On dual Rickart modules
and weak dual Rickart modules

Derya Keskin Tütüncü, Nil Orhan Ertaş
and Rachid Tribak

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Abstract. Let $R$ be a ring. A right $R$-module $M$ is called d-Rickart if for every endomorphism $\varphi$ of $M$, $\varphi(M)$ is a direct summand of $M$ and it is called wd-Rickart if for every nonzero endomorphism $\varphi$ of $M$, $\varphi(M)$ contains a nonzero direct summand of $M$. We begin with some basic properties of (w)d-Rickart modules. Then we study direct sums of (w)d-Rickart modules and the class of rings for which every finitely generated module is (w)d-Rickart. We conclude by some structure results.

1. Introduction

In [10], Lee, Rizvi and Roman introduced and studied a notion called d-Rickart modules. A module $M$ is said to be d-Rickart (or dual Rickart) if for every $\varphi \in \text{End}_R(M)$, $\text{Im} \varphi$ is a direct summand of $M$. Actually, this notion is dual to the notion of Rickart modules introduced by Lee, Rizvi and Roman in [9]. A module $M$ is called a Rickart module if for every endomorphism $\varphi$ of $M$, $\text{Ker} \varphi$ is a direct summand of $M$. Later in [13], Tribak introduced and investigated the notion called wd-Rickart modules, which is a generalization of the concept of d-Rickart modules. A module $M$ is said to be wd-Rickart (or weak dual Rickart) if for every...
nonzero endomorphism $\varphi$ of $M$, $\text{Im} \varphi$ contains a nonzero direct summand of $M$. Let $M$ and $N$ be two modules. Then $M$ is called $N$-wd-Rickart if for every nonzero homomorphism $\varphi : M \to N$, $\text{Im} \varphi$ contains a nonzero direct summand of $N$.

In Section 2, we investigate some basic properties of (w)d-Rickart modules.

In Section 3, we study direct sums of (w)d-Rickart modules. We provide a characterization for a direct sum of two d-Rickart modules to be d-Rickart. We also show that if $M_1, \ldots, M_n$ are modules such that $M_i$ is $M_j$-projective for all $j > i$ in $\{1, \ldots, n\}$. Then $\bigoplus_{i=1}^n M_i$ is a wd-Rickart module if and only if $M_i$ is $M_j$-wd-Rickart for all $i, j \in \{1, \ldots, n\}$.

Section 4 is devoted to the study of the class of rings over which finitely generated modules are (w)d-Rickart. Among other results, the class of commutative rings $R$ for which every finitely generated $R$-module is d-Rickart is shown to be precisely that of semisimple rings.

We conclude this paper by a short section in which we present some structure results.

Throughout this paper, $R$ is an associative ring with identity and all the modules are unital right $R$-modules. Let $M$ be a module. The notation $N \leq M$ means that $N$ is a submodule of $M$. By $\text{Soc}(M)$ and $\text{End}_R(M)$, we denote the socle of $M$ and the endomorphism ring of $M$, respectively. By $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ we denote the set of rational, integer and natural numbers, respectively.

2. Some properties of d-Rickart modules and wd-Rickart modules

Let $M$ and $N$ be two modules. Following [10, Definition 2.14], the module $M$ is called $N$-d-Rickart (or relatively d-Rickart to $N$) if for every homomorphism $\varphi : M \to N$, $\text{Im} \varphi$ is a direct summand of $N$. Therefore $M$ is a d-Rickart module if and only if $M$ is $M$-d-Rickart.

Recall that a module $M$ is called a $(C_3)$-module if whenever $A$ and $B$ are direct summands of $M$ with $A \cap B = 0$, then $A \oplus B$ is a direct summand of $M$. Note that every injective module is a $(C_3)$-module.

**Example 2.1.** Let $M_1$ be a semisimple module and let $M_2$ be a module such that the module $M = M_1 \oplus M_2$ is a $(C_3)$-module. Then $M_1$ and $M_2$ are relatively d-Rickart to each other by [2, Proposition 2.3].

If $M$ is a d-Rickart (wd-Rickart) module, then a factor module of $M$ may not be d-Rickart (wd-Rickart) as we see in the following example.
Example 2.2. Let $R$ be a von Neumann regular ring which is not a right $V$-ring (see [8, Example 3.74A]). By [10, Remark 2.2], $R_R$ is a d-Rickart module. Then by [10, Proposition 2.25], every finitely generated free $R$-module is a d-Rickart module. Since $R$ is not a right $V$-ring, there exists a finitely generated $R$-module $M$ such that $M$ is not a wd-Rickart module (Proposition 4.1). It is well known that every finitely generated $R$-module is a homomorphic image of a finitely generated free $R$-module. Therefore there exists a positive integer $n$ such that $M \cong R^{(n)}/K$ for some submodule $K$ of $R^{(n)}$. Hence $R^{(n)}/K$ is not a wd-Rickart (so $R^{(n)}/K$ is not a d-Rickart) module while $R^{(n)}$ is a d-Rickart module.

