast 0) \geq f(y) + f(z)$. Thus $f$ is an I-quasi-valuation map of $X$ by Lemma 3.21. Since

$$(((x \ast y) \ast y) \ast ((x \ast y) \ast y)) \ast 0 = 0$$

for all $x, y \in X$, we have $f(x \ast y) \geq f((x \ast y) \ast y) + f(0) = f((x \ast y) \ast y)$ by (3.2) and (c2). Therefore, by Theorem 3.11, $f$ is a PI-quasi-valuation map of $X$. 

**Proposition 3.23.** Every PI-quasi-valuation map of a BCK-algebra $X$ satisfies the following implication for all $x, y, z, a, b \in X$:

$$(((x \ast y) \ast z) \ast a) \ast b = 0 \Rightarrow f((x \ast z) \ast (y \ast z)) \geq f(a) + f(b). \quad (3.4)$$
Proof. Let \( x, y, z, a, b \in X \) be such that \( (((x \ast y) \ast z) \ast a) \ast b = 0 \). Using Propositions 3.17, Theorem 3.7 and [6, Proposition 3.14], we have

\[
f((x \ast z) \ast (y \ast z)) \geq f((x \ast y) \ast z) \geq f(a) + f(b)
\]

which is the desired result. \( \square \)

**Theorem 3.24.** Let \( X \) be a BCK-algebra. If a real-valued function \( f \) on \( X \) satisfies two conditions \((c2)\) and \((3.4)\), then \( f \) is a PI-quasi-valuation map of \( X \).

**Proof.** Let \( x, y, a, b \in X \) be such that \( (((x \ast y) \ast y) \ast a) \ast b = 0 \). Using \((a1)\), \((III)\) and \((3.4)\), we have

\[
f(x \ast y) = f((x \ast y) \ast 0) = f((x \ast y) \ast (y \ast y)) \geq f(a) + f(b).
\]

It follows from Theorem 3.22 that \( f \) is a PI-quasi-valuation map of \( X \). \( \square \)

**Theorem 3.25.** (Extension Property) Let \( f \) and \( g \) be I-quasi-valuation maps of a BCK-algebra \( X \) such that \( f(x) \geq g(x) \) for all \( x \in X \). If \( g \) is a PI-quasi-valuation map of \( X \), then so is \( f \).

**Proof.** Let \( x, y, z \in X \). Using \((a3)\), Proposition 3.17, \((III)\) and \((c2)\), we have

\[
f(((x \ast z) \ast (y \ast z)) \ast ((x \ast y) \ast z))
= f(((x \ast z) \ast ((x \ast y) \ast z)) \ast (y \ast z))
= f(((x \ast ((x \ast y) \ast z)) \ast z) \ast (y \ast z))
\geq g(((x \ast ((x \ast y) \ast z)) \ast z) \ast (y \ast z))
\geq g(((x \ast ((x \ast y) \ast z)) \ast y) \ast z)
= g(((x \ast y) \ast ((x \ast y) \ast z)) \ast z)
= g(((x \ast y) \ast z) \ast ((x \ast y) \ast z))
= g(0) = 0.
\]

It follows from \((c3)\) that

\[
f((x \ast z) \ast (y \ast z)) \geq f(((x \ast z) \ast (y \ast z)) \ast ((x \ast y) \ast z)) + f((x \ast y) \ast z)
= f((x \ast y) \ast z).
\]

So from Theorem 3.18 we have that \( f \) is a PI-quasi-valuation map of \( X \). \( \square \)
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