Decompositions of set-valued mappings

I. Protasov

On 100th anniversary of Professor V. S. Čarin

Abstract. Let $X$ be a set, $B_X$ denotes the family of all subsets of $X$ and $F : X \rightarrow B_X$ be a set-valued mapping such that $x \in F(x)$, $\sup_{x \in X} |F(x)| < \kappa$, $\sup_{x \in X} |F^{-1}(x)| < \kappa$ for all $x \in X$ and some infinite cardinal $\kappa$. Then there exists a family $F$ of bijective selectors of $F$ such that $|F| < \kappa$ and $F(x) = \{f(x) : f \in F\}$ for each $x \in X$. We apply this result to $G$-space representations of balleans.

1. Decompositions

For a set $X$, $B_X$ denotes the family of all subsets of $X$. Given a set-valued mapping $F : X \rightarrow B_X$, any function $f : X \rightarrow X$ such that, for each $x \in X$, $f(x) \in F(x)$ is called a selector of $F$. We say that a selector $f$ is bijective if $f : X \rightarrow X$ is a bijection. For $x \in X$, we denote $F^{-1}(x) = \{y \in X : x \in F(y)\}$.

In section 1 we prove the main result and apply it to $G$-space representations of balleans in section 2.

Theorem 1. Let $F : X \rightarrow B_X$ be a set-valued mapping such that $x \in F(x)$, $\sup_{x \in X} |F(x)| < \kappa$, $\sup_{x \in X} |F^{-1}(x)| < \kappa$ for each $x \in X$ and some infinite cardinal $\kappa$. Then there exists a family $F$ of bijective selectors of $X$ such that $|F| < \kappa$ and $F(x) = \{f(x) : f \in F\}$ for each $x \in X$.

*Victor Sil’vestrovich Čarin is known as the founder of topological algebra in Kyiv University, but his mathematical interests were not bounded by topological groups. He encouraged and supported the activity of students and collaborators in many areas, in particular, in combinatorics.

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Proof. We consider two cases.

Case $\kappa = \omega$. We put $\mathcal{P} = \{F(x) : x \in X\}$ and define a graph $\Gamma$ with the set of vertices $\mathcal{P}$ and the set of edges $\{\{F(x), F(y)\} : F(x) \cap F(y) \neq \emptyset\}$. We take a natural number $m$ such that $m > \sup_{x \in X} |F(x)|$, $m > \sup |F^{-1}(x)|$ and show that the local degree of each vertices of $\Gamma$ does not exceed $m^2 - 1$. Assume the contrary and choose $y \in X$ and distinct $y_1, \ldots, y_{m^2} \in X$ such that $F(y_i) \cap F(y_j) \neq \emptyset$ for every $i \in \{1, \ldots, m^2\}$. Then $y_i \in F^{-1}(F(y))$ but, by the choice of $m$, we have $|F^{-1}(F(y))| < m^2$.

We use the following simple fact [2]: if the local degree of each vertices of a graph $\Gamma'$ does not exceed $k$ then the chromatic number of $\Gamma'$ does not exceed $k + 1$.

Hence the set $\mathcal{P}$ of vertices of $\Gamma$ can be partition $\mathcal{P}_1, \ldots, \mathcal{P}_{m^2}$ so that any two vertices from each $\mathcal{P}_i$ are not incident.

To construct the family $\mathcal{F}$, we enumerate $\mathcal{P}_i = \{F(y_\alpha) : \alpha < \gamma\}$. Let $M = \sup_{x \in X} |F(x)|$. Then we enumerate each $F(y_\alpha)$ (with repetitions, if necessary) $F(y_\alpha) = \{y_{\alpha j} : j < M\}$, $y_{\alpha 0} = y_\alpha$. For each $j < M$, we define a bijective function $f_j$ such that $f_j$ acts as a transposition of $y_\alpha$ and $y_{\alpha j}$ at each $F(y_\alpha)$ and identically at all other elements of $X$. We put $\mathcal{F}_i = \{f_j : j < M\}$ and note that $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_{m^2}$ is the desired family of selectors of $F$.

Case $\kappa > \omega$. We take an infinite cardinal $\sigma$ such that $\sigma < \kappa$ and $|F(x)| \leq \sigma$, $|F^{-1}(x)| \leq \sigma$ for each $x \in X$. Then we define a partition $\mathcal{P}$ of $X$ such that each $P \in \mathcal{P}$ is the minimal by inclusion subset of $X$ satisfying $F(y) \in P$, $F^{-1}(y) \in P$ for each $y \in P$. Constructively, every $P$ can be obtained applying to $x \in P$ the sequence of operations $F$, $F^{-1} : F(x)$, $F^{-1}F(x)$, $FF^{-1}F(x)$, $\ldots$. Then $P$ is the union of all numbers of this sequence.

By the choice of $\sigma$, we have $|P| \leq \sigma$. We enumerate $\mathcal{P} = \{P_\alpha : \alpha < \gamma\}$, $P_\alpha = \{x_{\alpha j} : j < \gamma\}$. For each $j < \sigma$, we choose a family $\mathcal{F}_j$ of bijective selectors of $F$ such that $|F_j| \leq \sigma$ and $F(x_{\alpha j}) = \{f(x_{\alpha j}) : f \in \mathcal{F}_j\}$ for each $\alpha < \gamma$, see the case $\kappa = \omega$. Then $\bigcup_{j < \sigma} \mathcal{F}_j$ is the desired family $\mathcal{F}$ of bijective selectors of $F$. \hfill \square

2. Applications

Let $X$ be a set. A family $\mathcal{E}$ of subsets of $X \times X$ is called a coarse structure if

- each $E \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x, x) : x \in X\}$;
- if $E, E' \in \mathcal{E}$ then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x, y) : \exists z((x, z) \in E, (z, y) \in E')\}$, $E^{-1} = \{(y, x) : (x, y) \in E\}$;
• if $E \in \mathcal{E}$ and $\triangle X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;

• for any $x, y \in X$, there exists $E \in \mathcal{E}$ such that $(x, y) \in E$.