The following proposition provides a sufficient condition under which some factor modules of a d-Rickart module are d-Rickart.

**Proposition 2.3.** Let $M$ be a d-Rickart module and let $N$ be a fully invariant submodule of $M$. If every endomorphism of $M/N$ can be lifted to an endomorphism of $M$, then $M/N$ is also a d-Rickart module.

**Proof.** Let $\varphi$ be a nonzero endomorphism of $M/N$. By assumption, there exists an endomorphism $\psi$ of $M$ such that $\pi \psi = \varphi \pi$, where $\pi : M \to M/N$ is the canonical projection. It is clear that $\psi \neq 0$. As $M$ is d-Rickart, $\text{Im} \, \psi$ is a direct summand of $M$. Note that $\text{Im} \, \varphi = \varphi \pi (M) = \pi \psi (M) = (\psi (M) + N)/N$. Since $N$ is fully invariant in $M$, $\text{Im} \, \varphi$ is a direct summand of $M/N$. \qed

**Corollary 2.4.** Let $M$ be a quasi-projective d-Rickart module. If $N$ is a fully invariant submodule of $M$, then $M/N$ is a d-Rickart module.

**Proof.** By Proposition 2.3. \qed

Next, we investigate connections between a wd-Rickart module and its endomorphism ring.

A ring $R$ is called left $w$-Rickart if for every nonzero element $x \in R$, $l_R(x) = \{ r \in R \mid rx = 0 \}$ is contained in a proper direct summand of the left $R$-module $_RR$.

**Proposition 2.5.** If $M$ is a wd-Rickart module, then $S = \text{End}_R(M)$ is a left $w$-Rickart ring.

**Proof.** Let $\varphi$ be a nonzero endomorphism of $M$. Since $M$ is wd-Rickart, there exists a nonzero idempotent $e \in S$ with $e(M) \subseteq \varphi(M)$. Then clearly $l_S(\varphi) \subseteq S(1-e)$ and $S(1-e) \neq S$. This proves the proposition. \qed

The following example shows that the converse of the above proposition is not true, in general.
Example 2.6. The \( \mathbb{Z} \)-module \( \mathbb{Z} \) is not wd-Rickart, but \( \text{End}_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z} \) is a left w-Rickart ring.

Corollary 2.7. If \( R \) is a right wd-Rickart ring, then \( eRe \) is a left w-Rickart ring for any idempotent \( e \) in \( R \).

**Proof.** This follows from [13, Corollary 2.5] and Proposition 2.5. \( \square \)

Let \( M \) be an \( R \)-module and let \( S = \text{End}_R(M) \). We denote \( r_M(I) = \{ m \in M \mid Im = 0 \} \) for \( \emptyset \neq I \subseteq S \) and \( l_S(N) = \{ \varphi \in S \mid \varphi(N) = 0 \} \) for a submodule \( N \) of \( M \). In [1, Corollary 4.2], it is presented some examples of submodules \( K \) of a module \( M \) for which \( r_M(l_S(K)) = K \). Moreover, it is shown in [10, Corollary 3.7] that a module \( M \) is a d-Rickart module if and only if \( r_Ml_S(\varphi(M)) = \varphi(M) \) and \( r_Ml_S(\varphi(M)) \) is a direct summand of \( M \) for all \( \varphi \in S = \text{End}_R(M) \).

It is natural to ask when the converse of Proposition 2.5 holds. In this vein we give the next theorem. But first we need the following lemma.

**Lemma 2.8.** Let \( M \) be a module with \( S = \text{End}_R(M) \). Then \( S \) is a left w-Rickart ring if and only if \( r_Ml_S(\varphi(M)) \) contains a nonzero direct summand of \( M \) for all nonzero endomorphisms \( \varphi \) of \( M \).

**Proof.** \((\Rightarrow)\) Let \( \varphi : M \to M \) be a nonzero endomorphism of \( M \). Since \( S \) is left w-Rickart, there exists an idempotent \( f \) of \( S \) such that \( l_S(\varphi) \subseteq Sf \) and \( Sf \neq S \). Then \( r_M(Sf) \subseteq r_Ml_S(\varphi(M)) \). This implies that the nonzero direct summand \( (1-f)(M) \) of \( M \) is contained in \( r_Ml_S(\varphi(M)) \).

\((\Leftarrow)\) Let \( 0 \neq \varphi \in S \). By hypothesis, there exists \( 0 \neq e = e^2 \in S \) such that \( e(M) \subseteq r_Ml_S(\varphi(M)) \). Thus \( l_Sr_Ml_S(\varphi(M)) \subseteq l_S(e(M)) \). Hence \( l_S(\varphi(M)) \subseteq l_S(e(M)) \). So \( l_S(\varphi) \subseteq l_S(e) = S(1-e) \neq R \). This completes the proof. \( \square \)

**Theorem 2.9.** Let \( M \) be a module with the property that \( r_Ml_S(\varphi(M)) = \varphi(M) \) for every nonzero endomorphism \( \varphi \) of \( M \). Then \( M \) is a wd-Rickart module if and only if \( S = \text{End}_R(M) \) is a left w-Rickart ring.

