Spectra of locally matrix algebras

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Abstract. We describe spectra of associative (not necessarily unital and not necessarily countable-dimensional) locally matrix algebras. We determine all possible spectra of locally matrix algebras and give a new proof of Dixmier–Baranov Theorem. As an application of our description of spectra, we determine embeddings of locally matrix algebras.

Introduction

Let $F$ be a ground field. Recall that an associative $F$-algebra $A$ is called a locally matrix algebra (see [10]) if for an arbitrary finite subset of $A$ there exists a subalgebra $B \subset A$ containing this subset and such that $B$ is isomorphic to a matrix algebra $M_n(F)$ for some $n \geq 1$. In what follows we will sometimes identify $B$ and $M_n(F)$, that is, assume that $M_n(F) \subset A$. We call a locally matrix algebra unital if it contains 1.

Let $A$ be a countable-dimensional unital locally matrix algebra. In [7], J.G. Glimm defined the Steinitz number $\text{st}(A)$ of the algebra $A$ and proved that $A$ is uniquely determined by $\text{st}(A)$. J. Dixmier [5] showed that non-unital countable-dimensional locally matrix algebras over the field of complex numbers can be parameterized by pairs $(s, \alpha)$, where $s$ is a Steinitz number and $\alpha$ is a nonnegative real number. A.A. Baranov [1] extended this parametrization to locally matrix algebras over arbitrary fields.

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In [2], we defined the Steinitz number \(\text{st}(A)\) for a unital locally matrix algebra \(A\) of an arbitrary dimension. We showed that for a unital locally matrix algebra \(A\) of dimension \(> \aleph_0\) the Steinitz number \(\text{st}(A)\) no longer determines \(A\); see [3, 4]. However, it determines the universal elementary theory of \(A\) [3].

In this paper for an arbitrary (not necessarily unital and not necessarily countable-dimensional) locally matrix algebra \(A\), we define a subset of \(\mathbb{SN}\) that we call the spectrum of \(A\) and denote as \(\text{Spec}(A)\). We determine all possible spectra of locally matrix algebras and give a new proof of Dixmier–Baranov Theorem. As an application of our description of spectra, we determine embeddings of locally matrix algebras.

1. Spectra of locally matrix algebras

Let \(\mathbb{P}\) be the set of all primes and \(\mathbb{N}\) be the set of all positive integers. A Steinitz number (see [11]) is an infinite formal product of the form

\[
\prod_{p \in \mathbb{P}} p^{r_p},
\]

where \(r_p \in \mathbb{N} \cup \{0, \infty\}\) for all \(p \in \mathbb{P}\).

Denote by \(\mathbb{SN}\) the set of all Steinitz numbers. Notice, that the set of all positive integers \(\mathbb{N}\) is a subset of \(\mathbb{SN}\). The numbers \(\mathbb{SN} \setminus \mathbb{N}\) are called infinite Steinitz numbers.

Let \(A\) be a locally matrix algebra with a unit \(1\) over a field \(F\) and let \(D(A)\) be the set of all positive integers \(n\) such that there is a subalgebra \(A', 1 \in A' \subseteq A, A' \cong M_n(F)\). Then the least common multiple of the set \(D(A)\) is called the Steinitz number of the algebra \(A\) and denoted as \(\text{st}(A)\); see [2].

Now let \(A\) be a (not necessarily unital) locally matrix algebra. For an arbitrary idempotent \(0 \neq e \in A\) the subalgebra \(eAe\) is a unital locally matrix algebra. That is why we can talk about its Steinitz number \(\text{st}(eAe)\). The subset

\[
\text{Spec}(A) = \{ \text{st}(eAe) \mid e \in A, e \neq 0, e^2 = e \},
\]

where \(e\) runs through all nonzero idempotents of the algebra \(A\), is called the spectrum of the algebra \(A\).

For a Steinitz number \(s\) let \(\Omega(s)\) denote the set of all natural numbers \(n \in \mathbb{N}\) that divide \(s\).

For a Steinitz numbers \(s_1, s_2\) we say that \(s_1\) finitely divides \(s_2\) if there exists \(b \in \Omega(s_2)\) such that \(s_1 = s_2/b\) (we denote: \(s_1 \mid_{\text{fin}} s_2\)).
Steinitz numbers $s_1, s_2$ are rationally connected if $s_2 = q \cdot s_1$, where $q$ is some rational number.

We call a subset $S \subset \mathbb{SN}$ saturated if

1) any two Steinitz numbers from $S$ are rationally connected;
2) if $s_2 \in S$ and $s_1 \mid_{fin} s_2$ then $s_1 \in S$;
3) if $s, ns \in S$, where $n \in \mathbb{N}$, then $is \in S$ for any $i, 1 \leq i \leq n$.

**Theorem 1.** For an arbitrary locally matrix algebra $A$ its spectrum is a saturated subset of $\mathbb{SN}$.

Let us consider examples of saturated subsets of $\mathbb{SN}$.

**Example 1.** For an arbitrary natural number $n$ the set $\{1, 2, \ldots, n\}$ is saturated.

**Example 2.** Let $s$ be a Steinitz number. The set

$$S(\infty, s) := \left\{ \frac{a}{b} \cdot s \mid a \in \mathbb{N}, b \in \Omega(s) \right\}$$

is saturated. For an arbitrary Steinitz number $s' \in S(\infty, s)$ we have $S(\infty, s) = S(\infty, s')$. If $s \in \mathbb{N}$ then $S(\infty, s) = \mathbb{N}$.

**Example 3.** Let $r$ be a real number, $1 \leq r < \infty$. Let $s$ be an infinite Steinitz number. The set

$$S(r, s) = \left\{ \frac{a}{b} s \mid a, b \in \mathbb{N}; b \in \Omega(s), a \leq rb \right\}$$

is saturated.

**Example 4.** Let $s$ be an infinite Steinitz number and let $r = u/v$ be a rational number; $u, v \in \mathbb{N}, v \in \Omega(s)$. Then the set

$$S^+(r, s) = \left\{ \frac{a}{b} s \mid a, b \in \mathbb{N}; b \in \Omega(s), a < rb \right\}$$

is saturated.

**Theorem 2.** Every saturated subset of $\mathbb{SN}$ is one of the following sets:

1) $\{1, 2, \ldots, n\}, n \in \mathbb{N}, \text{ or } \mathbb{N}$;
2) $S(\infty, s), s \in \mathbb{SN} \setminus \mathbb{N}$;
3) $S(r, s), \text{ where } s \in \mathbb{SN} \setminus \mathbb{N}, r \in [1, \infty)$;
4) $S^+(r, s), \text{ where } s \in \mathbb{SN} \setminus \mathbb{N}, r = u/v, u \in \mathbb{N}, v \in \Omega(s)$.

**Remark 1.** The real number $r$ above is the inverse of the density invariant of Dixmier–Baranov.
Theorem 3. (1) For any saturated subset \( S \subseteq SN \) there exists a countable-dimensional locally matrix algebra \( A \) such that \( \text{Spec}(A) = S \).

(2) If \( A, B \) are countable-dimensional locally matrix algebras and \( \text{Spec}(A) = \text{Spec}(B) \) then \( A \cong B \).

Remark 2. The part (2) of Theorem 3 is a new proof of Dixmier–Baranov Theorem.

Which spectra above correspond to unital algebras?

