Generalization of primal superideals

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Abstract. Let $R$ be a commutative super-ring with unity $1 \neq 0$. A proper superideal of $R$ is a superideal $I$ of $R$ such that $I \neq R$. Let $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be any function, where $\mathcal{I}(R)$ denotes the set of all proper superideals of $R$. A homogeneous element $a \in R$ is $\phi$-prime to $I$ if $ra \in I - \phi(I)$ where $r$ is a homogeneous element in $R$, then $r \in I$. We denote by $\nu_\phi(I)$ the set of all homogeneous elements in $R$ that are not $\phi$-prime to $I$. We define $I$ to be $\phi$-primal if the set

$$
P = \begin{cases} 
[(\nu_\phi(I))_0 + (\nu_\phi(I))_1] + \phi(I) & : \text{if } \phi \neq \phi_\emptyset \\
(\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \text{if } \phi = \phi_\emptyset 
\end{cases}
$$

forms a superideal of $R$. For example if we take $\phi_\emptyset(I) = \emptyset$ (resp. $\phi_0(I) = 0$), a $\phi$-primal superideal is a primal superideal (resp., a weakly primal superideal). In this paper we study several generalizations of primal superideals of $R$ and their properties.

1. Introduction

A supercase on a ring is a $\mathbb{Z}_2$-grading on that ring. In general the grading on a ring, or a module, usually leads computation by allowing one to focus on the homogeneous elements, which are simpler and easier than random elements. However, to do this work you need to know that the constructions being studied are graded. One approach to this issue is to

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redefine the constructions entirely in terms of graded modules and avoid any consideration of non-graded modules or non-homogeneous elements. Unfortunately, while such an approach helps to understand the graded modules, it will only help to understand the original construction, where the graded version of the concept coincide with original one. Therefore, notably, the studying of the graded rings (or modules) is very important.

Because of the importance of the grading, the author made many researches in different subjects in mathematics in super-rings and graded rings few years ago. For example in [1,2,4], the author studied existence of superinvolutions and pseudo superinvolutions of kinds one and two, also in [3,5] he studied Division $\mathbb{Z}_3$-Algebra, and primitive $\mathbb{Z}_3$-algebra with $\mathbb{Z}_3$-involution. Moreover, in [7] he studied $\Delta$-supergraded submodules and in [6] he studied product of graded submodules. Finally, in [8] the author studied weakly primal graded superideals.

A few years ago Y. A. Bahturin and A. Giambruno in [12] studied Group Gradings on associative algebras with involution.

Let $R$ be any ring with unity, then $R$ is called a super-ring if $R$ is a $\mathbb{Z}_2$-graded ring such that if $a,b \in \mathbb{Z}_2$ then $R_a R_b \subseteq R_{a+b}$ where the subscripts are taken modulo 2. Let $h(R) = R_0 \cup R_1$. Then $h(R)$ is the set of homogeneous elements in $R$ and $1 \in R_0$.

Throughout, $R$ will be a commutative super-ring with unity. By a proper superideal of $R$ we mean a superideal $I$ of $R$ such that $I \neq R$. We will denote the set of all proper superideals of $R$ by $\mathcal{I}(R)$. If $I$ and $J$ are in $\mathcal{I}(R)$, then the superideal $\{ r \in R : rJ \subseteq I \}$ is denoted by $(I : J)$. Let $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be any function and let $I \in \mathcal{I}(R)$, we say that $I$ is a $\phi$-prime if whenever $x,y \in h(R)$ with $xy \in I - \phi(I)$, then $x \in I$ or $y \in I$. Since $I - \phi(I) = I - (\phi(I) \cap I)$, there is no loss of generality to assume that $\phi(I) \subseteq I$ for every proper superideal $I$ of $R$.

Given two functions $\psi_1, \psi_2 : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$, we define $\psi_1 \leq \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$ for each $I \in \mathcal{I}(R)$.

Let $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be any function, then an element $a \in h(R)$ is $\phi$-prime to $I$, if whenever $ra \in I - \phi(I)$, where $r \in h(R)$, then $r \in I$. That is $a \in h(R)$ is $\phi$-prime to $I$, if

$$h((I : a)) - h((\phi(I) : a)) \subseteq h(I).$$

Let $\nu_\phi(I)$ be the set of all homogeneous elements in $R$ that are not $\phi$-prime to $I$. We define $I$ to be $\phi$-primal if the set

$$P = \begin{cases} [\nu_\phi(I)_0 + (\nu_\phi(I)_1) \cup \{0\}] + \phi(I) : & \text{if } \phi \neq \phi_\emptyset \\ (\nu_\phi(I)_0 + (\nu_\phi(I)_1) : & \text{if } \phi = \phi_\emptyset \end{cases}$$
forms a superideal in $R$. In this case we say that $I$ is a $\phi$-$P$-primal superideal of $R$, and $P$ is the adjoint superideal of $I$.

In the next example we give some famous functions $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ and their corresponding $\phi$-primal superideals.