**Proof.** \((\Rightarrow)\) By Proposition 2.5.

\((\Leftarrow)\) This follows from Lemma 2.8. \( \square \)

Recall that a module \( M \) is called *retractable* if for every nonzero submodule \( N \leq M \), there exists a nonzero endomorphism \( \varphi \) of \( M \) such that \( \text{Im} \varphi \subseteq N \). It was shown in [10, Proposition 4.10] that if \( M \) is a retractable d-Rickart module, then every nonzero submodule of \( M \) contains a nonzero direct summand of \( M \). Now we give the following.
Proposition 2.10. Let $M$ be a wd-Rickart module. Then $M$ is retractable if and only if every nonzero submodule of $M$ contains a nonzero direct summand of $M$.

Proof. ($\Rightarrow$) By [13, Proposition 2.13].
($\Leftarrow$) This is clear.

Let $M$ and $N$ be two modules. The module $M$ is called $N$-wd-Rickart (or relatively wd-Rickart to $N$) if for every nonzero homomorphism $\varphi : M \to N$, $\text{Im} \varphi$ contains a nonzero direct summand of $N$. Therefore $M$ is a wd-Rickart module if and only if $M$ is $M$-wd-Rickart (see [13, Definition 2.1]).

Lemma 2.11. Let $M$ and $N$ be modules. Then $M$ is $N$-wd-Rickart ($N$-d-Rickart) if and only if $M/X$ is $N$-wd-Rickart ($N$-d-Rickart) for any submodule $X \leq M$.

Proof. ($\Rightarrow$) Assume that $M$ is $N$-wd-Rickart ($N$-d-Rickart). Let $\varphi : M/X \to N$ be a nonzero homomorphism. Consider the nonzero homomorphism $\varphi \pi : M \to M/X \to N$, where $\pi : M \to M/X$ is the natural epimorphism. By the assumption, there exists a nonzero direct summand $T$ of $N$ such that $T \subseteq \text{Im} \varphi \pi = \text{Im} \varphi$ ($\text{Im} \varphi \pi = \text{Im} \varphi$ is a direct summand of $N$).
($\Leftarrow$) The result follows by taking $X = 0$.

Theorem 2.12. The following conditions are equivalent for a module $M$:
(a) $M$ is a wd-Rickart module;
(b) For any submodule $N$ of $M$ and every direct summand $K$ of $M$, $M/N$ is $K$-wd-Rickart;
(c) For every pair of direct summands $K$ and $N$ of $M$, $N$ is $K$-wd-Rickart.

Proof. (a) $\Rightarrow$ (b) This is clear by Lemma 2.11 and [13, Proposition 2.4].
(b) $\Rightarrow$ (c) Clear.
(c) $\Rightarrow$ (a) Take $N = K = M$.

Definition 2.13. A module $M$ is called $w$-$C_2$ if for every nonzero submodule $N$ of $M$ and every direct summand $K$ of $M$, $N \cong K$ implies that $N$ contains a nonzero direct summand of $M$.

Proposition 2.14. A module $M$ is wd-Rickart if and only if $M$ has $w$-$C_2$ condition and for every nonzero $\varphi \in \text{End}_R(M)$, there exists a nonzero submodule $A$ of $M$ such that $A$ is isomorphic to a nonzero direct summand of $M$ and $A \subseteq \text{Im} \varphi$. 
Proof. This follows from [13, Proposition 2.3] and the definition of a wd-Rickart module. \qed

Theorem 2.15. The following are equivalent for a module $M$:

(a) $M$ is a wd-Rickart module;
(b) For every nonzero finitely generated right ideal $I$ of $S = \text{End}_R(M)$, 
   $\sum_{\varphi \in I} \varphi(M)$ contains a nonzero direct summand of $M$.

Proof. (a) $\Rightarrow$ (b) Let $I = < \varphi_1, \ldots, \varphi_n >$ be a finitely generated right ideal of $S$, where each $\varphi_i$ is a nonzero endomorphism of $M$. Note that $\sum_{\varphi \in I} \varphi(M) = \varphi_1(M) + \cdots + \varphi_n(M)$. Since $M$ is wd-Rickart, there exists a nonzero direct summand $T$ of $M$ such that $T \subseteq \varphi_1(M) \subseteq \sum_{\varphi \in I} \varphi(M)$.

(b) $\Rightarrow$ (a) This is clear. \qed

3. Direct sums of d-Rickart (wd-Rickart) modules

We begin with the following theorem which gives a characterization for a direct sum of two d-Rickart modules to be d-Rickart.