Theorem 4. A locally matrix algebra \( A \) is unital if and only if
\[
\text{Spec}(A) = \{1, 2, \ldots, n\}, \quad \text{where } n \in \mathbb{N},
\]
or
\[
\text{Spec}(A) = S(r, s), \quad \text{where } s \in SN \setminus \mathbb{N},
\]
\( r = u/v, \quad u, v \in \mathbb{N}, \quad v \in \Omega(s) \).

Proof of Theorem 1. In what follows, we assume that \( A \) is a locally matrix \( \mathbb{F} \)-algebra. Recall the partial order on the set of all idempotents of \( A \) : for idempotents \( e, f \in A \) we define \( e \geq f \) if \( f \in eAe \).

We claim that for arbitrary idempotents \( e_1, e_2 \in A \) there exists an idempotent \( e_3 \in A \) such that \( e_1 \leq e_3, e_2 \leq e_3 \). Indeed, there exists a subalgebra \( A' \subset A \) such that \( e_1, e_2 \in A' \) and \( A' \cong M_n(\mathbb{F}) \), \( n \geq 1 \). Let \( e_3 \) be the identity element of the subalgebra \( A' \). Then \( e_1 \leq e_3, e_2 \leq e_3 \).

Now suppose that the locally matrix algebra \( A \) is unital. Let \( a \in A \). Choose a subalgebra \( A' \subset A \) such that \( 1 \in A', a \in A' \) and \( A' \cong M_n(\mathbb{F}) \), \( n \geq 1 \). Let \( r \) be the range of the matrix \( a \) in \( A' \).

V.M. Kurochkin [9] noticed that the number \( r(a) \) does not depend on a choice of a subalgebra \( A' \). We call \( r(a) \) the relative range of the element \( a \). In [4], we showed that if \( A \) is a unital locally matrix algebra and \( e \in A \) is an idempotent, then
\[
\text{st}(eAe) = r(e) \cdot \text{st}(A).
\]

Now let \( A \) be a not necessarily unital locally matrix algebra. Let \( e_1, e_2 \in A \) be idempotents. Choose an idempotent \( e_3 \in A \) such that \( e_1 \leq e_3, e_2 \leq e_3 \), i.e. \( e_1, e_2 \in e_3Ae_3 \). Let \( q_1, q_2 \) be relative ranges of the idempotents \( e_1, e_2 \) in the unital locally matrix algebra \( e_3Ae_3 \). Then
\[
\text{st}(e_1 Ae_1) = q_1 \text{st}(e_3 Ae_3), \quad \text{st}(e_2 Ae_2) = q_2 \text{st}(e_3 Ae_3).
\]

This implies that the Steinitz numbers \( \text{st}(e_1 Ae_1), \text{st}(e_2 Ae_2) \) are rationally connected. We have checked the condition 1) from the definition of saturated sets.
Let \( 0 \neq e \in A \) be an idempotent. Let \( s_2 = \text{st}(eAe) \), \( k \in \Omega(s_2) \) and let \( s_1 = s_2/k \). The unital locally matrix algebra \( eAe \) contains a subalgebra \( e \in M_k(\mathbb{F}) \subset eAe \). Consider the matrix unit \( e_{11} \) of the algebra \( M_k(\mathbb{F}) \). The relative range of the idempotent \( e_{11} \) in the unital algebra \( eAe \) is equal to \( 1/k \). Hence

\[
\text{st}(e_{11}Ae_{11}) = \frac{1}{k} \text{st}(eAe) = s_1, \quad s_1 \in \text{Spec}(A).
\]

We have checked the condition 2).

Now let \( n \geq 1 \). Suppose that Steinitz numbers \( s \) and \( ns \) lie in Spec\((A)\). It means that there exist idempotents \( e_1, e_2 \in A \) such that \( s = \text{st}(e_1Ae_1) \), \( ns = \text{st}(e_2Ae_2) \). There exists a matrix subalgebra \( M_k(\mathbb{F}) \subset A \) that contains \( e_1 \) and \( e_2 \). As above, let \( e_3 \) be the identity element of the algebra \( M_k(\mathbb{F}) \). Let \( \text{rk}(e_i) \) be the range of the idempotent \( e_i \) in the matrix algebra \( M_k(\mathbb{F}) \). We have

\[
s = \frac{\text{rk}(e_1)}{k} \cdot \text{st}(e_3Ae_3), \quad ns = \frac{\text{rk}(e_2)}{k} \cdot \text{st}(e_3Ae_3),
\]

which implies \( \text{rk}(e_2) = n \cdot \text{rk}(e_1) \). In particular, \( n \cdot \text{rk}(e_1) \leq k \). Let \( 1 \leq i \leq n \). Consider the idempotent

\[
e = \text{diag} \left( 1, 1, \ldots, 1, 0, 0, \ldots, 0 \right)_i \cdot \text{rk}(e_1)
\]

in the matrix algebra \( M_k(\mathbb{F}) \). We have

\[
\text{st}(eAe) = \frac{i \cdot \text{rk}(e_1)}{k} \cdot \text{st}(e_3Ae_3) = i \cdot \text{st}(e_1Ae_1) = i \ s.
\]

We showed that \( is \in \text{Spec}(A) \). Hence Spec\((A)\) is a saturated subset of \( SN \). It completes the proof of Theorem 1. \( \square \)

2. Classification of saturated subsets of \( SN \)

Our aim in this section is to classify all saturated subsets of \( SN \). We remark that if at least one Steinitz number from a saturated set \( S \) is infinite then by the condition 1) all Steinitz numbers from \( S \) are infinite.

Let \( S \) be a saturated subset of \( SN \). For a Steinitz number \( s \in S \) and for a natural number \( b \in \Omega(s) \) let

\[
r_s(b) = \max \left\{ i \geq 1 \mid i \cdot \frac{s}{b} \in S \right\}.
\]
Lemma 1. If there exists a Steinitz number $s_0 \in S$ and a natural number $b_0 \in \Omega(s_0)$ such that $r_{s_0}(b_0) = \infty$ then for any $s \in S$ and any $b \in \Omega(s)$ we have $r_s(b) = \infty$.

Proof. Let us show at first that $r_{s_0}(b) = \infty$ for any $b \in \Omega(s_0).$ Indeed, there exists a natural number $c \in \Omega(s_0)$ such that both $b_0$ and $b$ divide $c$. Then for an arbitrary $i \geq 1$ we have

$$i \cdot \frac{s_0}{b_0} = \left( i \cdot \frac{c}{b_0} \right) \cdot \frac{s_0}{c} \in S.$$ 

This implies that $r_{s_0}(c) = \infty$. Hence,

$$i \cdot \frac{s_0}{b} = \left( i \cdot \frac{c}{b} \right) \cdot \frac{s_0}{c} \in S,$$

which proves the claim.

Now choose an arbitrary Steinitz number $s \in S$. By the condition 1), the Steinitz numbers $s$ and $s_0$ are rationally connected, i.e. there exist $a \in \mathbb{N}, b \in \Omega(s_0)$ such that $s = (a/b) \cdot s_0$. By the condition 2), $s_0/b \in S$. Choose a natural number $c \in \Omega(s_0/b)$. Then $c \in \Omega(s)$ and $bc \in \Omega(s_0)$. For an arbitrary $i \geq 1$ we have $i \cdot s/c = i \cdot a \cdot s_0/(bc) \in S$ since $r_{s_0}(bc) = \infty$. This implies $r_s(c) = \infty$ and completes the proof of the lemma.