**Example 1.1.**

- $\phi_{\emptyset}, \phi_{\emptyset}(I) = \emptyset \forall I \in \mathcal{I}(R)$ — primal superideal.
- $\phi_0, \phi_0(I) = \{0\} \forall I \in \mathcal{I}(R)$ — weakly primal superideal.
- $\phi_2, \phi_2(I) = I^2 \forall I \in \mathcal{I}(R)$ — almost primal superideal.
- $\phi_n, \phi_n(I) = I^n \forall I \in \mathcal{I}(R)$ — $n$-almost primal superideal.
- $\phi_\omega, \phi_\omega(I) = \bigcap_{n=1}^\infty I^n \forall I \in \mathcal{I}(R)$ — $\omega$-primal superideal.

Observe that $\phi_{\emptyset} \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2$.

For the nongraded case one can easily check that if $I$ is a $\phi$-$P$-primal ideal of $R$, with $\phi \neq \phi_{\emptyset}$, then $P = (\nu_\phi(I) \cup \{0\}) + \phi(I)$ if and only if $P = \nu_\phi(I) + \phi(I)$. But if $\phi = \phi_{\emptyset}$ then $P = \nu_\phi(I)$.

Y. Darani in [13] defined that for a commutative ring $R$ with unity and for a function $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ a proper ideal $I$ of $R$ is a $\phi$-$P$-primal ideal of $R$ if $P = \phi(I) \cup \nu_\phi(I)$ is an ideal in $R$, where $\nu_\phi(I)$ is the set of all elements in $R$ that are not $\phi$-prime to $I$.

By comparing the two definitions (in the trivial case and in the supercase), we can see that the definition of $\phi$-primal superideals is a generalization of the definition of the $\phi$-primal ideals to the supercase.

In section 2, we give some examples and properties of $\phi$-primal superideals of $R$. Also, we prove that if $R$ is $\phi$-torsion free super-ring, then every $\phi$-primary superideal of $R$ is $\phi$-primal and hence if $R$ is torsion free super-ring then every weakly primary (i.e., $\phi_0$-primary) superideal of $R$ is weakly primal.

In section 3, we introduce some conditions under which $\phi$-primal superideals are primal.

## 2. $\phi$-Primal superideals

Let $R$ be a commutative super-ring with unity $1 \neq 0 \in R_0$. Let $\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be any function and let $I$ be a proper superideal of $R$. Suppose that $\nu_\phi(I)$ is the set of all homogeneous elements in $R$ that are not $\phi$-prime to $I$, we recall that $I$ is a $\phi$-primal superideal of $R$ if the set

$$P = \begin{cases} 
[(\nu_\phi(I))_0 + (\nu_\phi(I))_1 \cup \{0\}] + \phi(I) & : \text{ if } \phi \neq \phi_{\emptyset} \\
(\nu_\phi(I))_0 + (\nu_\phi(I))_1 & : \text{ if } \phi = \phi_{\emptyset}
\end{cases}$$
forms a superideal in $R$. In this case $P$ is called the adjoint superideal of $I$.

In the next examples we show that the concepts *primal superideals* and *ϕ-primal superideals* are different.

**Example 2.1.** Let $R = \mathbb{Z}_{24} + u\mathbb{Z}_{24}$, where $u^2 = 0$, be a commutative super-ring and assume that $ϕ = ϕ_0$. Let $I = 8\mathbb{Z}_{24} + u\mathbb{Z}_{24}$.

1) Since $0 \notin 2 \cdot 4 \in I$ with $2, 4 \not\in I$, then we get that $2$ and $4$ are not $ϕ$-prime to $I$. Easy computations imply that $2 + 4 = 6$ is $ϕ$-prime to $I$. Thus we obtain that $I$ is not a $ϕ$-primal superideal of $R$.

2) Set $P = 2\mathbb{Z}_{24} + u\mathbb{Z}_{24}$. We show that $I$ is a primal superideal of $R$. It is easy to check that every element of $h(P)$ is not prime to $I$. Conversely, assume that $\bar{a} \in h(R) − h(P)$, then $\bar{a} \in \mathbb{Z}_{24}$ with $gcd(a, 8) = 1$. If $\bar{a} \cdot \bar{n} \in I$ for some $\bar{n} \in \mathbb{Z}_{24}$, then $8$ divides $n$; hence $\bar{n} \in I$. Therefore, $h(P)$ is exactly the set of elements in $h(R)$ which are not prime to $I$. Thus $I$ is a primal superideal of $R$.

**Example 2.2.** Let $ϕ = ϕ_0$, and let $T(R)$ be the collection of all homogeneous zero divisors of $R$. If $R$ is not a superdomain such that $Z(R) = T_0(R) + T_1(R)$ is not a superideal of $R$, then the trivial superideal of $R$ is a $ϕ$-primal superideal which is not primal.