Theorem 3.1. Let $M = M_1 \oplus M_2$ be a module. The following conditions are equivalent:

(a) $M$ is a d-Rickart module;
(b) (i) $M_i$ and $M_j$ are relatively d-Rickart for $i, j \in \{1, 2\}$, and
   (ii) for every $\varphi \in \text{End}_R(M)$ such that $\text{Im} \varphi + M_1$ is a direct summand of $M$, $\text{Im} \varphi$ is a direct summand of $M$.
(c) (i) $M_i$ and $M_j$ are relatively d-Rickart for $i, j \in \{1, 2\}$, and
   (ii) for every $\varphi \in \text{End}_R(M)$ with $(\text{Im} \varphi + M_1) \oplus N = M$ for some submodule $N \leq M_2$, $\text{Im} \varphi$ is a direct summand of $M$.

Proof. (a) $\Rightarrow$ (b) By [10, Theorem 2.19] and the definition of a d-Rickart module.

(b) $\Rightarrow$ (c) This is clear.

(c) $\Rightarrow$ (a) Let $\varphi : M \to M$ be a nonzero homomorphism. Let $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$ be the natural epimorphisms. Consider the homomorphisms $\varphi_1 = \pi_1 \varphi : M \to M_1$ and $\varphi_2 = \pi_2 \varphi : M \to M_2$. Note that $M$ is $M_1$-d-Rickart and $M$ is $M_2$-d-Rickart by [10, Corollary 5.4]. Then there exists a direct summand $M'_1$ of $M_1$ and a direct summand $M'_2$ of $M_2$ such that $M_1 = \varphi_1(M) \oplus M'_1$ and $M_2 = \varphi_2(M) \oplus M'_2$. It is easy to check that $\varphi(M) + M_1 = \varphi_1(M) \oplus \varphi(M) \oplus M'_1 = M_1 \oplus \varphi_2(M)$. So $(\varphi(M) + M_1) \oplus M'_2 = M$. By assumption, $\varphi(M)$ is a direct summand of $M$. Hence $M$ is a d-Rickart module. \qed
Recall that an element $c$ of a ring $R$ is called regular if $cr \neq 0$ and $rc \neq 0$ for all nonzero $r \in R$. Following [5, p. 104], an $R$-module $X$ is called divisible in case $X = Xc$ for every regular element $c$ of $R$. An $R$-module $Y$ is called torsion if for any $y \in Y$, there exists a regular element $c$ in $R$ such that $yc = 0$. On the other hand, an $R$-module $Z$ is called torsion-free if whenever $z \in Z$ satisfies $zd = 0$ for some regular element $d$ of $R$ then $z = 0$. The ring $R$ is called a right Goldie ring if $R^R$ has finite rank and $R$ has the acc on right annihilators. The following theorem provides many examples of d-Rickart modules.

**Theorem 3.2.** Let $R$ be a prime right Goldie ring such that $R$ is not right primitive and let an $R$-module $M$ be a direct sum of a torsion-free divisible submodule $X$ and a torsion semisimple submodule $Y$. Then $M$ is a d-Rickart module.

**Proof.** By [5, Propositions 6.12 and 6.13], $X$ is a nonsingular injective module. Hence $X$ is d-Rickart since $\text{End}_R(X)$ is von Neumann regular. Moreover, in the proof of [7, Corollary 2.16] it is shown that $\text{Hom}_R(X, Y) = 0$ and $\text{Hom}_R(Y, X) = 0$. Therefore $X$ and $Y$ are fully invariant submodules of $M$. Then $M$ is a d-Rickart module by [10, Proposition 5.14].

**Corollary 3.3.** Let $R$ be a prime PI-ring which is not artinian and let an $R$-module $M$ be a direct sum of a torsion-free divisible submodule $X$ and a torsion semisimple submodule $Y$. Then $M$ is a d-Rickart module.

**Proof.** By [7, Corollary 2.17] and [11, Corollary 13.6.6 and Theorem 13.3.8], $R$ is a right Goldie ring and $R$ is not right primitive. The result follows from Theorem 3.2.

The following proposition is inspired by [10, Proposition 5.2]. This result provides a rich source of examples showing that the wd-Rickart property does not go to direct sums of wd-Rickart modules. It extends [13, Example 2.6] to arbitrary modules.

**Proposition 3.4.** Let $M$ be an indecomposable module with a nonzero proper socle. Then $M \oplus \text{Soc}(M)$ is not a wd-Rickart module.

**Proof.** Assume that $M \oplus \text{Soc}(M)$ is wd-Rickart. By Theorem 2.12, $\text{Soc}(M)$ is $M$-wd-Rickart. Let $\mu : \text{Soc}(M) \to M$ be the inclusion map. Then there exists a nonzero direct summand $T$ of $M$ such that $T \subseteq \mu(\text{Soc}(M)) = \text{Soc}(M)$. Since $M$ is indecomposable, we have $T = M = \text{Soc}(M)$, which is a contradiction.
In [13, Proposition 2.7], it is studied when a direct sum \( \oplus_{i \in I} M_i \) of modules \( M_i \) (\( i \in I \)) is \( N \)-wd-Rickart for some module \( N \). Next, we provide a sufficient condition under which \( N \) is \( (\oplus_{i \in I} M_i) \)-wd-Rickart for some finite index set \( I \).