A subset $\mathcal{E}' \subseteq \mathcal{E}$ is called a base for $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $E \subseteq E'$. For $x \in X$, $A \subseteq X$ we denote $E[x] = \{y \in X : (x, y) \in E\}$, $E[A] = \cup_{a \in A} E[a]$ and say $E[x]$ and $E[A]$ are balls of radius $E$ around $x$ and $A$.

The pair $(X, \mathcal{E})$ is called a coarse space [6] or a ballean [5].

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be coarse spaces. A mapping $f : X \to X'$ is called macro-uniform if, for every $E \in \mathcal{E}$ there exists $E' \in \mathcal{E}'$ such that $E[x] \subseteq E'[f(x)]$. If $f$ is a bijection such that $f, f^{-1}$ are macro-uniform then $f$ is called an asymorphism.

Now we describe some general way of constructing balleans. Let $G$ be a group. A family $\mathcal{I}$ of subsets of $G$ is called an ideal if, for every $A, B \in \mathcal{I}$ and $A' \subseteq A$, we have $A \cup B \in \mathcal{I}$ and $A' \in \mathcal{I}$. An ideal $\mathcal{I}$ is called a group ideal if $F \in \mathcal{I}$ for every finite subset of $G$ and $A, B \in \mathcal{I}$ imply $AB^{-1} \in \mathcal{I}$.

Let a group $G$ acts transitively on a set $X$ by the rule $(g, x) \mapsto gx$, $g \in X$, $x \in X$. Every group ideal $\mathcal{I}$ on $G$ defines the ballean $(X, G, \mathcal{I})$ on $X$ with the base of entourages $\{(x, y) : y \in Ax \} : A \in \mathcal{I}$. By Theorem 1 from [3], for every ballean $(X, \mathcal{E})$, there exist a group $G$ of permutations of $X$ and a group ideal $\mathcal{I}$ on $G$ such that $(X, \mathcal{E})$ is asymorphic to $(X, G, \mathcal{I})$.

**Theorem 2.** Let $(X, \mathcal{E})$ be a ballean and let $\kappa$ be an infinite cardinal such that, for each $E \in \mathcal{E}$, $\sup_{x \in E} |E[x]| < \kappa$. Then there exist a group $G$ of permutations of $X$ and a group ideal $\mathcal{I}$ on $G$ such that $(X, \mathcal{E})$ is asymorphic to $(X, \mathcal{E}, \mathcal{I})$ and $|A| < \kappa$ for each $A \in \mathcal{I}$.

**Proof.** For each $E \in \mathcal{E}$, we define a mapping $F_E : X \to B_X$ by $F_E(x) = E[x]$. By Theorem 1, there exists a family $\mathcal{F}_E$ of permutations of $X$ such that $|\mathcal{F}_E| < \kappa$ and $F_E(x) = \{f(x) : f \in \mathcal{F}_E\}$ for each $x \in X$. We denote by $\mathcal{I}$ the minimal by inclusion group ideal of $G$ such that $\mathcal{F}_E \in \mathcal{I}$ for each $E \in \mathcal{E}$. Then $(X, \mathcal{E})$ is asymorphic to $(X, G, \mathcal{I})$. \hfill $\Box$

In the case $\kappa = \omega$, Theorem 2 was proved in [4]. For its applications see Remark 3.5 in [1].

A ballean $(X, \mathcal{E})$ is called cellular if $\mathcal{E}$ has a base consisting of equivalence relations. By Theorem 3 from [3], every cellular ballean is asymorphic to some ballean $(X, G, \mathcal{I})$ such that $\mathcal{I}$ has a base consisting of subgroups of $G$.

A ballean $(X, \mathcal{E})$ is called finitary if, for every $E \in \mathcal{E}$ there exists a natural number $m$ such $|E[x]| < m$ for each $x \in X$. The finitary ballean
of a $G$ space $X$ is the ballean $(X, G, \mathcal{I})$, where $\mathcal{I}$ is the ideal of all finite subsets of $G$.

**Theorem 3.** For every finitary cellular ballean $(X, \mathcal{E})$ there exists a locally finite group of permutations of $X$ such that $(X, \mathcal{E})$ is asymorphic to the finitary ballean of $G$-space $X$.

**Proof.** We take a base $\mathcal{E}'$ of consisting of partitions of $X$. For every $P \in \mathcal{E}$ we pick a natural number $n_P$ such that $|P| \leq n_P$ for each $P \in \mathcal{P}$. We denote by $G_P$ the direct product of the family of symmetric groups \{\(S_m : m \leq n_P\)} and note that $G_P$ acts on each $P \in \mathcal{P}$ so that $G_P x = P$ for each $x \in P$. Then the group $G$ generated by the family $\{G_P : P \in \mathcal{E}'\}$ satisfies the conclusion of Theorem 3. \( \square \)

**References**


**Contact Information**

Igor Protasov
Faculty of Computer Science and Cybernetics, Kyiv University, Academic Glushkov pr. 4d, 03680 Kyiv, Ukraine
E-Mail(s): i.v.protasov@gmail.com

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