According to Examples 2.1 and 2.2 a primal superideal of $R$ need not to be $ϕ$-primal and a $ϕ$-primal superideal of $R$ need not to be primal.

In the next lemma we show that if $I$ is a $ϕ$-primal superideal in $R$, then $I \subseteq P$. The same result for the non graded case has been proved in [13].

**Lemma 2.3.** Let $I$ be a superideal of $R$, and let $ϕ : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}$ be any function. Suppose that $I$ is $ϕ$-primal superideal of $R$ with the adjoint superideal $P$. Then

1) $I \subseteq P$.

2) $h(P) = h(ϕ(I)) \cup ν_ϕ(I)$.

**Proof.** (1) Let $r$ be any homogeneous element in $I$, if $r \in ϕ(I)$, then $r \in P$. If $r \in h(I) − h(ϕ(I))$, then $1 \cdot r \in I − ϕ(I)$ with $1 \not\in I$, hence $r \in P$. Thus, $I \subseteq P$.

(2) It is trivial that $ν_ϕ(I) \subseteq h(P) − h(ϕ(I))$. For the reverse inclusion, let $x \in h(P) − h(ϕ(I))$ then $x = x_α + y_α$, where $x_α \not\in ν_ϕ(I)$ and $y_α \in (ϕ(I))_α$, for some $α$ in $\mathbb{Z}_2$. Since $x_α \not\in ν_ϕ(I)$, there exists $r \in h(R) − h(I)$ with $rx_α \in I − ϕ(I)$. Thus, $rx = rx_α + ry_α \in I − ϕ(I)$ since $ry_α \in ϕ(I)$. Hence $x \in ν_ϕ(I)$. □
Proposition 2.4. Let $I, P$ be proper superideals of $R$. Then the following statements are equivalent.

(1) $I$ is a $\phi$-primal superideal of $R$ with the adjoint superideal $P$.

(2) For $x \in h(R)$ with $x \notin h(P) - h(\phi(I))$ we have $h((I : x)) = h(I) \cup h((\phi(I) : x))$. If $x \in h(P) - h(\phi(I))$ then $h((I : x)) \supseteq h(I) \cup h((\phi(I) : x))$.

Proof. (1) $\Rightarrow$ (2) If $x \in h(P) - h(\phi(I))$, then $x \notin \nu_\phi(I)$, so there exists $r \in h(R) - h(I)$ with $rx \in I - \phi(I)$. Thus $r \in h((I : x))$ and $r \notin h(I) \cup h((\phi(I) : x))$. Since it is easy to see that $h((I : x)) \supseteq h(I) \cup h((\phi(I) : x))$, we have that $h((I : x)) \supseteq h(I) \cup h((\phi(I) : x))$.

Now let $x \notin h(P) - h(\phi(I))$, where $x \in h(R)$, then $x \notin \nu_\phi(I)$ hence $x$ is $\phi$-prime to $I$. Let $r \in h((I : x))$, if $rx \notin \phi(I)$ then $r \in h(I)$. If $rx \in \phi(I)$ then $r \in h((\phi(I) : x))$. Hence

$h((I : x)) \subseteq h(I) \cup h((\phi(I) : x)) \subseteq h((I : x))$.

(2) $\Rightarrow$ (1) From part (2) we have $h(P) - h(\phi(I)) = \nu_\phi(I)$. Thus $I$ is a $\phi$-primal superideal of $R$.

Theorem 2.5. If $I$ is a $\phi$-primal superideal of $R$, then

$$P = \begin{cases} 
([\nu_\phi(I)]_0 + ([\nu_\phi(I)]_1 \cup \{0\}) + \phi(I) & : \text{if } \phi \neq \phi_\emptyset \\
([\nu_\phi(I)]_0 + ([\nu_\phi(I)]_1 & : \text{if } \phi = \phi_\emptyset 
\end{cases}$$

is a $\phi$-prime superideal of $R$.

Proof. Suppose that $a, b \in h(R) - h(P)$ we show that $ab \in \phi(P)$ or $ab \notin P$. Assume that $ab \notin \phi(P)$, then $ab \notin \phi(I)$, since $\phi(I) \subseteq \phi(P)$. Let $rab \in I - \phi(I)$ for some $r \in h(R)$. Then by Proposition 2.4, we have $ra \in h((I : b)) = h(I) \cup h((\phi(I) : b))$, but $ra \notin (\phi(I) : b)$; hence $ra \in h(I)$. Moreover $ra \notin h(\phi(I))$, for if $ra \in h(\phi(I))$, then $rab \in h(\phi(I))$, which is a contradiction. Therefore, $ra \in h((I : b)) = h(I) \cup h((\phi(I) : b))$. Since $ra \notin \phi(I)$, we have $r \notin h((\phi(I) : a))$, so $r \in h(I)$. Hence $ab$ is $\phi$-prime to $I$ which implies that $ab \notin P$.