**Proposition 3.5.** Let \( M = M_1 \oplus M_2 \) such that \( M_2 \) is \( M_1 \)-projective and let \( N \) be a module. Then \( N \) is \( M \)-wd-Rickart if and only if \( N \) is \( M_i \)-wd-Rickart for all \( i = 1, 2 \).

**Proof.** (\( \Rightarrow \)) By Theorem 2.12.

(\( \Leftarrow \)) Let \( \varphi : N \to M \) be a nonzero homomorphism. Let \( \pi_2 : M \to M_2 \) be the projection on \( M_2 \) along \( M_1 \). Let \( \varphi_2 = \pi_2 \varphi : N \to M_2 \).

**Case 1:** Assume that \( \varphi_2 \) is nonzero. Since \( N \) is \( M_2 \)-wd-Rickart, there exists a nonzero direct summand \( K_2 \) of \( M_2 \) such that \( K_2 \subseteq \text{Im} \varphi_2 = (\text{Im} \varphi + M_1) \cap M_2 \). Then \( K_2 = (\text{Im} \varphi + M_1) \cap K_2 \). Let \( L_2 \) be a submodule of \( M_2 \) such that \( M_2 = L_2 \oplus K_2 \). Note that \( K_2 \) is \( M_1 \)-projective by [15, 18.1]. On the other hand, \( K_2 \oplus M_1 = [\text{Im} \varphi \cap (K_2 \oplus M_1)] + M_1 \). Then by [15, 41. 14], \( K_2 \oplus M_1 = C \oplus M_1 \) for some submodule \( C \subseteq \text{Im} \varphi \cap (K_2 \oplus M_1) \). Clearly, \( C \) is a nonzero direct summand of \( M \) which is contained in \( \text{Im} \varphi \).

**Case 2:** Assume that \( \varphi_2 = 0 \). Then \( (\text{Im} \varphi + M_1) \cap M_2 = 0 \). This implies that \( \text{Im} \varphi + M_1 = M_1 \) and hence \( \text{Im} \varphi \subseteq M_1 \). Since \( N \) is \( M_1 \)-wd-Rickart, \( \text{Im} \varphi \) contains a nonzero direct summand of \( M \).

**Theorem 3.6.** Let \( M = \oplus_{i=1}^n M_i \) such that \( M_j \) is \( M_i \)-projective for all \( j > i \) in \( \{1, \ldots, n\} \), and let \( N \) be a module. Then \( N \) is \( M \)-wd-Rickart if and only if \( N \) is \( M_i \)-wd-Rickart for all \( i = 1, \ldots, n \).

**Proof.** The proof is by induction on \( n \) and using Proposition 3.5, Theorem 2.12 and [15, 18.2(2)].

**Corollary 3.7.** Assume that \( M_1, \ldots, M_n \) are \( R \)-modules such that \( M_i \) is \( M_j \)-projective for all \( j > i \) in \( \{1, \ldots, n\} \). Then \( \oplus_{i=1}^n M_i \) is a \( \text{wd-Rickart module if and only if } M_i \) is \( M_j \)-wd-Rickart for all \( i, j \in \{1, \ldots, n\} \).

**Proof.** (\( \Rightarrow \)) Clear by Theorem 2.12.

(\( \Leftarrow \)) By [13, Proposition 2.7], \( \oplus_{i=1}^n M_i \) is \( M_j \)-wd-Rickart for all \( j \in \{1, \ldots, n\} \). Therefore \( \oplus_{i=1}^n M_i \) is a \( \text{wd-Rickart module by Theorem 3.6} \).

4. **Rings whose finitely generated modules are \( \text{d-Rickart (wd-Rickart)} \)**

We begin with a result which gives some information about the class of rings over which every finitely generated module is \( \text{wd-Rickart} \).
Proposition 4.1. Let $R$ be a ring such that every finitely generated $R$-module is a wd-Rickart module. Then

(i) $R$ is a right $V$-ring.

(ii) Every indecomposable finitely generated $R$-module is a simple injective module.

(iii) Every uniform module is a simple injective module.

Proof. (i) Assume that there is a simple $R$-module $S$ with $E(S) \neq S$. Take a nonzero element $x \in E(S)$ which is not in $S$. Clearly, we have $\text{Soc}(xR) = S$. By hypothesis, the finitely generated right $R$-module $xR \oplus \text{Soc}(xR) = xR \oplus S$ is wd-Rickart. This is impossible (see Proposition 3.4).

