Group of continuous transformations of real interval preserving tails of $G_2$-representation of numbers

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Abstract. In the paper, we consider a two-symbol system of encoding for real numbers with two bases having different signs $g_0 < 1$ and $g_1 = g_0 - 1$. Transformations (bijections of the set to itself) of interval $[0, g_0]$ preserving tails of this representation of numbers are studied. We prove constructively that the set of all continuous transformations from this class with respect to composition of functions forms an infinite non-abelian group such that increasing transformations form its proper subgroup. This group is a proper subgroup of the group of transformations preserving frequencies of digits of representations of numbers.

Introduction

Many two-symbol systems of encoding (representation) of fractional part of real numbers are known. They use the alphabet $A = \{0, 1\}$ and are based on expansions of numbers in series, infinite products, continued fractions, etc. These systems identify a number as a sequence of zeros and ones.
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ones, i.e., as element of space $L = A \times A \times \ldots$ of sequences of zeros and ones. Every such a system has some advantages and restrictednesses as well as certain conveniences for solving some problems of number-theoretical, topological-metric and probabilistic kind. A system with two positive bases ($q_0 \in (0, 1)$, $q_1 = 1 - q_0$) is among them. It generalizes classic binary system, has a self-similar geometry and zero redundancy. This is a system of $Q_2$-representation of numbers based on representation of a number by the series

$$x = \alpha_1 q_1^{-\alpha_1} + \sum_{k=2}^{\infty} \left( \alpha_k q_1^{-\alpha_k} \prod_{j=1}^{k-1} q_{\alpha_j} \right) \equiv \Delta_{Q_2}^{\alpha_1 \alpha_2 \ldots \alpha_k \ldots}.$$  

For $q_0 = 0.5$, it is a classic binary system.

In expression $\prod_{j=1}^{k} q_{\alpha_j} = q_1^{N_0(x,k)} q_1^{N_1(x,k)}$, where $N_1(x, k) = \alpha_1 + \alpha_2 + \ldots + \alpha_k$ and $N_0(x, k) = k - N_1(x, k)$, one can see that this is a system with two bases. This system of encoding of numbers has various applications in metric and probabilistic theory of numbers, function theory and measure theory, fractal analysis and fractal geometry. For this system, the left shift operator is a discontinuous function, linear and increasing on cylinders of rank 1: $\Delta_{Q_2}^{0} = [0, q_0]$ and $\Delta_{Q_2}^{1} = [q_0, 1]$.

In the papers [7], [8], an analogue of $Q_2$-representation of numbers, namely, system of encoding of numbers in interval $[0, g_0]$ with bases $g_0 \in (0, 1)$ and $g_1 = g_0 - 1$ having different signs is introduced. It is based on expansion of a number in the series

$$x = \alpha_1 g_1^{-\alpha_1} + \sum_{k=2}^{\infty} \left( \alpha_k g_1^{-\alpha_k} \prod_{j=1}^{k-1} g_{\alpha_j} \right) \equiv \Delta_{G_2}^{\alpha_1 \alpha_2 \ldots \alpha_k \ldots}.$$  

This system also has a zero redundancy (any number has at most two representations, and set of numbers having two representations is countable).

Generally speaking, series (1) has positive as well as negative terms. So $G_2$-representation is not topologically equivalent to $Q_2$-representation and it is not a simple reencoding of $Q_2$-representation. At the same time metric theories of these representations are similar. A vivid peculiarity of $G_2$-representation is the fact that left shift operator of $G_2$-representation is a continuous function on a whole interval $[0, g_0]$. This is a fundamental difference between this system and previously studied systems.

Real numbers in interval $[0, g_0]$ having two $G_2$-representations are called $G_2$-binary numbers, and numbers having a unique representation
are called $G_2$-unary numbers. It is known \cite{10} that the set $B$ of all $G_2$-binary numbers is countable and consists of numbers with the following $G_2$-representation: $\Delta_{c_1 \ldots c_m 01(0)}^{G_2} = \Delta_{c_1 \ldots c_m 11(0)}^{G_2}$.