Remark 2.6. Let $I$ is a $\phi$-primal superideal of $R$ then by Theorem 2.5,

$$P = \begin{cases} 
([\nu_\phi(I)]_0 + ([\nu_\phi(I)]_1 \cup \{0\}) + \phi(I) & : \text{if } \phi \neq \phi_\emptyset \\
([\nu_\phi(I)]_0 + ([\nu_\phi(I)]_1 & : \text{if } \phi = \phi_\emptyset 
\end{cases}$$

is a $\phi$-prime superideal of $R$. In this case $P$ is called the $\phi$-prime adjoint superideal (simply adjoint superideal) of $I$, and $I$ is called a $\phi$-$P$-primal superideal of $R$. 


The next result shows that every $\phi$-prime superideal of $R$ is $\phi$-primal.

**Theorem 2.7.** Every $\phi$-prime superideal of $R$ is $\phi$-primal.

**Proof.** Let $P$ be a $\phi$-prime superideal of $R$, we show that $P$ is a $\phi$-$P$-primal superideal of $R$. Thus we must prove that

$$P = \begin{cases} ([\nu_\phi(P)]_0 + (\nu_\phi(P))_1 \cup \{0\}) + \phi(P) & \text{if } \phi \neq \phi_\circ \\ (\nu_\phi(P))_0 + (\nu_\phi(P))_1 & \text{if } \phi = \phi_\circ \end{cases}$$

Case 1. Suppose that $P \neq \phi(P)$. We show that $h(P) - h(\phi(P)) = \nu_\phi(P)$. Let $a \in h(P) - h(\phi(P))$. Then $a.1 \in P - \phi(P)$ with $1 \not\in P$, so $a \in \nu_\phi(P)$. On the other hand, if $a \in h(P) - h(\phi(P))$. If $a \in h(\phi(P))$, then $a \in \phi(P)$. If $a \not\in h(\phi(P))$, then $a \not\in P$, so for any $r_\alpha \in R_\alpha$ with $r_\alpha a \in P - \phi(P)$ we have $r_\alpha \in P_\alpha$, since $P$ is $\phi$-prime. Thus $a$ is $\phi$-prime to $P$, hence $a \not\in \nu_\phi(P)$. Therefore, $h(P) - h(\phi(P)) = \nu_\phi(P)$ which implies that $P$ is a $\phi$-$P$-primal superideal of $R$.

Case 2. Suppose that $P = \phi(P)$ then it is easy to check that $\nu_\phi(P) = \emptyset$, hence $P$ is a $\phi$-$P$-primal superideal of $R$. \hfill $\Box$

In the next example we introduce a $\phi$-$P$-primal superideal $I$ of $R$ such that $I$ itself is not $\phi$-prime.

**Example 2.8.** Let $\phi = \phi_0$ and let $R = \mathbb{Z}_8 + u\mathbb{Z}_8$ where $u^2 = 0$. Then $R$ is a commutative super-ring with unity. If $I = 4\mathbb{Z}_8 + u\mathbb{Z}_8$, then $I$ is not a $\phi$-prime superideal of $R$, since $2 \cdot 2 \not\in I$, but $2 \not\in I$. Let $P = 2\mathbb{Z}_8 + u\mathbb{Z}_8$, we show that $I$ is a $\phi$-$P$-primal superideal of $R$. It is enough to show that $\nu(I) = h(P) - \{0\}$. Let $\bar{a} \in h(P)$, if $\bar{a} \in 2\mathbb{Z}_8$ then $\bar{a} = 2k \in \mathbb{Z}_8$. If $k$ is an odd number, then $0 \neq 2\bar{a} \in I$, but $2 \not\in I$, and if $k$ is an even number $0 \neq \bar{a} \in I$ with $\bar{1} \not\in I$; hence $\bar{a} \in \nu(I)$. If $\bar{a} \in u\mathbb{Z}_8$ then $\bar{a} \in I \subseteq \nu(I)$. On the other hand, if $\bar{a} \in h(R) - h(P)$, then $\bar{a}$ is an odd number in $\mathbb{Z}_8$. If $0 \neq \bar{a} \bar{m} \in I$ for some $\bar{m} \in \mathbb{Z}_8$ then $4$ divides $\alpha \bar{m}$ and so, $4$ divides $\bar{m}$ since $(4, a) = 1$; hence $\bar{m} \in I$. Thus $I$ is a $\phi$-$P$-primal superideal of $R$.

Let $\phi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$ be any function. We assume that for any $I, J \in \mathcal{J}(R)$, $\phi(J) \subseteq \phi(I)$ if $J \subseteq I$. We produced in Example 2.2 a $\psi_2$-primal which is not $\psi_1$-primal, where $\psi_1 \leq \psi_2$. In the next theorem we give the condition on $\psi_2$-$P$-primal superideal to be $\psi_1$-$P$-primal.

