Sets of prime power order generators of finite groups

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Abstract. A subset $X$ of prime power order elements of a finite group $G$ is called pp-independent if there is no proper subset $Y$ of $X$ such that $\langle Y, \Phi(G) \rangle = \langle X, \Phi(G) \rangle$, where $\Phi(G)$ is the Frattini subgroup of $G$. A group $G$ has property $B_{pp}$ if all pp-independent generating sets of $G$ have the same size. $G$ has the pp-basis exchange property if for any pp-independent generating sets $B_1, B_2$ of $G$ and $x \in B_1$ there exists $y \in B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\}$ is a pp-independent generating set of $G$. In this paper we describe all finite solvable groups with property $B_{pp}$ and all finite solvable groups with the pp-basis exchange property.

1. Introduction

Throughout this paper, all groups are finite. Let $G$ be a group. We denote by $\Phi(G)$ the Frattini subgroup of $G$ and we call a group with the trivial Frattini subgroup a Frattini-free group. For other notation, terminology and results one can consult for example [3, 4].

In this paper, our purpose is to extend the famous theorem of Burnside known as Burnside basis theorem. This theorem provides that the Frattini quotient of every $p$-group is an elementary abelian $p$-group. Hence it can

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be view as a vector space over the field of order \( p \). So generating sets of \( p \)-groups share properties with generating sets of vector spaces. However generating sets outside the class of \( p \)-groups do not have such properties, even generating sets of cyclic groups whose order is divisible by at least two different primes.

Obviously all elements of \( p \)-groups have prime power orders. So also in arbitrary groups we want to consider sets of prime power order generators. In this purpose we introduce the concept of a \( pp \)-element which simplifies our considerations. So we say that an element \( g \in G \) is a \( pp \)-element if it has prime power order, while by \( p \)-element, as usual, we mean an element of order being a power of a prime \( p \). Many authors have studied similar problems concerning sets of not only \( pp \)-generators, see for instance [1, 8, 9, 11] and the reference therein. In particular in [1] groups in which all minimal generating sets have the same size are classified.

A subset \( X \) of \( pp \)-elements of a group \( G \) will be called \( pp \)-independent if \( \langle Y, \Phi(G) \rangle \neq \langle X, \Phi(G) \rangle \) for every \( Y \subset X \) and a \( pp \)-base of \( G \) if \( X \) is a \( pp \)-independent generating set of \( G \). We say that a finite group \( G \)

- has property \( B_{pp} \) (is a \( B_{pp} \)-group for short) if all \( pp \)-bases of \( G \) have the same size;
- has the \( pp \)-embedding property if every \( pp \)-independent set of \( G \) can be embedded to a \( pp \)-base of \( G \);
- has the \( pp \)-basis exchange property if for any two \( pp \)-basis \( B_1, B_2 \) and \( x \in B_1 \) there exists \( y \in B_2 \) such that \( (B_1 \setminus \{x\}) \cup \{y\} \) is a \( pp \)-base of \( G \).
- is a \( pp \)-matroid group if \( G \) has property \( B_{pp} \) and the \( pp \)-embedding property.

In view of the above definitions, Burnside basis theorem provides that all finite \( p \)-groups are \( pp \)-matroid and have the \( pp \)-basis exchange property. Another example, outside the class of \( p \)-groups, is a group called a scalar extension. After [6] we say that \( G \) is a scalar extension if \( G = P \rtimes Q \), where \( P \) is an elementary abelian \( p \)-group, \( Q \) is a non-trivial cyclic \( q \)-group for distinct primes \( p \neq q \) such that \( Q \) acts faithfully on \( P \) and the \( \mathbb{F}_p \[Q\]-module \( P \) is a direct sum of isomorphic copies of one simple module. This construction will be constantly use in our further considerations. A scalar extension is not always a \( pp \)-matroid group (only if \( Q \) has prime order, see [11]) but every scalar extension is a \( B_{pp} \)-group (see [8]).