If a saturated set satisfies the assumptions of Lemma 1 then it is referred to as a set of infinite type. Otherwise, we talk about a saturated set of finite type.

Lemma 2. 1) For an arbitrary Steinitz number $s_0 \in \mathbb{SN}$ the set

$$S(\infty, s_0) := \left\{ \frac{a}{b} \cdot s_0 \mid a \in \mathbb{N}, b \in \Omega(s_0) \right\}$$

is a saturated set of infinite type.

2) If $S$ is a saturated set of infinite type, then for an arbitrary Steinitz number $s \in S$ we have $S = S(\infty, s)$.

Proof. We have to show that the set $S(\infty, s_0)$ satisfies the conditions 1), 2), 3). The condition 1) is obvious. Let $s = (a/b) \cdot s_0, b \in \Omega(s_0).$ Without loss of generality, we assume that $a$ and $b$ are coprime. Let $c \in \Omega(s)$ and let $d = \gcd(c, a)$ be the greatest common divisor of $a$ and $c$, $a = a'd$, $c = c'd$, the numbers $a', c'$ are coprime. Then $a \cdot s_0/(bc) = a' \cdot s_0/(bc')$, which implies that $dc' \in \Omega(s_0).$ Hence

$$\frac{s}{c} = \frac{a}{bc} \cdot s_0 = \frac{a'}{bc'} \cdot s_0 \in S(\infty, s_0).$$
We have checked the condition 2).

Let us check the condition 3). Choose $s = (a/b) \cdot s_0 \in S(\infty, s_0)$, $b \in \Omega(s_0)$. Let $c \in \Omega(s)$. We need to check that for any $i \geq 1$

$$i \cdot \frac{s}{c} = \frac{ia}{bc} \cdot s_0 \in S(\infty, s_0).$$

Let $a/(bc) = a'/b'$, where the natural numbers $a'$, $b'$ are coprime. Since

$$\frac{a}{bc} \cdot s_0 = \frac{s}{c} \in \mathbb{SN},$$

it follows that $b' \in \Omega(s_0)$. Hence, $i \cdot (a'/b') \cdot s_0 \in S(\infty, s_0)$, which implies that $S(\infty, s_0)$ satisfies the condition 3) and, therefore, is saturated.

Let $S$ be a saturated subset of $\mathbb{SN}$ of infinite type. Choose $s_0 \in S$. Our aim is to show that $S = S(\infty, s_0)$. Since the subset $S$ is of infinite type it follows that $r_s(b) = \infty$ for any $s \in S$, $b \in \Omega(s)$. In particular,

$$S(\infty, s_0) = \left\{ \frac{a}{b} \cdot s_0 \mid s \in \Omega(s_0) \right\} \subseteq S.$$

An arbitrary Steinitz number $s \in S$ is rationally connected to $s_0$, hence there exist $a, b \in \mathbb{N}$ such that $s = (a/b) \cdot s_0$. Without loss of generality, we assume that $a$ and $b$ are coprime, which implies $b \in \Omega(s_0)$. We proved that $s \in S(\infty, s_0)$.

Now let $S \subset \mathbb{SN}$ be a saturated subset of finite type, that is, for any $s \in S$, $d \in \Omega(s)$ we have

$$r_s(b) = \max \left\{ i \in \mathbb{N} \mid i \cdot \frac{s}{b} \in S \right\} < \infty.$$

By the condition 3),

$$\left\{ i \in \mathbb{N} \mid i \cdot \frac{s}{b} \in S \right\} = \left[ 1, r_s(b) \right].$$

Since $b \cdot (s/b) \in S$ it follows that $b \leq r_s(b)$. Choose a Steinitz number $s \in S$ and two natural numbers $b, c \in \Omega(s)$ such that $b$ divides $c$. If $i \cdot (s/b) \in S$ then $(ic/b) \cdot (s/c) \in S$. Hence $r_s(b) \cdot (c/b) \leq r_s(c)$. In other words,

$$\frac{r_s(b)}{b} \leq \frac{r_s(c)}{c}. \tag{1}$$

Let $i \in \mathbb{N}$, $s/c \in S$ and let $k$ be a maximal nonnegative integer such that $k \cdot (c/b) \leq i$. By the condition 3), $k \cdot (c/b) \cdot (s/c) \in S$, hence $k \cdot (s/b) \in S$. So, $k \leq r_s(b)$. We proved that

$$\left\lfloor \frac{r_s(c)}{c/b} \right\rfloor \leq r_s(b). \tag{2}$$
The inequalities (1), (2) imply
\[ \left[ \frac{r_s(c)}{c/b} \right] \leq r_s(b) \leq \frac{r_s(c)}{c/b}. \]
Hence
\[ r_s(b) = \left[ \frac{r_s(c)}{c/b} \right]. \tag{3} \]
In particular,
\[ \frac{r_s(c)}{c/b} - 1 < r_s(b), \quad \frac{r_s(c)}{c/b} < r_s(b) + 1. \]
Dividing by \( b \), we get
\[ \frac{r_s(b)}{b} \leq \frac{r_s(c)}{c} < \frac{r_s(b)}{b} + \frac{1}{b}. \tag{4} \]

**Lemma 3.** Let \( S \subset SN \) be a saturated subset of finite type and let \( s \in S \) be an infinite Steinitz number. Then there exists a limit
\[ r_S(s) = \lim_{b \in \Omega(s)} \frac{r_s(b)}{b}, \quad 1 \leq r_S(s) < \infty. \]
If the set \( S \) is fixed then we denote \( r_S(s) = r(s) \).

**Remark 3.** The limit \( r(s) \) is equal to the inverse of the density invariant of Dixmier–Baranov [1, 5].

**The proof of Lemma 3.** The set \( \{ r_s(b)/b \mid b \in \Omega(s) \} \) is bounded from above. Indeed, choose \( b_0 \in \Omega(s) \). For an arbitrary \( b \in \Omega(s) \) there exists \( c \in \Omega(s) \) that is a common multiple for \( b_0 \) and \( b \). Then by (1) and (4),
\[ \frac{r(b)}{b} \leq \frac{r(c)}{c} < \frac{r(b_0)}{b_0} + \frac{1}{b_0}. \]
Let
\[ r = r(s) = \sup \left\{ \frac{r_s(b)}{b} \mid b \in \Omega(s) \right\}. \]
Clearly, \( 1 \leq r < \infty \). Choose \( \varepsilon > 0 \). Let \( N(\varepsilon) = \lfloor 2r/\varepsilon \rfloor + 1 \). We will show that for any \( b \in \Omega(s), b \geq N(\varepsilon), \) we have \( r - \varepsilon < r_s(b)/b \).

Indeed, let \( b \in \Omega(s), b \geq N(\varepsilon) > 2r/\varepsilon \). Then \( 1/b < \varepsilon/(2r) \leq \varepsilon/2 \). There exists a natural number \( b_0 \in \Omega(s) \) such that \( r - \varepsilon/2 < r_s(b_0)/b_0 \). Let \( c \in \Omega(s) \) be a common multiple of \( b_0 \) and \( b \). Then (4) implies
\[ \frac{r(b)}{b} > \frac{r(c)}{c} - \frac{1}{b} \geq \frac{r(b_0)}{b_0} - \frac{1}{b} > r - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = r - \varepsilon. \]
So,

\[ r = \lim_{b \to \infty} \frac{r_s(b)}{b} \]

and this completes the proof of the lemma. \(\square\)

**Lemma 4.** Let \(s, s' \in S\) be infinite Steinitz numbers, \(s' = (a/b) \cdot s; a, b \in \mathbb{N}; b \in \Omega(s)\). Then \(r(s') = (a/b) \cdot r(s)\).