Algorithm for comparison of numbers in terms of their $G_2$-representations is the following.

**Theorem 1** (\cite{10}). Numbers $\Delta_{c_1c_2 \ldots c_md_1d_2 \ldots}^{G_2} \neq \Delta_{c_1c_2 \ldots c_md_1'd_2' \ldots}^{G_2}$ are in the relation

\[ x_1 \geq x_2 \quad \text{if} \quad \sigma_m \equiv c_1 + c_2 + \ldots + c_m = 2k, \]
\[ x_1 \leq x_2 \quad \text{if} \quad \sigma_m \equiv c_1 + c_2 + \ldots + c_m = 2k - 1. \]

In this paper, we consider continuous transformations of interval $[0, g_0]$, i.e., bijections of interval to itself, preserving “tails of $G_2$-representation of numbers”, namely, we study group properties of the family of these transformations. Groups of transformations preserving tails of different representations of numbers were studied in the papers \cite{2}, \cite{3}, \cite{4}. Specific properties of $G_2$-representation generate a peculiar corresponding transformation group having a non-trivial subgroup of increasing functions unlike other systems.

1. **Tails and tail sets**

**Definition 1.** We say that $G_2$-representations of numbers $x = \Delta_{\alpha_1 \ldots \alpha_n \ldots}^{G_2}$ and $y = \Delta_{\beta_1 \beta_2 \ldots \beta_n \ldots}^{G_2}$ have the same tail if there exist positive integers $k$ and $m$ such that

\[ \alpha_{k+j} = \beta_{m+j} \quad \text{(2)} \]

for any $j \in N$. We denote it symbolically by $x \sim y$.

**Definition 2.** Suppose that $k$ and $m$ are the smallest numbers satisfying condition (2). Then $G_2$-representation and corresponding number

\[ z \equiv [x \wedge y] = \Delta_{\alpha_{k+1} \alpha_{k+2} \ldots}^{G_2} = \Delta_{\beta_{m+1} \beta_{m+2} \ldots}^{G_2} \]

is called a common tail of representations of numbers $x$ and $y$.

It is evident that binary relation $\sim$ (“has the same tail”) is an equivalence relation, i.e., it is reflexive, symmetric and transitive. Thus it provides a partition of the set $Z$ of all $G_2$-representations of numbers in interval $[0, g_0]$ into the equivalence classes, and they form together a quotient set $W = Z / \sim$. Any element of the set $W$ is called a tail set, this set is uniquely determined by an arbitrary its element.
Theorem 2. Any tail set is a countable everywhere dense in interval 
\([0, g_0]\) set. The set \(W\) of all tail sets is a continuum set.

Proof. Suppose \(K_x\) is a tail set containing the number \(x\), \(K_1\) is a set of all numbers \(y\) such that \([y \land x] = x\), and \(K_n\) is a set of all numbers \(y\) such that \([y \land \Delta_{\alpha_{n+1}(x)\alpha_{n+2}(x)}^{G_2}] = \Delta_{\alpha_{n+1}(x)\alpha_{n+2}(x)}^{G_2}\).

It is evident that every set \(K_n\) \((n \in \mathbb{N})\) is countable and \(K_x = \bigcup_n K_n\). Thus \(K_x\) is a countable set because it is a countable union of countable sets.

An arbitrary cylinder \(\Delta_{c_1...c_n}^{G_2}\) contains points of tail set \(K_x\) because points \(\Delta_{c_1...c_n\alpha_1(x)\alpha_2(x)}^{G_2}\), \(\Delta_{c_1...c_n\alpha_2(x)\alpha_3(x)}^{G_2}\), and \(\Delta_{c_1...c_n\alpha_k(x)\alpha_{k+1}(x)}^{G_2}\) belong to this cylinder. This proves that the set \(K_x\) is everywhere dense in interval \([0, g_0]\).