(ii) Let $M$ be an indecomposable finitely generated $R$-module. Let $0 \neq x \in M$. Since $xR \oplus M$ is wd-Rickart, $xR$ is $M$-wd-Rickart by [13, Corollary 2.8(ii)]. Therefore $xR$ contains a nonzero direct summand of $M$. As $M$ is indecomposable, $xR = M$. Hence $M$ is a simple module.

(iii) Let $U$ be a uniform $R$-module and let $0 \neq x \in U$. So $xR$ is indecomposable. Thus $xR$ is simple by (ii). It follows that $U$ is a semisimple module. But $U$ is indecomposable. Then $U$ is a simple module. \qed

The following example shows that, in general, a right $V$-ring may have a finitely generated module which is not wd-Rickart. Note that there exist right noetherian right $V$-rings which are not von Neumann regular (see [4]).

Example 4.2. Let $R$ be a right noetherian right $V$-ring which is not von Neumann regular. Then $RR$ is not a d-Rickart module by [10, Remark 2.2]. Therefore $RR$ is not a wd-Rickart module by [13, Corollary 3.5].

Next, we focus on the class of rings over which every finitely generated module is d-Rickart.

A module $M$ is said to be regular if every cyclic submodule of $M$ is a direct summand of $M$. Equivalently, every finitely generated submodule of $M$ is a direct summand of $M$ (see [14, Remark 6.1]).

Lemma 4.3. (i) If $M$ is an $R$-module such that $R \oplus M$ is a d-Rickart $R$-module, then $M$ is a von Neumann regular module and $R$ is a von Neumann regular ring.

(ii) If $N$ is a finitely generated $R$-module and $M$ is a regular $R$-module, then $N$ is $M$-d-Rickart.

Proof. (i) Let $a \in M$ and consider the $R$-homomorphism $\varphi_a : R \to M$ defined by $\varphi_a(x) = ax$ for all $x \in R$. By (i) and [10, Theorem 2.19], $R$ is $M$-d-Rickart. Therefore $\text{Im} \varphi_a = aR$ is a direct summand of $M$. So $M$
is a von Neumann regular module. Similarly, we can see that $R$ is a von Neumann regular ring.

(ii) Let $\varphi : N \to M$ be an $R$-homomorphism. Then $\text{Im} \varphi$ is finitely generated. Hence $\text{Im} \varphi$ is a direct summand of $M$ since $M$ is a regular module. It follows that $N$ is $M$-d-Rickart.

**Proposition 4.4.** The following conditions are equivalent for a finitely generated $R$-module $M$:

(i) $R \oplus M$ is a d-Rickart module;

(ii) $M$ is a von Neumann regular module and $R$ is a von Neumann regular ring.

**Proof.** (i) $\Rightarrow$ (ii) By Lemma 4.3(i).

(ii) $\Rightarrow$ (i) Applying Lemma 4.3(ii), we conclude that $M$ is d-Rickart, $R_R$ is $M$-d-Rickart, $M$ is $R_R$-d-Rickart and $R_R$ is d-Rickart. By [10, Corollary 5.6], it follows that $R \oplus M$ is a d-Rickart module.

**Corollary 4.5.** The following are equivalent for a ring $R$:

(i) Every finitely generated $R$-module is a d-Rickart module;

(ii) For any finitely generated $R$-module $M$, $R \oplus M$ is a d-Rickart module;

(iii) Every finitely generated $R$-module is a regular module.

**Proof.** By Lemma 4.3 and Proposition 4.4.

A ring $R$ is called a right FGC-ring if every finitely generated right $R$-module is a direct sum of cyclic submodules.

**Proposition 4.6.** Let $R$ be a ring such that every finitely generated $R$-module is d-Rickart. Then the following hold:

(i) $R$ is a von Neumann regular ring,

(ii) $R$ is a right V-ring,

(iii) $R$ is an FGC-ring,

(iv) Every indecomposable finitely generated $R$-module is a simple injective module, and

(v) For any right ideal $I$ of $R$ and any $x \in R$, there exists a right ideal $I'$ of $R$ such that $I \subseteq I'$, $xR \cap I' \subseteq I$ and $xR + I' = R$.

**Proof.** (i) By Corollary 4.5 (see also [10, Remark 2.2]).

(ii) By Proposition 4.1.

(iii) By Corollary 4.5 and [14, Remark 6.2(2)].

(iv) By Proposition 4.1.

(v) Let $I$ be a right ideal of $R$ and let $x \in R$. By Corollary 4.5, $R/I$ is a regular $R$-module. So $(xR + I)/I$ is a direct summand of $R/I$. Let $I'$ be
a right ideal of $R$ which contains $I$ such that $((xR + I)/I) \oplus (I'/I) = R/I$. Then $xR + I' = R$ and $xR \cap I' \subseteq I$. This completes the proof. 

**Proposition 4.7.** Let $R$ be a right noetherian ring. Then the following are equivalent:

(i) Every finitely generated $R$-module is a $d$-Rickart module;

(ii) $R$ is a semisimple ring.