**Proof.** It is sufficient to show that if \(s, ms \in S, m \in \mathbb{N}\), then \(m \cdot r(ms) = r(s)\).

Suppose that \(b \in \Omega(s)\) and \(i \cdot (ms/b) \in S\). Then \(i \cdot m \cdot (s/b) \in S\). Hence \(r_{ms}(b) \cdot m \leq r_s(b)\) and, therefore, \(r(ms) \cdot m \leq r(s)\).

On the other hand, if \(i \cdot (s/b) \in S\) then \([i/m] \cdot m \leq i\) and, therefore, \([i/m] \cdot m \cdot (s/b) \in S\). We showed that

\[
\left[ \frac{r_s(b)}{m} \right] \leq r_{ms}(b), \quad \frac{r_s(b)}{m} - 1 < r_{ms}(b),
\]

\[
\frac{1}{m} \cdot \frac{r_s(b)}{b} - \frac{1}{b} < \frac{r_{ms}(b)}{b}.
\]

Assuming \(b \to \infty\) we get \((1/m) \cdot r(s) \leq r(ms)\), which completes the proof of the lemma. \(\square\)

In the inequality (4), let \(c \to \infty\). Then

\[
\frac{r_s(b)}{b} \leq r(s) \leq \frac{r_s(b)}{b} + \frac{1}{b}, \quad r_s(b) \leq r(s) b \leq r_s(b) + 1.
\]

If the number \(r(s)\) is irrational then \(r_s(b) = [r(s)b]\) for all \(b \in \Omega(s)\).

Now suppose that the number \(r = r_s(b)\) is rational; \(r = u/v; u, v\) are coprime. If a number \(b \in \Omega(s)\) is not a multiple of \(v\) then, as above, \(r_s(b) = [(u/v) \cdot b]\). If \(b\) is a multiple of \(v\) then

\[ r_s(b) = \left\lfloor \frac{r \cdot b}{b - 1} \right\rfloor \text{ or } \left\lfloor \frac{r \cdot b}{b} \right\rfloor. \]

**Lemma 5.** If at least for one number \(b_0 \in \Omega(s) \cap v\mathbb{N}\) we have \(r_s(b_0) = rb_0\) then for all \(b \in \Omega(s) \cap v\mathbb{N}\) we have \(r_s(b) = rb\).

**Proof.** Let \(b, c \in \Omega(s) \cap v\mathbb{N}\) and \(b\) divides \(c\). If \(r_s(b) = rb\) then, by the inequality (1), we have

\[
r = \frac{r_s(b)}{b} \leq \frac{r_s(c)}{c},
\]
which implies \( r_s(c) = rc \). On the other hand, if \( r_s(c) = rc \) then, by the inequality (4),

\[
    r = \frac{r_s(c)}{c} < \frac{r_s(b)}{b} + \frac{1}{b},
\]

which implies \( r_s(b) > rb - 1 \). Hence \( r_s(b) = rb \). We showed that \( r_s(b) = rb \) if and only if \( r_s(c) = rc \).

Now choose \( b_1, b_2 \in \Omega(s) \cap \mathbb{N} \) and suppose that \( r_s(b_1) = rb_1 \). There exists \( c \in \Omega(s) \cap \mathbb{N} \) such that both \( b_1 \) and \( b_2 \) divide \( c \). In view of the above, \( r_s(b_1) = rb_1 \) implies \( r_s(c) = rc \), which implies \( r_s(b_2) = rb_2 \). This completes the proof of the lemma.

Recall that for an infinite Steinitz number \( s \) and a real number \( r \), \( 1 \leq r < \infty \),

\[
    S(r,s) = \left\{ \frac{a}{b} \ s \mid a, b \in \mathbb{N}; b \in \Omega(s), a \leq rb \right\},
\]

\[
    S^+(r,s) = \left\{ \frac{a}{b} \ s \mid a, b \in \mathbb{N}; b \in \Omega(s), a < rb \right\}.
\]

If \( r \) is an irrational number or \( r = u/v \), the integers \( u, v \) are coprime and \( v \notin \Omega(s) \) then \( S(r,s) = S^+(r,s) \). If \( r = u/v \), \( v \in \Omega(s) \) then \( S^+(r,s) \nsubseteq S(r,s) \).

**Lemma 6.** The subsets \( S(r,s) \) and \( S^+(r,s) \) are saturated.

**Proof.** The condition 1) in the definition of saturated subsets is obviously satisfied. Let us check the condition 2). Let \( (a/b) \cdot s \in S(r,s) \) (respectively, \( (a/b) \cdot s \in S^+(r,s) \)), where \( a, b \) are coprime natural numbers, \( b \in \Omega(s) \). Then \( a \leq rb \) (respectively, \( a < rb \)). Suppose that \( c \in \Omega(s) \). We need to show that \( (a \cdot s)/(b \cdot c) \in S(r,s) \) (respectively, \( (a \cdot s)/(b \cdot c) \in S^+(r,s) \)). Let \( d = \gcd(a,c) \), \( a = da', c = dc' \). Then

\[
    \frac{a}{b} \frac{s}{c} = \frac{a'}{b'c'} s \in \mathbb{SN}.
\]

Since the number \( bc' \) is coprime with \( a' \) it follows that \( bc' \in \Omega(s) \). The inequality \( a' \leq rbc' \) (respectively, \( a' < rbc' \)) is equivalent to the inequality \( a \leq rbc \) (respectively, \( a < rbc \)). The latter inequality follows from \( a \leq rb \) (respectively, \( a < rb \)). The condition 2) is verified.

Let us check the condition 3). As above, we assume that \( a, b \) are coprime natural numbers, \( b \in \Omega(s) \) and \( a/b \in S(r,s) \) (respectively, \( a/b \in S^+(r,s) \)). Let \( c \in \Omega((a/b) \cdot s) \), \( \gcd(a,c) = d \), \( a = da' \), \( c = dc' \). We have shown above that \( bc' \in \Omega(s) \). Let \( n \in \mathbb{N} \) and \( n \cdot (as/(bc)) \in S(r,s) \)
(respectively, \( n \cdot (as/(bc)) \in S^+(r,s) \)). Then \( na' \leq rbc' \) (respectively, \( na' < rbc' \)). This immediately implies that for any \( i, 1 \leq i \leq n \), we have \( ia' \leq rbc' \) (respectively, \( ia' < rbc' \)). Hence, \( i \cdot (as/b) \in S(r,s) \) (respectively, \( i \cdot (as/b) \in S^+(r,s) \)).

**Lemma 7.** Let \( r = u/v \), where \( u, v \) are coprime natural numbers. Let \( s \) be an infinite Steinitz number and \( v \in \Omega \left( s \right) \). Then the set \( S^+(r,s) \) is not equal to any of the sets \( S(r', s') \), \( r' \in [1, \infty) \), \( s' \in \mathbb{SN} \).

**Proof.** Let \( s_2 \in S(r,s_1) \) (respectively, \( s_2 \in S^+(r,s_1) \)). Then \( s_2 = (a/b) \cdot s_1 \), where \( a, b \in \mathbb{N} \), \( b \in \Omega(s_1) \). By Lemma 4,

\[
S(r,s_1) = S \left( r \frac{b}{a}, s_2 \right)
\]

(respectively, \( S^+(r,s_1) = S^+ \left( r \frac{b}{a}, s_2 \right) \)).