Our focus of interest is to study the structure of groups which have one of the properties listed above. Solvable groups with the \( pp \)-embedding property were studied in [10, 11]. In [7] all \( pp \)-matroid groups were de-
scribed. Moreover in [7] it was proved that pp-matroid groups have the pp-basis exchange property.

The properties of pp-matroid groups imply that every maximal pp-independent set of a pp-matroid group $G$ is a pp-base of $G$. Let $I$ be the family of all pp-independent sets of $G$. Then the pair $(I, G)$ forms a matroid where every pp-base of $G$ is a base of a matroid $(I, G)$ (see [12]). Hence pp-matroid groups can be view as a generalization of $p$-groups in the sense of generating sets. Thus the aim of this paper is to describe the structure of solvable groups with property $B_{pp}$ and the structure of solvable groups with the pp-basis exchange property.

By [6, Theorem 4.2], we know that every pp-independent set (pp-base) of $G/\Phi(G)$ may be lifted to a pp-independent set (pp-base) of $G$. Hence using properties of the Frattini subgroup we obtain the following

**Theorem 1.1.** A group $G$ has property $B_{pp}$, the pp-embedding property, the pp-basis exchange property if and only if $G/\Phi(G)$ has, respectively, property $B_{pp}$, the pp-embedding property and the pp-basis exchange property. In particular $G$ is pp-matroid if and only if $G/\Phi(G)$ is pp-matroid.

Based on the above theorem we may restrict our consideration to Frattini-free groups. The structure of the paper is as follows. We present our concepts and main results in Sections 1. In Section 2 we present the classification of all solvable groups with property $B_{pp}$. The proof of Theorem 1.2 is presented in Section 3.

**Theorem 1.2.** Let $G$ be a Frattini-free solvable group. Then $G$ has property $B_{pp}$ if and only if it is one of the following groups:

1) an elementary abelian $p$-group;
2) a scalar extension;
3) a direct product of groups given in (1) and (2) with pairwise coprime orders.

Using the above theorem we describe in Section 3 solvable groups with the pp-basis exchange property. The proof of Theorem 1.3 is presented in Section 4.

**Theorem 1.3.** Let $G$ be a Frattini-free solvable groups. Then $G$ has the pp-basis exchange property if and only if $G$ is a $B_{pp}$-group such that all $pp$-elements of $G$ have prime orders.
2. Groups with property $\mathcal{B}_{pp}$

In this section we present the classification of groups with property $\mathcal{B}_{pp}$. First results concerning a $\mathcal{B}_{pp}$-groups appear in [6, 8]. We recall some of these results which we will apply in further proofs.

**Theorem 2.1 ([6]).** Let $G = P \rtimes Q$ be a non-trivial semidirect product, where $P$ is a $p$-group and $Q$ is a cyclic $q$-group, for distinct primes $p \neq q$. Then the following conditions are equivalent:

1) $G$ is a $\mathcal{B}_{pp}$-group.
2) $G/\Phi(G)$ is a scalar extension.

Furthermore, suppose that the above conditions hold. Then all minimal generating sets of $G$ have the same size.

**Theorem 2.2 ([8]).** Let $G$ be a group and $G/\Phi(G)$ be a scalar extension. Then

1) $G$ has a unique Sylow $p$-subgroup $P$;
2) $G = P \rtimes Q$ for a Sylow $q$-subgroup $Q$ and all Sylow $q$-subgroups of $G$ are cyclic;
3) $\Phi(G) = \Phi(P) \times \langle x^q \rangle$, where $x$ is a generator of $Q$. Moreover, $x^q$ centralizes $P$.

**Theorem 2.3 ([6]).** If $G$ is a $\mathcal{B}_{pp}$-group, then every homomorphic image of $G$ is also a $\mathcal{B}_{pp}$-group.

**Theorem 2.4 ([6]).** Let $G_1$ and $G_2$ be groups with coprime orders. Then $G_1$ and $G_2$ are $\mathcal{B}_{pp}$-groups if and only if $G_1 \times G_2$ is a $\mathcal{B}_{pp}$-group.