**Theorem 2.9.** Suppose that $\psi_1 \leq \psi_2$, where $\psi_1$ and $\psi_2$ are maps from $\mathcal{J}(R)$ into $\mathcal{J}(R) \cup \{\emptyset\}$, and let $I$ be a $\psi_2$-$P$-primal superideal of $R$, with $\imath_\alpha I_\alpha \neq \psi_2(I)_\alpha$ for all $\alpha \in \mathbb{Z}_2$. If $P$ is a prime superideal of $R$, then $I$ is $\psi_1$-$P$-primal.
Proof. Since $I$ is a $\psi_2$-$P$-primal superideal of $R$, then

$$P = \begin{cases} 
[(\nu_{\psi_2}(I))_0 + (\nu_{\psi_2}(I))_1 \cup \{0\}] + \psi_2(I) & \text{if } \psi_2 \neq \phi_R \\
(\nu_{\psi_2}(I))_0 + (\nu_{\psi_2}(I))_1 & \text{if } \psi_2 = \phi_R 
\end{cases}$$

To show that $I$ is a $\psi_1$-$P$-primal superideal of $R$ we must prove that

$$P = \begin{cases} 
[(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \cup \{0\}] + \psi_1(I) & \text{if } \psi_1 \neq \phi_R \\
(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 & \text{if } \psi_1 = \phi_R 
\end{cases}$$

If $\psi_2 = \phi_R$, then $\psi_1 = \psi_2$ and hence we have that $P = \nu_{\psi_1}(I)_0 + (\nu_{\psi_1}(I))_1$ which implies that $I$ is a $\psi_1$-$P$-primal superideal of $R$. Now we may assume that $\psi_2 \neq \phi_R$, so we need to prove that

$$P = \begin{cases} 
[(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \cup \{0\}] + \psi_1(I) & \text{if } \psi_1 \neq \phi_R \\
(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 & \text{if } \psi_1 = \phi_R 
\end{cases}$$

Let $a \in \nu_{\psi_2}(I)$, then there exists $r \in h(R) - h(I)$ with $rs \in I - \psi_2(I) \subseteq I - \psi_1(I)$, so $a \in \nu_{\psi_1}(I)$ which implies that

$$(\nu_{\psi_2}(I))_0 + (\nu_{\psi_2}(I))_1 \subseteq (\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1$$

(1)

Now, let $a \in h(\psi_2(I))$, if $a \not\in \psi_1(I)$ then $a \in I - \psi_1(I)$, so $1.a \in I - \psi_1(I)$ with $a \not\in I$, hence $a \in \nu_{\psi_1}(I)$. Therefore,

$$\psi_2(I) \subseteq \begin{cases} 
[(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \cup \{0\}] + \psi_1(I) & \text{if } \psi_1 \neq \phi_R \\
(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 & \text{if } \psi_1 = \phi_R 
\end{cases}$$

(2)

From (1) and (2) we have that

$$P \subseteq \begin{cases} 
[(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \cup \{0\}] + \psi_1(I) & \text{if } \psi_1 \neq \phi_R \\
(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 & \text{if } \psi_1 = \phi_R 
\end{cases}$$

(3)

Since $\psi_1(I) \subseteq \psi_2(I) \subseteq P$, by (3)

$$P = \begin{cases} 
[(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \cup \{0\}] + \psi_1(I) & \text{if } \psi_1 \neq \phi_R \\
(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 & \text{if } \psi_1 = \phi_R 
\end{cases}$$

if $\nu_{\psi_1}(I) \subseteq P$.

Let $a \in (\nu_{\psi_1}(I))_\alpha$. Then there exists $r_\beta \in R_\beta - I_\beta$ with $ar_\beta \in I - \psi_1(I)$, if $ar_\beta \in I - \psi_2(I)$, then $a \in \nu_{\psi_2}(I) \subseteq P$. So we may assume that $ar_\beta \not\in$
\(I - \psi_2(I)\), hence \(ar_\beta \in \psi_2(I)\). First suppose that \(aI_\beta \not\subset (\psi_2(I))_{\alpha\beta}\), say \(as_\beta \in I_{\alpha\beta} - (\psi_2(I))_{\alpha\beta}\) with \(s_\beta \in I_\beta\). Then \(a(r_\beta + s_\beta) = ar_\beta + as_\beta \not\in \psi_2(I)\) with \(r_\beta + s_\beta \in R_\beta - I_\beta\), hence \(a \in \nu_{\psi_2}(I) \subset P\). Therefore, we may assume that \(aI_\beta \subseteq (\psi_2(I))_{\alpha\beta}\).