We showed that the set \( S(r,s) \) (respectively, \( S^+(r,s) \)) is determined by any Steinitz number \( s' \in S(r,s) \) (respectively, \( s' \in S^+(r,s) \)) with an appropriate recalibration of \( r \).

Let \( S = S(r_1,s_1) = S^+(r_2,s_2) \). Choosing an arbitrary Steinitz number \( s \in S \) we get \( S(r'_1,s) = S^+(r'_2,s) \) for some \( r'_1, r'_2 \in [1, \infty) \). The number \( r'_2 = u/v \) is rational, \( \gcd(u,v) = 1 \) and \( v \in \Omega(s) \).

The number \( r \) is uniquely determined by a saturated subset \( S \) and a choice of \( s \in S \). Hence \( r'_1 = r'_2 \). Now it remains to notice that for a rational number \( r = u/v \), \( \gcd(u,v) = 1 \), and an infinite Steinitz number \( s \) such that \( v \in \Omega(s) \) we have \( S(r,s) \neq S^+(r,s) \). This completes the proof of the lemma.

**Lemma 8.** Let \( S \subset \mathbb{SN} \setminus \mathbb{N} \) be a saturated subset of finite type, \( s \in S \), \( r = rs(s) \in [1, \infty) \). Then \( S = S(r,s) \) or \( S = S^+(r,s) \).

**Proof.** Recall that for a natural number \( b \in \Omega(s) \) we defined

\[
r_s(b) = \max \left\{ i \in \mathbb{N} \mid i \frac{s}{b} \in S \right\}
\]

We showed that if \( r \) is an irrational number or \( r = u/v; u, v \) are coprime and \( v \notin \Omega \left( s \right) \), then \( r_s(b) = \lfloor rb \rfloor \) for an arbitrary \( b \in \Omega(s) \).

An arbitrary Steinitz number \( s' \in S \) is representable as \( s' = (a/b) \cdot s \), where \( a, b \) are coprime natural numbers. Clearly, \( b \in \Omega(s) \) and \( a \leq r_s(b) = \lfloor rb \rfloor \). That is why in the case when \( r \) is irrational or \( r = u/v \), \( \gcd(u,v) = 1 \), \( v \notin \Omega(s) \), we have \( S = S(r,s) = S^+(r,s) \).

Suppose now that \( r = u/v; \gcd(u,v) = 1 \), \( v \in \Omega(s) \). If \( b \in \Omega(s) \setminus v\mathbb{N} \) then as above \( r_s(b) = \lfloor rb \rfloor \). By Lemma 5, either for all \( b \in \Omega(s) \cap v\mathbb{N} \) we have \( r_s(b) = rb \) or for all \( b \in \Omega(s) \cap v\mathbb{N} \) we have \( r_s(b) = rb - 1 \). In the first case \( S = S(r,s) \), in the second case \( S = S^+(r,s) \).
Lemma 9. Let $S \subseteq \mathbb{N}$ be a saturated subset. Then either $S = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$ or $S = \mathbb{N}$.

Proof. First, notice that the subsets $\{1, 2, \ldots, n\}$ and $\mathbb{N}$ are saturated.

Now let $S \subseteq \mathbb{N}$ be a saturated subset. If $n \in S$ then $n \in \Omega(n)$ and $n \cdot (n/n) \in S$. By the condition 3), all natural numbers $i = i \cdot (n/n)$, $1 \leq i \leq n$, lie in $S$. This implies the assertion of the lemma. □

Now, Theorem 2 follows from Lemmas 8, 9.

3. Countable-dimensional locally matrix algebras

For an algebra $A$ and an idempotent $0 \neq e \in A$ we call the subalgebra $eAe$ a corner of the algebra $A$.

Let $A_1 \subset A_2 \subset \cdots$ be an ascending chain of unital locally matrix algebras, $A_i$ is a corner of the algebra $A_{i+1}$, $i \geq 1$,

$$A = \bigcup_{i=1}^{\infty} A_i.$$ 

Clearly, $\text{Spec}(A_1) \subseteq \text{Spec}(A_2) \subseteq \cdots$.

Lemma 10.

$$\text{Spec}(A) = \bigcup_{i=1}^{\infty} \text{Spec}(A_i).$$

Proof. For an arbitrary idempotent $e \in A_i$ we have $eA_ie = eAe$, hence $\text{Spec}(A_i) \subseteq \text{Spec}(A)$. On the other hand, an arbitrary idempotent $e \in A$ lies in one of the subalgebras $A_i$. Hence $\text{st}(eAe) = \text{st}(eA_ie) \in \text{Spec}(A_i)$. □

Proof of Theorem 3 (1). To start with we notice that $\{1, 2, \ldots, n\} = \text{Spec}(M_n(\mathbb{F}))$. Let $s$ be a Steinitz number. In [2], we showed that there exists a unital locally matrix algebra $A$, $\dim_\mathbb{F} A \leq \aleph_0$, such that $\text{st}(A) = s$. Consider the algebra $M_\infty(A)$ of infinite $\mathbb{N} \times \mathbb{N}$-matrices, having finitely many nonzero entries. The algebra $M_n(A)$ of $n \times n$-matrices over $A$ is embedded in $M_\infty(A)$ as a north-west corner,

$$M_1(A) \subset M_2(A) \subset \cdots, \quad M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A).$$
In particular, it implies that $M_\infty(A)$ is a locally matrix algebra. We will show that
\[ \operatorname{Spec}(M_\infty(A)) = S(\infty, s). \tag{5} \]
Indeed, by Lemma 10,
\[ \operatorname{Spec}(M_\infty(A)) = \bigcup_{n=1}^\infty \operatorname{Spec}(M_n(A)). \]
We have $\text{st}(M_n(A)) = ns$. In [4], we showed that
\[ \operatorname{Spec}(M_n(A)) = \left\{ \frac{a}{b} \middle| n s, a, b \in \mathbb{N}; 1 \leq a \leq b \right\}. \]
This implies $\operatorname{Spec}(M_n(A)) \subseteq S(\infty, s)$. A Steinitz number $(a/b) \cdot s, b \in \Omega(s)$, lies in $\operatorname{Spec}(M_n(A))$ provided that $a/b \leq n$. This completes the proof of (5). In particular, $\operatorname{Spec}(M_\infty(\mathbb{F})) = \mathbb{N}$.