The set \(W\) is a continuum set. Indeed, suppose that it is a countable set. Then we see that interval \([0, g_0]\) is a countable set because it is a countable union of countable sets. This contradicts the fact that interval is a continuum set. \(\square\)

Note that all \(G_2\)-binary numbers belong to the same tail set.

2. Left and right shift operators of \(G_2\)-representation of numbers

Theorem 3 (\([8]\)). Left shift operator \(\omega\) of \(G_2\)-representation of numbers in interval \([0, g_0]\) defined by equality

\[
\omega(\Delta_{\alpha_1\alpha_2...\alpha_n}^{G_2}) = \Delta_{\alpha_1\alpha_2...\alpha_n}^{G_2}
\]

in space of \(G_2\)-representations, is analytically expressed in the form

\[
\omega(x) = \frac{1}{g_{\alpha_1}(x)} x - \frac{\delta_{\alpha_1(x)}}{g_{\alpha_1}(x)},
\]

is a continuous well-defined function on \([0, g_0]\), is linear on every cylinder of rank 1, is increasing on \(\Delta_0\) and decreasing on \(\Delta_1\).

Numbers

\[
0 = \Delta_{(0)}^{G_2} \quad \text{and} \quad \Delta_{(1)}^{G_2} = g_0 + g_0 g_1 + g_0 g_1^2 + \ldots = \frac{g_0}{2 + g_0}
\]

are invariant points of left shift operator \(\omega\).

Remark 1. The last theorem shows an essential difference between \(G_2\)-representation and other known two-symbol representations, in particular, \(Q_2^*\)-representation, \(\tilde{Q}\)-representation \([9]\) and \(A_2\)-continued fractions \([3]\).
Let \( n \) be a positive integer that is greater than 1. Put
\[
\omega^n(x) = \omega(\omega^{n-1}(x)) = \Delta_{\alpha_n+1(x)\alpha_{n+2}(x)}^{G_2} \omega(x),
\]
Since
\[
x = \delta_{\alpha_1(x)} + \sum_{k=2}^{n} \left( \delta_{\alpha_k(x)} \prod_{j=1}^{k-1} g_{\alpha_j(x)} \right) \omega^n(x),
\]
we have
\[
\omega^n(x) = \frac{x}{P_n(x)} - \frac{1}{P_n(x)} \left( \delta_{\alpha_1(x)} + \sum_{k=2}^{n} \left( \delta_{\alpha_k(x)} \prod_{j=1}^{k-1} g_{\alpha_j(x)} \right) \right). 
\]

**Theorem 4 ([10]).** Function \( \omega^n \) is well defined by equality
\[
\omega^n(x) = \omega^n(\Delta_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)}) \equiv \Delta_{\alpha_n+1(x)\alpha_{n+2}(x)}^{G_2},
\]
is analytically expressed in the form \( \omega^n(x) = \frac{1}{P_n} x - \frac{B_n}{P_n} \), where \( P_n = \prod_{j=1}^{n} g_{\alpha_j(x)} \),
\( B_n = \delta_{\alpha_1(x)} + \sum_{k=2}^{n} \left( \delta_{\alpha_k(x)} \prod_{j=1}^{k-1} g_{\alpha_j(x)} \right) \), is continuous on interval \([0, g_0]\) and linear on every cylinder of rank \( n \).

**Remark 2.** In terms of dynamical systems, relation “has the same tail” can be defined in the following form:
\[
x = \Delta_{\alpha_1\alpha_2\ldots\alpha_n}^{G_2} \sim y = \Delta_{\beta_1\beta_2\ldots\beta_n}^{G_2} \iff O_x \cap O_y \neq \emptyset,
\]
where \( O_u = \{ u, \omega^1(u), \omega^2(u), \ldots \} \) is an orbit of point \( u \) under mapping \( \omega \), and \([x \wedge y] = O_x \cap O_y\).