Now suppose that \(r_\beta I_0 \not\subset (\psi_2(I))_{\beta}\), then there exists \(c \in I_0\) with \(r_\beta c \in I_\beta - (\psi_2(I))_{\beta}\). Since \(a^2 \in R_0\), we have that \((a^2 + c)r_\beta \in I_\beta - (\psi_2(I))_{\beta}\) with \(r_\beta \not\in I_\beta\), so \(a^2 + c \in P_0\), but \(c \in I_0 \subseteq P_0\), therefore \(a^2 \in P\) and hence \(a \in P\), since \(P\) is a prime superideal. So we may assume that \(r_\beta I_0 \subseteq (\psi_2(I))_{\beta}\). Since \((I_0I_\beta) \neq (\psi_2(I))_{\beta}\) there exists \(\overline{a} \in I_0\) and \(\overline{b} \in I_\beta\) with \(\overline{a}\overline{b} \not\in (\psi_2(I))_{\beta}\). Thus, \((a^2 + \overline{a})(r_\beta + \overline{b}) = a^2r_\beta + a^2\overline{b} + \overline{a}r_\beta + \overline{a}\overline{b} \not\in \psi_2(I)\), so \((a^2 + \overline{a})(r_\beta + \overline{b}) \in I - \psi_2(I)\) with \(r_\beta + \overline{b} \in R_\beta - I_\beta\) which implies that \(a^2 + \overline{a} \in (\nu_{\psi_2}(I))_0 \subset P_0\), hence \(a^2 \in P_0 \subset P\) and then \(a \in P\), since \(P\) is a prime superideal of \(R\). Therefore, \(\nu_{\psi_1}(I) \subset P\), so

\[
P = \begin{cases} 
[(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 \cup \{0\}] + \psi_1(I) & : \text{if } \psi_1 \neq \phi_{\emptyset} \\
(\nu_{\psi_1}(I))_0 + (\nu_{\psi_1}(I))_1 & : \text{if } \psi_1 = \phi_{\emptyset}
\end{cases}
\]

and hence \(I\) is a \(\psi_1\)-P-primal superideal of \(R\). \(\square\)

We end the section by proving the following results about the relationship between \(\phi\)-primary and \(\phi\)-primal superideals. For more properties about primary and primal superideals see [8, section 4].

**Definition 2.10.** Let \(\phi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}\) be any function such that \(\phi \neq \phi_{\emptyset}\), then \(R\) is a \(\phi\)-torsion free if \(ab \in \phi(P)\) where \(P \in \mathcal{J}(R)\), then \(a \in \phi(P)\) or \(b \in \phi(P)\).

For example if \(\phi = \phi_0\), then a \(\phi\)-torsion free super-ring is just a torsion free super-ring.

**Theorem 2.11.** Let \(\phi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}\) be any function, where \(\phi \neq \phi_{\emptyset}\), and let \(R\) be a \(\phi\)-torsion free. Then every \(\phi\)-primary superideal of \(R\) is \(\phi\)-primal.

**Proof.** Let \(I\) be a \(\phi\)-primary superideal of \(R\). We show that

\[
\sqrt{I} = [(\nu_{\phi}(I))_0 + (\nu_{\phi}(I))_1 \cup \{0\}] + \phi(I).
\]

\((\supseteq)\) Let \(r \in \nu_{\phi}(I)\), then there exists \(a \in h(R) - h(I)\) with \(ra \in I - \phi(I)\) which implies that \(r \in \sqrt{I}\), since \(I\) is \(\phi\)-primary. Moreover, \(\phi(I) \subset I \subset \sqrt{I}\).

\((\subseteq)\) Let \(b \in h(\sqrt{I})\). If \(b \in \phi(I)\), then done. So, we may assume that \(b \not\in \phi(I)\). Let \(n\) be the smallest positive integer such that \(b^n \in I\). Suppose
\[ n = 1. \text{ If } b \in \phi(I), \text{ then done. If } b \not\in \phi(I), \text{ then } 1.b \in I - \phi(I) \text{ and } 1 \not\in I \text{ so } b \in \nu_{\phi}(I). \text{ Therefore we may assume that } n > 1. \text{ If } b^n \in \phi(I), \text{ then } b^n = b^{n-1}b \in \phi(I) \text{ and } b^{n-1} \not\in \phi(I), \text{ since } b^{n-1} \not\in I \text{ and } \phi(I) \subseteq I, \text{ which is a contradiction since } R \text{ is } \phi\text{-torsion free. So, } b^n = b^{n-1}b \in I - \phi(I) \text{ and } b^{n-1} \not\in I, \text{ hence } b \in \nu_{\phi}(I). \]

**Corollary 2.12.** If \( R \) is a torsion free, then every weakly primary superideal of \( R \) is weakly primal.

\[ \square \]

3. **Conditions on \( \phi \)-primal superideals**

In this section, we introduce some conditions under which \( \phi \)-primal superideals are primal.

Let \( \phi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{ \emptyset \} \) be any function. We have to remind that if \( I \) is a \( \phi \)-\( P \)-primal superideal of \( R \), then

\[
P = \begin{cases} 
[(\nu_{\phi}(I))_0 + (\nu_{\phi}(I))_1 \cup \{0\}] + \phi(I) & : \text{ if } \phi \neq \phi_{\emptyset} \\
(\nu_{\phi}(I))_0 + (\nu_{\phi}(I))_1 & : \text{ if } \phi = \phi_{\emptyset}
\end{cases}
\]

is a \( \phi \)-prime superideal of \( R \).