Consider now a saturated subset $S = S(r, s)$ or $S = S^+(r, s), 1 \leq r < \infty$, where $s$ is an infinite Steinitz number. Choose a sequence $b_1, b_2, \ldots \in \Omega(s)$ such that $b_i$ divides $b_{i+1}, i \geq 1$, and $s$ is the least common multiple of $b_i, i \geq 1$. There exists a unique (up to isomorphism) unital countable-dimensional locally matrix algebra $A_0/s/b_i$ such that $\text{st}(A_0/s/b_i) = s/b_i$. Let $A_i = M_{r_0(b_i)}(A_0/s/b_i)$. We have
\[ \text{st}(A_0/s/b_i) = \text{st}\left( (M_{b_i+1}/b_i)(A_0/s/b_i+1) \right) = s/b_i+1. \]
Hence, by Glimm’s Theorem, $A_0/s/b_i \cong M_{b_i+1}/b_i(A_0/s/b_i+1)$ and, therefore,
\[ A_i = M_{r_0(b_i)}(A_0/s/b_i) \cong M_{r_0(b_i) \cdot s_{b_i+1}/b_i} (A_0/s/b_i+1). \]
By the inequality (1), $r_0(b_i) \cdot (b_i+1/b_i) \leq r_0(b_{i+1})$. Hence, the algebra $A_i$ is embeddable in the algebra $A_{i+1}$ as a north-west corner. Let
\[ A = \bigcup_{i=1}^\infty A_i. \]
We will show that $\operatorname{Spec}(A) = S$. Let $0 \neq e \in A$ be an idempotent. Then $e \in A_i$ for some $i \geq 1$. In [4], we showed that
\[ \text{st}(eA_i e) = \frac{a}{b} \text{st}(A_i), \]
where \(a, b \in \mathbb{N}; a, b\) are coprime natural numbers; \(b \in \Omega(\text{st}(A_i))\), \(a \leq b\). Furthermore,

\[
\text{st}(A_i) = r_s(b_i) \cdot \frac{s}{b_i}, \quad \text{st}(e A_i e) = \frac{a}{b} \cdot r_s(b_i) \cdot \frac{s}{b_i}.
\]

Let \(d = \gcd(b, r_s(b_i))\), \(b = db', r_s(b_i) = d \cdot r_s(b_i)'\). So,

\[
\text{st}(e A_i e) = \frac{a \cdot r_s(b_i)'}{b'} \cdot \frac{s}{b_i} \in \mathbb{S}N.
\]

This implies that \(b' \in \Omega(s/b_i)\). Therefore, \(b'b_i \in \Omega(s)\). To show that \(\text{st}(e A_i e)\) lies in \(S(r, s)\) (respectively, \(S^+(r, s)\)) we need to verify that \(a \cdot r_s(b_i)' \leq r b'b_i\) (respectively, \(a \cdot r_s(b_i)' < r b'b_i\)). Multiplying both sides of the inequality by \(d\) we get \(a \cdot r_s(b_i) \leq r b b_i\) (respectively, \(a \cdot r_s(b_i) < r b b_i\)). This inequality holds since \(a \leq b\) and \(r_s(b_i) \leq r \cdot b_i\) (respectively, \(a \leq b\) and \(r_s(b_i) < r \cdot b_i\)). We proved that \(\text{Spec}(A) \subseteq S\).

Let us show that \(S \subseteq \text{Spec}(A)\). Consider a Steinitz number \((a/b) \cdot s \in S\), where \(a, b \in \mathbb{N}; b \in \Omega(s)\), \(a \leq r b\) in the case \(S = S(r, s)\) or \(a < r b\) in the case \(S = S^+(r, s)\).

There exists a member of our sequence \(b_i\) such that \(b\) divides \(b_i\), \(b_i = k \cdot b, k \in \mathbb{N}\). Then \((a/b) \cdot s = (a k/b_i) \cdot s\).

We will show that \(a k \leq r_s(b_i)\). Indeed, multiplying both sides of the inequality by \(b\) we get \(a b_i \leq r_s(b_i)b\). Let \(S = S(r, s)\). Then \(a \leq r b\). Since \(a \in \mathbb{N}\) it implies \(a \leq [r b]\). Furthermore, \(r_s(b_i) = [rb_i] = [rbk]\). So, it is sufficient to show that \([rbk]k \leq [rbk]\). This inequality holds since \([r b]k\) is an integer and \([r b]k \leq [rbk]\).

Now suppose that \(S = S^+(r, s)\). Then \(a < r b\),

\[
r_s(b_i) = \begin{cases} [r b_i], & \text{if } r b_i \notin \mathbb{N}, \\ r b_i - 1, & \text{if } r b_i \in \mathbb{N}. \end{cases}
\]

There are three possibilities:

1) \(r b \in \mathbb{N}\) and, therefore, \(r b_i \in \mathbb{N}\). In this case \(a \leq r b - 1\), \(r_s(b_i) = rb_i - 1\). We have \(a b_i \leq (r b - 1)b_i \leq (rb_i - 1)b = r_s(b_i)b\);

2) \(r b \notin \mathbb{N}\), but \(r b_i \in \mathbb{N}\). In this case \(a \leq [rb], r_s(b_i) = rb_i - 1\), we have \(a b_i \leq [rb]b_i, r_s(b_i)b = (rb_i - 1)b\). Hence, it is sufficient to show that \([rbk] \leq rb_i - 1 = rbk - 1\). The number \([rbk]k\) is an integer and \([rbk]k < rbk\) since \([rb] < rb\). This implies the claimed inequality;

3) \(r b_i \notin \mathbb{N}\) and, therefore, \(r b \notin \mathbb{N}\). In this case \(a b_i \leq [rb]bk, r_s(b_i)b = [rbk]b\) and it remains to notice that \([rbk] \leq [rbk]\).
We showed that both for $S = S(r, s)$ and for $S = S^+(r, s)$ there holds the inequality $ak \leq r_s(b_i)$.

Recall that $A_i = M_{r_s(b_i)}(A_{s/b_i})$. Consider the north-east corner $M_{ak}(A_{s/b_i})$ of the algebra $A_i$. We have

$$\text{st} \left( M_{ak} \left( A_{s/b_i} \right) \right) = ak \cdot \frac{s}{b_i} = \frac{a}{b} s,$$

and, therefore, $S \subseteq \text{Spec}(A)$. This completes the proof of Theorem 3 (1). \qed

For the proof of Theorem 3 (2) we will need several lemmas on extensions of isomorphisms.

**Lemma 11.** Let $A$ be a locally matrix algebra and let $A_1$ be a subalgebra of $A$ such that $A_1 \cong M_n(\mathbb{F})$. Then every automorphism of the algebra $A_1$ extends to an automorphism of the algebra $A$.

**Proof.** Let $e$ be the identity element of the subalgebra $A_1$. Then the corner $eAe$ is a unital locally matrix algebra. Let $C$ be the centralizer of the subalgebra $A_1$ in $eAe$. By Wedderburn’s Theorem (see [6, 8]), we have $eAe = A_1 \otimes \mathbb{F} C$. An arbitrary automorphism $\varphi$ of the subalgebra $A_1$ is inner, that is, there exists an invertible element $x$ of the subalgebra $A_1$ such that $\varphi(a) = x^{-1}ax$ for all elements $a \in A_1$. The conjugation by the element $x \otimes e$ extends $\varphi$ to an automorphism of the algebra $eAe$. Consider the Peirce decomposition

$$A = eAe + eA(1-e) + (1-e)Ae + (1-e)A(1-e),$$

and the mapping

$$\tilde{\varphi}: A \ni a \mapsto x^{-1}ax + x^{-1}a(1-e) + (1-e)ax + (1-e)a(1-e).$$

The mapping $\tilde{\varphi}$ extends $\varphi$ and $\tilde{\varphi} \in \text{Aut}(A)$. This completes the proof of the lemma. \qed

**Lemma 12.** Let $A$ be a unital locally matrix algebra with an idempotent $e \neq 0$. Then an arbitrary automorphism of the corner $eAe$ extends to an automorphism of the algebra $A$.

**Proof.** Suppose at first that an automorphism $\varphi$ of the algebra $eAe$ is inner, and there exists an element $x_e \in eAe$ that is invertible in the algebra $eAe$ such that $\varphi(a) = x_e^{-1}ax_e$ for an arbitrary element $a \in eAe$. The
element \( x = x_e + (1 - e) \) is invertible in the algebra \( A \). So, conjugation by the element \( x \) extends \( \varphi \).