**Definition 3.** Function \( \tau_i \) defined on \([0, g_0]\) by equality
\[
\tau_i(x) = \tau_i \left( \Delta_{\alpha_1\alpha_2\ldots\alpha_n(x)}^{G_2} \right) = \Delta_{\alpha_i\alpha_1\alpha_2\ldots\alpha_n(x)}^{G_2},
\]
where \( i \in \{0, 1\} \), is called a right shift operator of \( G_2 \)-representation of numbers with parameter \( i \) (in the sequel, we just say “right shift operator”).

Since \( \tau_i \left( \Delta_{c_1c_2\ldots c_m01(0)}^{G_2} \right) = \tau_i \left( \Delta_{c_1c_2\ldots c_m11(0)}^{G_2} \right) \), we see that function \( \tau_i \) is well defined. It is evident that the set of values of function \( \tau_i \) is cylinder \( \Delta_{i}^{G_2} \). In particular,
\[
\tau_0(0) = \tau \left( \Delta_{(0)}^{G_2} \right) = 0, \quad \tau_1(0) = \tau_1 \left( \Delta_{(0)}^{G_2} \right) = \Delta_{1(0)}^{G_2} = g_0.
\]
Lemma 1. Function $\tau_i$ is continuous at any point of interval $[0, g_0]$ and analytically expressed in the form $\tau_i(x) = \delta_i + g_i x$.

Indeed, since
\[
\tau_i \left( \Delta_{a_1a_2...a_n}\right) = \Delta_{a_1a_2...a_n}\]
we have $\tau_i(x) = \delta_i + g_i x$, i.e., $\tau_0(x) = g_0 x$, $\tau_1(x) = g_0 + g_1 x$. So, it is evident that function $\tau_i$ is continuous on cylinders of rank 1. From equality $\tau_i \left( \Delta_{c_1c_2...c_m}\right) = \tau_i \left( \Delta_{c_1c_2...c_m11}\right)$ it follows that the function is continuous at $G_2$-binary points that are endpoints of cylinders.

Corollary 1. Function $\tau_0$ is increasing and function $\tau_1$ is decreasing. Moreover, $\tau_0(g_0) = \tau_1(g_0)$.

The following equalities are evident: $\omega(\delta_i(x)) = x$ and $\tau_\alpha(x) = x$.

Equation $\tau_i(x) = \omega(x)$ has two solutions: $x = \Delta_{(ji)}$, where $j \in \{0, 1\}$.

Equation $\tau_i(x) = \omega^m(x)$ has $2^m$ solutions: $x = \Delta_{(j_1j_2...j_mi)}$, where $j_k \in \{0, 1\}$, $k = 1, m$.

Let $(i_1, i_2, \ldots, i_n)$ be a tuple of zeros and ones. Function $\tau_{i_1i_2...i_n}$ defined by equality $\tau_{i_1i_2...i_n}(x) = \Delta_{i_1i_2...i_\alpha_1(x)i_\alpha_2(x)...}$ is called a right shift operator with parameters $(i_1, i_2, \ldots, i_n)$. By induction, from equality $\tau_{i_1i_2...i_n}(x) = \tau_{i_1}(\tau_{i_2...i_n}(x))$ it follows that operator $\tau_{i_1i_2...i_n}(x)$ is well defined.

Operator $\tau_{i_1i_2...i_n}$ is analytically expressed in the form
\[
\tau_{i_1i_2...i_n}(x) = \delta_{i_1} + \sum_{k=1}^{n} \left( \delta_{i_k} \prod_{j=1}^{k-1} g_{i_j} \right) + \left( \prod_{j=1}^{n} g_{i_j} \right) x,
\]

is a linear function, is increasing if $P_n = \prod_{j=1}^{n} g_{i_j} > 0$ (this is equivalent to $i_1 + i_2 + \ldots + i_n$ is even number) and decreasing if $P_n < 0$ (this is equivalent to $i_1 + i_2 + \ldots + i_n$ is odd number).