**Definition 3.1.** Let \( r \) be a homogeneous element in \( R \), then \( |r| = \alpha \) if \( r \in R_{\alpha} \) for some \( \alpha \in \mathbb{Z}_2 \).

In the next theorem we provide some conditions under which a \( \phi \)-primal superideal is primal.

**Theorem 3.2.** Let \( R \) be a commutative super-ring with unity and let \( \phi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{ \emptyset \} \) be any function. Suppose that \( I \) is a \( \phi \)-\( P \)-primal superideal of \( R \) with \( I_{\gamma}I_{\delta} \not\subseteq \phi(I) \) for each \( \gamma, \delta \in \mathbb{Z}_2 \). If \( P \) is a prime superideal of \( R \), then \( I \) is \( P \)-primal.

**Proof.** Assume that \( a \) is a homogeneous element in \( P \). Then \( a \in \phi(I) \) or \( a \in (\nu_{\phi}(I))_\alpha \) for some \( \alpha \in \mathbb{Z}_2 \) or \( a = b_\beta + c_\beta \) where \( b_\beta \in (\nu_{\phi}(I))_\beta \) and \( c_\beta \in \phi(I) \) for some \( \beta \in \mathbb{Z}_2 \). If the first two cases hold, then \( a \) is not prime to \( I \), since it is not \( \phi \)-prime to \( I \). In the last case, let \( d \) be a homogeneous element in \( R \) such that \( d \not\in I \) with \( b_\beta d \in I - \phi(I) \). Then \( ad = b_\beta d + c_\beta d \in I - \phi(I) \), because \( ad \in \phi(I) \) implies that \( b_\beta d \in \phi(I) \), since \( c_\beta d \in \phi(I) \) which is a contradiction. Thus \( a \) is not \( \phi \)-prime to \( I \) and hence \( a \) is not prime to \( I \). Now assume that \( b \in h(R) \) is not prime to \( I \), so \( rb \in I \) for some homogeneous element \( r \in R - I \). If \( rb \not\in \phi(I) \), then \( b \) is not \( \phi \)-prime to \( I \), so \( b \in P \). Thus assume that \( rb \in \phi(I) \). Suppose that
\[|r| = \alpha.\] First suppose that \(bI_\alpha \not\subseteq \phi(I)\). Then, there exists \(r' \in I_\alpha\) such that \(br' \not\subseteq \phi(I)\). So \(b(r + r') \in I - \phi(I)\), where \(r + r'\) is a homogeneous element in \(R - I\), implies that \(b\) is not \(\phi\)-prime to \(I\), that is \(b \in P\). Therefore, we may assume that \(bI_\alpha \subseteq \phi(I)\). Let \(|b| = \beta\). If \(rI_\beta \not\subseteq \phi(I)\), then \(rc \not\subseteq \phi(I)\) for some \(c \in I_\beta\). In this case \(r(b + c) \in I - \phi(I)\) with \(r \in R - I\), that is \(b + c \in P\) and hence \(b \in P\), since \(c \in I \subseteq P\). So we may assume that \(rI_\beta \subseteq \phi(I)\). Since \(I_\alpha I_\beta \not\subseteq \phi(I)\), there are \(b' \in I_\alpha\) and \(a' \in I_\beta\) with \(b'a' \not\subseteq \phi(I)\). Then \((b + a')(r + b') \in I - \phi(I)\), where \(r + b'\) is a homogeneous element in \(R - I\), implies that \(b + a'\) is a homogeneous element in \(P\). On the other hand \(a' \in I \subseteq P\), so that \(b \in P\). We have already shown that \(P\) is exactly the set of all elements of \(R\) that are not prime to \(I\). Hence \(I\) is \(P\)-primal.

Let \(R\) and \(S\) be commutative super-rings. It is easy to prove that the prime superideals of \(R \times S\) have the forms \(P \times S\) or \(R \times Q\) where \(P\) is a prime superideal of \(R\) and \(Q\) is a prime superideal of \(S\). Also we have the following two propositions about primal superideals of \(R \times S\). We leave the easy proof for the next two results to the reader. For the trivial case (i.e., \(R_1 = \{0\}\)) they have proved in [10, Lemma 13] and [9, Theorem 16].

**Proposition 3.3.** Let \(R\) and \(S\) be commutative super-rings. If \(P\) is a primal superideal of \(R\) and \(Q\) is a primal superideal of \(S\), then \(P \times S\) and \(R \times Q\) are primal superideals of \(R \times S\).

**Proposition 3.4.** Let \(R_1\) and \(R_2\) be commutative super-rings with unities and let \(\psi_1 : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}\) be functions. Let \(\phi = \psi_1 \times \psi_2\). Then \(\phi\)-primes of \(R_1 \times R_2\) have exactly one of the following three types:

1. \(I_1 \times I_2\) where \(I_i\) is a proper superideal of \(R_i\) with \(\psi_1(I_i) = I_i\);
2. \(I_1 \times R_2\) where \(I_1\) is a \(\psi_1\)-prime of \(R_1\) which must be prime if \(\psi_2(R_2) \neq R_2\);
3. \(R_1 \times I_2\) where \(I_2\) is a \(\psi_2\)-prime of \(R_2\) which must be prime if \(\psi_1(R_1) \neq R_1\).