Now let \( \varphi \) be an arbitrary automorphism of the corner \( eAe \). Let \( A_1 \subseteq A \) be a subalgebra such that \( 1, e \in A_1 \) and \( A_1 \cong M_m(\mathbb{F}) \) for some \( m \geq 1 \). Consider \( A_2 \subseteq A \) such that \( A_1 \subseteq A_2 \), \( \varphi(eA_1e) \subseteq A_2 \) and \( A_2 \cong M_n(\mathbb{F}) \) for some \( n \geq 1 \). Consider the embedding

\[
\varphi : eA_1e \to \varphi(eA_1) \subseteq eA_2e
\]

that preserves the identity element \( e \). By Skolem–Noether Theorem (see [6]), there exists an invertible element \( x_e \in eA_2e \) such that \( \varphi(a) = x_e^{-1}ax_e \) for an arbitrary element \( a \in eA_1e \).

As noticed above, there exists an automorphism \( \psi \) of the algebra \( A \) that extends the automorphism \( eAe \to eAe, a \mapsto x_e^{-1}ax_e \). The composition \( \psi^{-1} \circ \varphi \) leaves all elements of the algebra \( eA_1e \) fixed. Since it is sufficient to prove that the automorphism \( \psi^{-1} \circ \varphi \in \text{Aut}(eAe) \) extends to an automorphism of \( A \) we will assume without loss of generality that the automorphism \( \varphi \in \text{Aut}(eAe) \) fixes all elements of \( eA_1e \).

Let \( C \) be the centralizer of the subalgebra \( A_1 \) in \( A \). Then \( A = A_1 \otimes_F C \) and \( eAe = eA_1e \otimes_F C \). Since the subalgebra \( e \otimes_F C \) is the centralizer of \( eA_1e \otimes_F C \) in the algebra \( eAe \) it follows that \( e \otimes_F C \) is invariant with respect to the automorphism \( \varphi \). Hence, there exists an automorphism \( \theta \in \text{Aut}(C) \) such that \( \varphi(a \otimes c) = a \otimes \theta(c) \) for all elements \( a \in eAe, c \in C \). So, the automorphism \( \tilde{\varphi}(a \otimes c) = a \otimes \theta(c), a \in A_1, c \in C, \) extends \( \varphi \). This completes the proof of the lemma. \( \square \)

**Lemma 13.** Let \( A \) be a unital locally matrix algebra with nonzero idempotents \( e_1, e_2 \). An arbitrary isomorphism \( \varphi : e_1Ae_1 \to e_2Ae_2 \) extends to an automorphism of the algebra \( A \).

**Proof.** There exists a subalgebra \( A_1 \subseteq A \) such that \( 1, e_1, e_2 \in A_1 \) and \( A_1 \cong M_n(\mathbb{F}) \) for some \( n \geq 1 \). Let \( r_i \) be the matrix range of the idempotent \( e_i \) in \( A_1, i = 1, 2 \). In [4], it was shown that

\[
\text{st}(e_1Ae_1) = \frac{r_1}{n} \cdot \text{st}(A), \quad \text{st}(e_2Ae_2) = \frac{r_2}{n} \cdot \text{st}(A).
\]

Since \( e_1Ae_1 \cong e_2Ae_2 \) it follows that \( r_1 = r_2 \). In the matrix algebra \( M_n(\mathbb{F}) \) any two idempotents of the same range are conjugate via an automorphism. Hence, the idempotents \( e_1, e_2 \) are conjugate via an automorphism of \( A_1 \). By Lemma 11, an arbitrary automorphism of \( A_1 \) extends to an automorphism of the algebra \( A \). Now the assertion of the lemma follows from Lemma 12. \( \square \)
Lemma 14. Let $A$, $B$ be isomorphic unital locally matrix algebras with nonzero idempotents $e \in A$, $f \in B$. An arbitrary isomorphism $e A e \to f B f$ extends to an isomorphism $A \to B$.

Proof. Let $\varphi : A \to B$, $\psi : e A e \to f B f$ be isomorphisms. Then

$$\varphi^{-1} \circ \psi : e A e \to \varphi^{-1}(f) A \varphi^{-1}(f)$$

is an isomorphism of two corners of the algebra $A$. By Lemma 13, $\varphi^{-1} \circ \psi$ extends to an automorphism $\chi$ of the algebra $A$, the isomorphism $\varphi \circ \chi$ extends $\psi$.

Lemma 15. Let $A$ be a unital locally matrix algebra and let $s_1$, $s_2$ be Steinitz numbers from $\text{Spec}(A)$. Suppose that $s_2/s_1 > 1$. Let $e_1 \in A$ be an idempotent such that $\text{st}(e_1 A e_1) = s_1$. Then there exists an idempotent $e_2 > e_1$ such that $\text{st}(e_2 A e_2) = s_2$.

Proof. Since $s_2 \in \text{Spec}(A)$ there exists an idempotent $e' \in A$ such that $\text{st}(e' A e') = s_2$. Choose a subalgebra $A_1 \subseteq A$ such that $e_1, e' \in A_1$ and $A_1 \cong M_n(\mathbb{F})$.

Let $r_1$, $r_2$ be the matrix ranges of $e_1$, $e'$ in $M_n(\mathbb{F})$, respectively. In [4], it was shown that

$$\text{st}(e_1 A e_1) = s_1 = \frac{r_1}{n} \text{st}(A), \quad \text{st}(e' A e') = s_2 = \frac{r_2}{n} \text{st}(A).$$

Hence $r_2 > r_1$. Since every idempotent in the algebra $M_n(\mathbb{F})$ is diagonalizable there exist automorphisms $\varphi$, $\psi$ of the algebra $A_1$ such that $\psi(e') > \varphi(e_1)$. By Lemma 11, the automorphisms $\varphi$, $\psi$ extend to automorphisms $\tilde{\varphi}$, $\tilde{\psi}$ of the algebra $A$, respectively.

Let $e_2 = \tilde{\varphi}^{-1}(\psi(e'))$. Then $e_2 > e_1$ and $\text{st}(e_2 A e_2) = s_2$, which completes the proof of the lemma. \qed

Proof of Theorem 3 (2). Let $A$, $B$ be countable-dimensional locally matrix algebras, $\text{Spec}(A) = \text{Spec}(B)$. Choose bases $a_1$, $a_2$, ... and $b_1$, $b_2$, ... in the algebras $A$, $B$, respectively.

We will construct ascending chains of corners $\{0\} = A_0 \subset A_1 \subset A_2 \subset \cdots$ in the algebra $A$ and $\{0\} = B_0 \subset B_1 \subset B_2 \subset \cdots$ in the algebra $B$, such that

$$\bigcup_{i=0}^{\infty} A_i = A, \quad \bigcup_{i=0}^{\infty} B_i = B$$

and $a_1$, ..., $a_i \in A_i$, $b_1$, ..., $b_i \in B_i$, $\text{st}(A_i) = \text{st}(B_i)$ for all $i \geq 1$. 

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Suppose that corners \( \{0\} = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n \), \( \{0\} = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_n \) have already been selected, \( n \geq 0 \). There exist corners \( A' \subset A \), \( B' \subset B \) in the algebras \( A \), \( B \), respectively, such that \( A_n \subset A' \), \( a_{n+1} \in A' \) and \( B_n \subset B' \), \( b_{n+1} \in B' \). The Steinitz numbers \( \text{st}(A') \), \( \text{st}(B') \) lie in the same saturated subset of \( \mathbb{S} \), therefore, they are rationally connected.