**Proof.** (i) $\Rightarrow$ (ii) Let $I$ be a right ideal of $R$. Since $R$ is right noetherian, $I$ is finitely generated. Then by Corollary 4.5, $I$ is a direct summand of $R_R$. Thus $R$ is a semisimple ring.

(ii) $\Rightarrow$ (i) This is clear. 

Note that there exists a commutative noetherian local ring $R$ that may have an $R$-module which is not wd-Rickart, and hence not $d$-Rickart.

**Example 4.8.** Let $F$ be a field. Consider $F[[x]]$, the formal power series ring over $F$. It is not hard to see that $F[[x]]$ is a commutative local noetherian ring (it is also a domain). Let $F((x))$ be the quotient field of $F[[x]]$. Take the cyclic $F[[x]]$-module $K = \{ q \in F((x)) \mid xq \in F[[x]] \}$. Note that $F[[x]] \subseteq K$. Consider the nonzero $F[[x]]$-monomorphism $\alpha : K \to K$ defined by $q \mapsto xq$. Clearly, $\text{Im} \, \alpha \subseteq F[[x]]$. If $\text{Im} \, \alpha$ contains a nonzero direct summand of $K$, then $\text{Im} \, \alpha = F[[x]]$, which is a contradiction. This means that $K$ is not a wd-Rickart $F[[x]]$-module.

Now we characterize commutative semisimple rings in terms of finitely generated $d$-Rickart modules.

**Proposition 4.9.** The following are equivalent for a commutative ring $R$:

(i) Every finitely generated $R$-module is a $d$-Rickart module;

(ii) $R$ is a semisimple ring.

**Proof.** (i) $\Rightarrow$ (ii) By Proposition 4.6, $R$ is an FGC-ring which is von Neumann regular. Thus $R$ is a direct sum of indecomposable rings by [3, Theorem 9.1]. Since $R$ is von Neumann regular, it follows that $R$ is a semisimple ring.

(ii) $\Rightarrow$ (i) This is clear. 

Note that there exists a non-commutative artinian local ring $R$ that may have a finitely generated injective $R$-module which is not wd-Rickart, and hence not $d$-Rickart.
Example 4.10. Let $R$ be a local artinian ring with radical $W$ such that $W^2 = 0$, $Q = R/W$ is commutative, $\dim(QW) = 2$ and $\dim(W_Q) = 1$. Then the indecomposable injective 2-generated right $R$-module $U = [(R \oplus R)/D]R$ with $D = \{(ur, -vr) \mid r \in R\}$ and $W = Ru + Rv$ is not regular. For, let $N$ be a cyclic submodule of $U$ with length 2. Then $N \neq U$ since $U$ has length 3. Therefore $N$ cannot be a direct summand of $U$. On the other hand, note that $U/N$ is simple and let $\pi : U \to U/N$ denote the canonical epimorphism. Since $R$ is an artinian ring, we have $\text{Soc}(U) \neq 0$. Let $S$ be a simple submodule of $U$. Therefore there exists an isomorphism $\alpha : U/N \to S$ as $R$ is a local ring. Let $\mu : S \to U$ be the inclusion map. It follows that $f = \mu \alpha \pi : U \to U$ is an endomorphism of $U$ such that $\text{Im} f = S$ is not a direct summand of $U$. This implies that $U$ is not a d-Rickart module. Since $U$ is indecomposable, $U$ is not wd-Rickart, either.

5. Some structure results

Recall that a module $M$ is said to be dual Baer if for every submodule $N \leq M$, there exists an idempotent $e \in S = \text{End}_R(M)$ such that $D(N) = eS$, where $D(N) = \{\varphi \in S \mid \text{Im} \varphi \subseteq N\}$. This notion was introduced by Keskin Tütüncü-Tribak in 2010 [6].

In this section, we present some structure results for some subclasses of wd-Rickart modules.

Since the properties of d-Rickart and wd-Rickart coincide for every noetherian module by [13, Corollary 3.5], the following three results can be obtained immediately from [10, Propositions 4.12 and 4.13 and Theorem 4.14], respectively.

**Proposition 5.1.** Let $M$ be a noetherian wd-Rickart module. Then there exists a decomposition $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ where for each $i$, $M_i$ is an indecomposable noetherian wd-Rickart module with $\text{End}_R(M_i)$ a division ring. Moreover, $n \in \mathbb{N}$ is uniquely determined, and the sequence of isomorphism types of $M_1, M_2, \ldots, M_n$ is uniquely determined up to permutation.

**Proposition 5.2.** Let $M$ be a noetherian module over a commutative ring $R$. Then the following are equivalent for $M$:
(a) $M$ is a d-Rickart module;
(b) $M$ is a wd-Rickart module;
(c) $M$ is a dual Baer module;
(d) $M$ is a semisimple module.
Theorem 5.3. Let $M$ be an $n$-generated module over a commutative noetherian ring $R$ for $n \in \mathbb{N}$. Then the following are equivalent for $M$:

(a) $M$ is a d-Rickart module;
(b) $M$ is a wd-Rickart module;
(c) $M$ is a dual Baer module;
(d) $M \cong R/m_1 \oplus R/m_2 \oplus \cdots \oplus R/m_n$, where $m_i$ are maximal ideals of $R$ with $1 \leq i \leq n$.