Now let \(R_1\), \(R_2\) be commutative super-rings with unities and let \(R = R_1 \times R_2\). Let \(\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}\) be a function. In the next theorem, we provide some conditions under which a \(\phi\)-primal superideal of \(R\) is primal, but first we start with the following remark.

**Remark 3.5.** Let \(I\) be a proper superideal of a commutative super-ring \(R\) and let \(\phi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{\emptyset\}\) be a function. If a homogeneous element \(a\) is not \(\phi\)-prime to \(I\), then there is a homogeneous element \(r\) in \(R - I\) such that \(ar \in I - \phi(I) \subseteq I\) so \(a\) is not prime to \(I\).
Theorem 3.6. Let $R_1, R_2$ be commutative super-rings with unities and let $R = R_1 \times R_2$. Let $\psi_i : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be functions with $\psi_i(R_i) \neq R_i$ for $i = 1, 2$. Let $\phi = \psi_1 \times \psi_2$. Assume that $P$ is a superideal of $R$ with $\phi(P) \neq P$. If $I$ is a $\phi$-$P$-primal superideal of $R$, then either $I = \phi(I)$ or $I$ is primal.

Proof. Suppose $\phi(I) \neq I$. By Theorem 2.5, $P$ is a $\phi$-prime superideal of $R$. Therefore, by Proposition 3.4, $P$ has one of the following three cases.

Case 1. $P = P_1 \times P_2$ where $P_i$ is a proper superideal of $R_i$ with $\psi_i(P_i) = P_i$ for $i = 1, 2$. In this case $\phi(P) = P$, a contradiction.

Case 2. $P = P_1 \times P_2$ where $P_i$ is a $\psi_1$-prime superideal of $R_1$. Since $\psi_2(R_2) \neq R_2$, by Proposition 3.4(2), $P_1$ is a prime superideal of $R_1$ and so $P$ is a prime superideal of $R$.

We will show that $I_2 = R_2$. Since $I \neq \phi(I)$, there exists a homogeneous element $(a, b)$ in $I - \phi(I)$. So $(a, 1)(1, b) = (a, b) \in I - \phi(I)$. If $(a, 1) \notin I$, then $(1, b)$ is not $\phi$-prime to $I$, hence $(1, b) \in P = P_1 \times P_2$, so $1 \in P_1$ a contradiction. Thus $(a, 1) \in I = I_1 \times I_2$ i.e., $1 \in I_2$ that is $I_2 = R_2$.

Now we prove that $I_1$ is a $P_1$-primal superideal of $R_1$. Let $a_1$ be a homogeneous element in $P_1$. Then $(a_1, 0) \in P_1 \times R_2 = P$. If $(a_1, 0) \in \phi(I) = \psi_1(I_1) \times \psi_2(R_2)$, then $a_1 \in \psi_1(I_1) \subseteq I_1$ so $a_1$ is not prime to $I_1$. Therefore, we may assume that $(a_1, 0) \in \nu_\phi(I)$. In this case there exists a homogeneous element $(r_1, r_2) \in R - I$ such that $(a_1, 0)(r_1, r_2) \in I - \phi(I)$ so $a_1r_1 \in I_1 - \psi_1(I_1)$ with $r_1 \in R_1 - I_1$, since $R - I = (R_1 - I_1) \times R_2$, implies that $a_1$ is not $\psi_1$-prime to $I_1$, hence by Remark 3.5, $a_1$ is not prime to $I_1$. Conversely, let $b_1$ be a homogeneous element in $R_1$ such that $b_1$ is not prime to $I_1$. Then there exists a homogeneous element $c_1$ in $R_1 - I_1$ with $b_1c_1 \in I_1$. Since $\psi_2(R_2) \neq R_2$, $(b_1, 1)(c_1, 1) = (b_1c_1, 1) \in I_1 \times R_2 - (I_1 \times \psi_2(R_2)) \subseteq I - \phi(I)$ with $(c_1, 1) \in R - I$. Hence $(b_1, 1)$ is not $\phi$-prime to $I$ which implies that $(b_1, 1) \in P = P_1 \times R_2$ and so $b_1 \in P_1$.

We have already shown that the set of homogeneous elements in $P_1$ consists exactly of the homogeneous elements of $R_1$ that are not prime to $I_1$. Hence $I_1$ is $P_1$-primal superideal of $R_1$ so by Proposition 3.3, $I$ is a $P$-primal superideal of $R$.

Case 3. $P = R_1 \times P_2$ where $P_2$ is a $\psi_2$-primal superideal of $R_2$. The proof of case(3) is similar to that of case(2). 

References


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