For example, consider $n = 2$ and corresponding functions $\tau_{00}$, $\tau_{01}$, $\tau_{10}$, $\tau_{11}$. Functions $\tau_{00} = g_0^2 x$ and $\tau_{11} = g_0^2 x + g_0^2$ are linear increasing, but functions $\tau_{01} = g_0 g_1 x + g_0^2$ and $\tau_{10} = g_0 g_1 x + g_0$ are linear decreasing.

3. Continuous functions and transformations of interval $[0, g_0]$ preserving tails of $G_2$-representation of numbers

We say that function $y = f(x)$ preserves tails of $G_2$-representation of numbers in interval $[0, g_0]$ (or is a tail function) if any number $x \in [0, g_0]$ and its image $y = f(x)$ have the same tail.
Left and right shift operators $\omega^n$, $\tau_{i_1i_2\ldots i_n}$ for any positive integer $n$ and for any tuple $(i_1, i_2, \ldots, i_n)$ of zeros and ones are simple examples of continuous functions preserving tails of $G_2$-representation of numbers. Various "joinings" of these functions are the same. For example, function

$$f(x) = \begin{cases} 
\omega(x) & \text{if } 0 \leq x \leq x_1, \\
\tau_1(x) & \text{if } x_1 \leq x \leq x_2, \\
\omega(x) & \text{if } x_2 \leq x \leq g_0,
\end{cases}$$

where $x_1$ and $x_2$ are solutions of equation $\omega(x) = \tau_1(x)$, i.e., $x_1 = \Delta_{(01)}^{G_2}$, $x_2 = \Delta_{(1)}^{G_2}$, preserves tails of $G_2$-representation.

But not every continuous function defined on interval $[0, g_0]$ is its transformation, i.e., bijection of the interval to itself. It is clear that above mentioned functions are not transformations.

It is clear that continuous transformations of interval $[0, g_0]$ can be only strictly monotonic (increasing and decreasing) functions such that their domain and set of values coincide with this interval.

**Lemma 2.** Decreasing function

$$f_1(x) = \begin{cases} 
\tau_1(x) & \text{if } x \leq x_1 = \Delta_{(1)}^{G_2} = \frac{g_0}{2 - g_0}, \\
\omega(x) & \text{if } x \geq x_1 = \Delta_{(1)}^{G_2} = \frac{g_0}{2 - g_0},
\end{cases}$$

is a continuous tail transformation of interval $[0, g_0]$.

**Proof.** Number $x_1$ is a solution of equation $\tau_1(x) = \omega(x)$ being equivalent to system of equations $1 = \alpha_2(x) = \alpha_4(x) = \ldots$, $\alpha_1(x) = \alpha_3(x) = \alpha_5(x) = \ldots$. So this equation has two solutions: $x = \Delta_{(a_1)}^{G_2}$, $a_1 \in \{0, 1\}$.

First solution $x_0 = \Delta_{(01)}^{G_2}$ belongs to interval of decrease of function $\tau_1$ and to interval of increase of function $\omega$, and second solution $x_1$ belongs to interval of decrease of function $\tau_1$ and to interval of decrease of function $\omega$. Thus $f_1$ is a continuous and strictly decreasing function. Moreover, $f_1(0) = g_0$ and $f_1(g_0) = 0$. Hence $f_1$ is a continuous transformation of interval $[0, g_0]$.