3. Solvable $\mathcal{B}_{pp}$-groups

In this section we investigate the finite solvable $\mathcal{B}_{pp}$-groups. The following lemmas will be needed for proving Theorem 1.2.

**Remark 3.1.** Let $G$ be a solvable group and $G = P \rtimes H$, where $P$ is a minimal normal subgroup of $G$ and $C_H(P) = 1$. Assume that $d \geq 2$ is a size of a minimal generating set of $G$. Theorem 7 of [2] follows that there exists a minimal generating set $\{h_1, \ldots, h_d\}$ of $H$ such that $\langle h_1^{x_1}, \ldots, h_d^{x_d} \rangle = G$ for some $x_1, \ldots, x_d \in P$. We say then, after the authors, that $(P, H)$ does not satisfy the strong complement property.

**Lemma 3.2.** Let $G$ be a solvable group with a minimal normal $p$-subgroup $P$, and let $H$ be a complement to $P$ in $G$, where $p$ is a prime and $p$ does not divide $|H|$. Assume that $H$ is a non-cyclic $q$-group for some prime $q$
or \( H/\Phi(H) \) is a scalar extension. If \( H \) acts non-trivially on \( P \), then \( G \) is not a \( B_{pp} \)-group.

**Proof.** By assumption, \( H \) is a \( q \)-group or \( H/\Phi(H) \) is a scalar extension. Hence, by Theorem 2.1, all minimal generating sets of \( H \) have the same size, say \( d \). Assume \( d \geq 2 \). Since \( H \) acts non-trivially on \( P \), by Remark 3.1 there exists a minimal generating set \( \{h_1, \ldots, h_d\} \) of \( H \) such that \( \langle h_1^{x_1}, \ldots, h_d^{x_d} \rangle = G \) for some \( x_1, \ldots, x_d \in P \). Observe that \( \{h_1^{x_1}, \ldots, h_d^{x_d}\} \) is a generating set of pp-elements of \( G \). Hence there exists a pp-base \( B' \subseteq \{h_1^{x_1}, \ldots, h_d^{x_d}\} \) of \( G \) such that \( |B'| \leq n \).

On the other hand \( P = \langle a \rangle H \), for every \( 1 \neq a \in P \). Hence \( \{a, h_1, \ldots, h_n\} \) is a pp-base of \( G \). Thus \( G \) is not a \( B_{pp} \)-group. \( \square \)

**Lemma 3.3.** Let \( G \) be a solvable group with a minimal normal \( p \)-subgroup \( P \), and let \( H \) be a nilpotent complement to \( P \) in \( G \), where \( p \) is a prime and \( p \) does not divide \( |H| \). If \( G \) is an indecomposable \( B_{pp} \)-group, then \( H \) is a cyclic \( q \)-group for some prime divisor \( q \) of \( |H| \).

**Proof.** Assume that \( H = P_1 \times \ldots \times P_n \), where \( P_i \) is a Sylow \( p_i \)-subgroup of \( H \) and \( [P_i, P_i] \neq 1 \), for \( i = 1, \ldots, n \). Let \( 1 \neq a \in P \) and \( B_i \) be a pp-base of \( P_i \), for \( i = 1, \ldots, n \). Then \( \{a\} \cup B_1 \cup \ldots \cup B_n \) is a pp-base of \( G \). Moreover assume that \( B_1 = \{x_1, \ldots, x_k\} \) and \( B_2 = \{y_1, \ldots, y_l\} \). Hence there exist \( c_1, \ldots, c_k \in P \) such that \( x_1^a = c_1 x_1, \ldots, x_k^a = c_k x_k \). Observe that \( (x_j^a)^{y_j} (x_i^a)^{-1} = (c_i x_i)^{y_j} (c_i x_i)^{-1} = c_i^{y_j} c_i^{-1} \neq 1 \) for at least one \( j \in \{1, \ldots, l\} \). So \( \{x_1^a, \ldots, x_k^a, y_1, \ldots, y_l\} \cup B_3 \cup \ldots \cup B_k \) is a pp-base of \( G \), a contradiction. Hence only one \( P_i \) acts non-trivially on \( P \). Without loss of generality we can set \( i = 1 \). Then \( G = (P \times P_1) \times (P_2 \times \ldots \times P_n) \), in contradiction to our assumption. This contradiction implies that \( H = P_1 \). Thus, by Lemma 3.2, \( H \) is a cyclic \( q \)-group with \( q = p_1 \). \( \square \)