Suppose that \( \text{st}(B') \geq \text{st}(A') \). Let \( e' \) be an idempotent of the algebra \( A \) such that \( A' = e' A e' \). The Steinitz number \( \text{st}(B') \) lies in \( \text{Spec}(A) \). Hence, by Lemma 15, there exists an idempotent \( e \in A \) such that \( e \geq e' \) and \( \text{st}(e A e) = \text{st}(B') \). Choose \( A_{n+1} = e A e \), \( B_{n+1} = B' \). The chains \( \{0\} = A_0 \subset A_1 \subset A_2 \subset \cdots \) and \( \{0\} = B_0 \subset B_1 \subset B_2 \subset \cdots \) have been constructed.

By Lemma 14, every isomorphism \( A_i \to B_i \) extends to an isomorphism \( A_{i+1} \to B_{i+1} \). This gives rise to a sequence of isomorphisms \( \phi_i : A_i \to B_i \), \( i \geq 0 \), where each \( \phi_{i+1} \) extends \( \phi_i \). Taking the union \( \cup_{i \geq 0} \phi_i \) we get an isomorphism from the algebra \( A \) to the algebra \( B \). This completes the proof of the theorem.

\[ \square \]

Proof of Theorem 4. It is easy to see that a locally matrix algebra \( A \) is a unital if and only if the set of idempotents of \( A \) has a largest element: an identity. This is equivalent to \( \text{Spec}(A) \) containing a largest Steinitz number. Among saturated sets of Steinitz numbers only \( \{1, 2, \ldots, n\} \) and \( S(r, s) \), \( s \in \mathbb{S} \setminus \mathbb{N} \), \( r = u/v \); \( u \) and \( v \) are coprime natural numbers, \( v \in \Omega(s) \), satisfy this assumption.

\[ \square \]

4. Embeddings of locally matrix algebras

Lemma 16. Let \( S_1 \), \( S_2 \) be saturated sets of Steinitz numbers. Then either \( S_1 \cap S_2 = \emptyset \) or one of the sets \( S_1 \), \( S_2 \) contains the other one.

Proof. Let \( s \in S_1 \cap S_2 \). If \( s \in \mathbb{N} \) then, by Lemma 9, each set \( S_i \) is either a segment \([1, n]\), \( n \geq 1 \), or the whole \( \mathbb{N} \). In this case the assertion of the lemma is obvious.

Suppose that the number \( s \) is infinite. Then by Theorem 2, \( S_i = S(r_i, s) \) or \( S_i = S^+(r_i, s) \), where \( r_i = r_{S_i}(s) \in [1, \infty) \cup \{\infty\} \), \( i = 1, 2 \). Clearly, if \( r_{S_1}(s) < r_{S_2}(s) \) then \( S_1 \subsetneq S_2 \). If \( r_{S_1}(s) = r_{S_2}(s) \) then

\[
S_1, S_2 = \begin{bmatrix}
S(r, s) \\
S^+(r, s) \\
S(\infty, s)
\end{bmatrix}
\]
and \( S^+(r, s) \subseteq S(r, s) \subseteq S(\infty, s) \) for any \( r \in [1, \infty) \). This completes the proof of the lemma. \( \square \)

Let \( A \) be a locally matrix algebra. A subalgebra \( B \subseteq A \) is called an \textit{approximative corner} of \( A \) if \( B \) is the union of an increasing chain of corners. In other words, there exist idempotents \( e_0, e_1, e_2, \ldots \) such that

\[
e_0 \ A \ e_0 \subseteq e_1 \ A \ e_1 \subseteq e_2 \ A \ e_2 \subseteq \cdots, \quad B = \bigcup_{i=0}^{\infty} e_i \ A \ e_i.
\]

It is easy to see that an approximative corner of a locally matrix algebra is a locally matrix algebra.

**Theorem 5.** Let \( A, B \) be countable-dimensional locally matrix algebras. Then \( B \) is embeddable in \( A \) as an approximative corner if and only if \( \text{Spec}(B) \subseteq \text{Spec}(A) \).

**Proof.** If \( B \) is an approximative corner of \( A \) then every corner of \( B \) is a corner of \( A \), hence \( \text{Spec}(B) \subseteq \text{Spec}(A) \).

Suppose now that \( \text{Spec}(B) \subseteq \text{Spec}(A) \). If the algebra \( B \) is unital then it embeds in the algebra \( A \) as a corner. Indeed, the embedding \( \text{Spec}(B) \subseteq \text{Spec}(A) \) implies that there exists an idempotent \( e \in A \) such that \( \text{st}(B) = \text{st}(eAe) \). By Glimm’s Theorem [7], we have \( B \cong eAe \).

Suppose now that the algebra \( B \) is not unital. Then there exists a sequence of idempotents \( 0 = f_0, f_1, f_2, \ldots \) of algebra \( B \) such that

\[
\{0\} = f_0 \ B \ f_0 \subsetneq f_1 \ B \ f_1 \subsetneq f_2 \ B \ f_2 \subsetneq \cdots, \quad \bigcup_{i=0}^{\infty} f_i \ B \ f_i = B.
\]

We will construct a sequence of idempotents \( e_0, e_1, e_2, \ldots \) in the algebra \( A \) such that

\[
e_0 \ A \ e_0 \subsetneq e_1 \ A \ e_1 \subsetneq e_2 \ A \ e_2 \subsetneq \cdots, \quad \text{st}(f_i \ B \ f_i) = \text{st}(e_i \ A \ e_i)
\]

for an arbitrary \( i \geq 0 \). Let \( e_0 = 0 \). Suppose that we have already selected idempotents \( e_0, e_1, \ldots, e_n \in A \) such that \( e_0 A e_0 \subsetneq e_1 A e_1 \subsetneq \cdots \subset e_n A e_n \) and \( \text{st}(e_i A e_i) = \text{st}(f_i B f_i) \), \( 0 \leq i \leq n \). We have

\[
\text{st}(f_{n+1} \ B \ f_{n+1}) = \text{st}(f_n \ B \ f_n) = \text{st}(e_n \ A \ e_n)
\]

and \( \text{st}(f_{n+1} B f_{n+1}) \in \text{Spec}(A) \). By Lemma 15, there exists an idempotent \( e_{n+1} \in A \) such that \( e_n A e_n \subsetneq e_{n+1} A e_{n+1} \) and \( \text{st}(e_{n+1} A e_{n+1}) = \text{st}(f_{n+1} B f_{n+1}) \), which proves existence of a sequence \( e_0, e_1, e_2, \ldots \).
The union
\[ A' = \bigcup_{i=0}^{\infty} e_i A e_i \]
is an approximative corner of the algebra \( A \). By Glimm’s Theorem [7], \( e_i A e_i \cong f_i B f_i, \ i \geq 1 \). By Lemma 10,
\[ \text{Spec}(A') = \bigcup_{i=1}^{\infty} \text{Spec}(e_i A e_i) \quad \text{and} \quad \text{Spec}(B) = \bigcup_{i=1}^{\infty} \text{Spec}(f_i B f_i). \]
Hence \( \text{Spec}(B) = \text{Spec}(A') \). By Theorem 3 (2), we have \( A' \cong B \), which completes the proof of the theorem. \( \square \)

References

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