Let $R$ be a Dedekind domain which is not a field. Then for each nonzero prime ideal $P$ of $R$, $R(P^\infty)$ will denote the $P$-primary component of the torsion $R$-module $K/R$, where $K$ is the quotient field of $R$.

Theorem 5.4. Let $R$ be a Dedekind domain which is not a field. Let $K$ be the quotient field of $R$. The following are equivalent for an $R$-module $M = \oplus_{i \in I} M_i$, where $M_i$ is indecomposable for each $i \in I$:

(i) $M$ is a dual Baer module;
(ii) $M$ is a d-Rickart module;
(iii) $M$ is a wd-Rickart module;
(iv) $M$ is a direct sum of copies of $K$, $(R(P_i^\infty))_{i \in I}$ and $(R/Q_j)_{j \in J}$, where $(P_i)_{i \in I}$ and $(Q_j)_{j \in J}$ are nonzero prime ideals of $R$ with $P_i \neq Q_j$ for every couple $(i, j) \in I \times J$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear by definitions.

(iii) $\Rightarrow$ (iv) By [13, Corollaries 2.5 and 3.4], each $M_i$ ($i \in I$) is an indecomposable dual Baer module. Applying [6, Theorem 3.4], we see that each $M_i$ is either isomorphic to $K$ or $R(P_i^\infty)$ or $R/Q_i$ for some nonzero prime ideals $P_i$ and $Q_i$ of $R$. Moreover, by [13, Example 2.6], it follows that for every nonzero prime ideal $P$ of $R$, the $R$-module $R(P^\infty) \oplus R/P$ is not a wd-Rickart module. The result follows.

(iv) $\Rightarrow$ (i) By [6, Theorem 3.4].

Corollary 5.5. For a $\mathbb{Z}$-module $M = \oplus_{i \in I} M_i$, where $M_i$ is indecomposable for each $i \in I$, the following are equivalent:

(i) $M$ is a dual Baer module;
(ii) $M$ is a d-Rickart module;
(iii) $M$ is a wd-Rickart module;
(iv) $M$ is isomorphic to a direct sum of arbitrarily many copies of $\mathbb{Q}$ and $(\mathbb{Z}(p_i^\infty))_{i \in I}$ and $(\mathbb{Z}/q_j\mathbb{Z})_{j \in J}$, where $p_i(i \in I)$ and $q_j(j \in J)$ are primes with $p_i \neq q_j$ for every couple $(i, j) \in I \times J$.

Recall that a module $M$ is called lifting if for every submodule $N$ of $M$, there exists a direct summand $K$ of $M$ such that $K \leq N$ and $N/K$ is small in $M/K$. 

Theorem 5.6. Let $R$ be a non-local Dedekind domain. The following are equivalent for an $R$-module $M = \bigoplus_{i \in I} M_i$, where $M_i$ is indecomposable for each $i \in I$:

(i) $M$ is a dual Baer lifting module;
(ii) $M$ is a $d$-Rickart lifting module;
(iii) $M$ is a $wd$-Rickart lifting module;
(iv) $M$ is torsion and every $P$-primary component of $M$ is isomorphic either to $[R(P^\infty)]^{nP}$ or $[R/P]^{(I_P)}$ for some natural number $nP$ and index set $I_P$.

Proof. By Theorem 5.4 and [12, Propositions A.7 and A.8].

Corollary 5.7. For a $\mathbb{Z}$-module $M = \bigoplus_{i \in I} M_i$, where $M_i$ is indecomposable for each $i \in I$, the following are equivalent:

(i) $M$ is dual Baer lifting;
(ii) $M$ is $d$-Rickart lifting;
(iii) $M$ is $wd$-Rickart lifting;
(iv) $M$ is torsion and each $p$-primary component $M_p$ is isomorphic either to $[\mathbb{Z}(p^\infty)]^{nP}$ or $[\mathbb{Z}/p\mathbb{Z}]^{(I_P)}$ for some natural number $nP$ and index set $I_P$.

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References


Contact Information

Derya Keskin Tütüncü
Department of Mathematics,
Hacettepe University
06800 Beytepe, Ankara, Turkey
E-Mail(s): keskin@hacettepe.edu.tr

Nil Orhan Ertaş
Department of Mathematics,
Karabük University
78050 Karabük, Turkey
E-Mail(s): orhannil@yahoo.com

Rachid Tribak
Centre Régional des Métiers de L’Education et de la Formation (CRMEF)-Tanger
Avenue My Abdelaziz, Souani, B.P.:3117
Tangier 90000, Morocco
E-Mail(s): tribak12@yahoo.com

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