**Example 1.** Function

$$f_2(x) = \begin{cases} 
\tau_1(x) & \text{if } 0 \leq x \leq \Delta_{(101)}^{G_2}, \\
\omega^2(x) & \text{if } \Delta_{(101)}^{G_2} \leq x \leq g_0,
\end{cases}$$

is a continuous decreasing tail transformation.
Indeed, $\tau_1$ is a continuous decreasing tail function and $\Delta_{(101)}^{G_2}$ is a solution of equation $\tau_1(x) = \omega^2(x)$ belonging to the last interval of decrease of function $\omega^2(x)$.

**Example 2.** Decreasing function

$$f_3(x) = \begin{cases} 
\tau_10\ldots0(x) & \text{if } 0 \leq x \leq x_k \equiv \Delta_{l10\ldots0}^{G_2}, \\
\omega(x) & \text{if } x_k \leq x \leq g_0, \ i \in A,
\end{cases}$$

is a continuous tail transformation of interval $[0, g_0]$.

Indeed, $\omega$ and $\tau_{10\ldots0}$ are continuous tail functions, number $x_k$ is a solution of equation $\tau_{10\ldots0}(x) = \omega(x)$ belonging to intervals of decrease of both functions. Hence $f_3$ is a continuous decreasing tail function.

**Theorem 5.** The set $C$ of all continuous bijections of interval $[0, g_0]$ preserving tails of $G_2$-representation of numbers with respect to composition (superposition) $\circ$ forms an infinite non-abelian group such that increasing functions form its non-trivial subgroup.

**Proof.** It is known that the set of all bijections of interval forms a group such that an identity transformation is its neutral element and an inverse transformation is its symmetric element. It is evident that composition of tail transformations is a tail transformation. The same is true for inverse transformation. Thus, by the subgroup test, $(C, \circ)$ is a group. From example 3 (where $k$ is an arbitrary positive integer) it follows that this group is infinite.

To prove that group $(C, \circ)$ is non-abelian it is enough to provide two transformations in the set $C$ that are not commute. To this end we consider function $f_1$ and $f_2$ from examples 1 and 2 and number $x_0$ that is less than $x_1$. Then we have

$$f_2(f_1(\Delta_{01(0)}^{G_2})) = f_2(\tau_1(\Delta_{01(0)}^{G_2})) = f_2(\Delta_{101(0)}^{G_2})$$

$$= \tau_1(\Delta_{101(0)}^{G_2}) = \Delta_{1101(0)}^{G_2}$$

because of $\Delta_{101(0)}^{G_2} < \Delta_{(101)}^{G_2}$;

$$f_1(f_2(\Delta_{01(0)}^{G_2})) = f_1(\tau_1(\Delta_{01(0)}^{G_2})) = f_1(\Delta_{101(0)}^{G_2})$$

$$= \omega(\Delta_{101(0)}^{G_2}) = \Delta_{01(0)}^{G_2},$$
because of $\Delta_{(011)}^{G_2} > \Delta_{(1)}^{G_2}$. Hence $f_2(f_1(x_0)) \neq f_1(f_2(x_0))$.

To prove that there exists a non-trivial subgroup of increasing functions it is enough to give an example of non-trivial increasing bijection $f \in C$. A such function is the following:

$$f_5(x) = \begin{cases} 
\omega(x) & \text{if } 0 \leq x \leq \Delta_{(011)}^{G_2}, \\
\tau_{11}(x) & \text{if } \Delta_{(011)}^{G_2} \leq x \leq \Delta_{(1)}^{G_2}, \\
x & \text{if } \Delta_{(1)}^{G_2} \leq x \leq g_0,
\end{cases}$$

because functions $\omega$, $\tau_{11}$ and $f(x) = x$ are increasing on given intervals, $\Delta_{(011)}^{G_2}$ is a solution of equation $\omega(x) = \tau_{11}(x)$, and $\Delta_{(1)}^{G_2}$ is a solution of equation $\tau_{11}(x) = x$.

Remark 3. The group $(C, \circ)$ is a proper subgroup of the group of transformations of interval $[0, g_0]$ preserving frequencies of digits of representation.

References


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