**Lemma 3.4.** Let \( G \) be a solvable group with a minimal normal \( p \)-subgroup \( P_1 \), where \( p \) is a prime and let \( H \) be a complement to \( P \) in \( G \). Assume that \( H = Q \times P_2 \), where \( Q \) is a \( q \)-group for some prime \( q \neq p \) and \( P_2 \) is a cyclic \( p \)-group such that \( H/\Phi(H) \) is a scalar extension. Then \( G \) is not a \( B_{pp} \)-group.

**Proof.** Since \( P_1 \) is a minimal normal subgroup of \( G \) and \( G \) is solvable, \( P_1 \) is elementary abelian and \( \langle g \rangle^H = P_1 \) for all \( 1 \neq g \in P_1 \). Let \( P_2 = \langle y \rangle \) and \( \{x_1, \ldots, x_n\} \subseteq Q \) be a minimal set such that \( \langle x_1, \ldots, x_n \rangle^{P_2} = Q \). Hence, by the assumption, \( \{g, x_1, \ldots, x_n, y\} \) is a pp-base of \( G \). We need to consider the following cases:

1. \( [P_1, Q] \neq 1, [P_1, P_2] \neq 1 \);
2. \([P_1, Q] = 1, [P_1, P_2] \neq 1;\)
3. \([P_1, Q] \neq 1, [P_1, P_2] = 1;\)
4. \([P_1, Q] = 1, [P_1, P_2] = 1.\)

1. In this case there exists \(a \in P_1\) such that \(Q^a \neq Q.\) From [4, Theorem 2.3], we know that \(P_1 = C_{P_1}(Q) \times [P_1, Q].\) Assume first that \(C_{P_1}(Q) \neq 1.\) Since \(P_1\) is a minimal normal subgroup of \(G,\) there exists \(a \in C_{P_1}(Q)\) such that \(a^{y^{-1}} \notin C_{P_1}(Q).\) Set \(b = a^{y^{-1}}.\) Then \(b^y = a \in C_{P_1}(Q)\) and \(b^{-1}b^{y^{-1}} \notin C_{P_1}(Q).\) It follows that \(Q^b \neq Q^{b^{y^{-1}}}.\) Since \(H/\Phi(H)\) is a scalar extension,
\[
Q/\Phi(Q) = Q_1\Phi(Q)/\Phi(Q) \times \ldots \times Q_n\Phi(Q)/\Phi(Q),
\]
where \(Q_i\Phi(Q)/\Phi(Q)\) is a simple \(\mathbb{F}_q[Q]\)-module. Hence \(\langle x_i \rangle = Q_i,\) for every \(x_i \in Q_i \setminus \Phi(Q).\) Moreover \(Q = Q_1 \ldots, Q_n.\) It follows that there exists \(Q_i\) such that \(Q_i^b \neq Q_i,\) for some \(i = 1, \ldots, n.\) Thus at least one element, say \(x_i,\) satisfies \(x_i^b \notin Q_i.\) Consider the set \(X = \{y, x_1^b, \ldots, x_n^b\}.\) Observe that
\[
x_i^{by} = x^{yy^{-1}by} = (x_i^y)^{by}.
\]
Since \(x_i^y \in Q\) and \(b^y \in C_{P_1}(Q),\) we have \(x_i^{by} = x_i^y \in \langle X \rangle.\) Hence \(x_i \in \langle X \rangle\) and further \(c = x_i^{-1}x_i^b \in P_1 \cap \langle X \rangle,\) where \(c \neq 1.\) It follows that \(G = \langle X \rangle\) and \(X\) is a generating set of pp-elements of \(G.\) So there exists a pp-base \(B \subseteq X\) of \(G\) such that \(|B| < n + 2 = |\{g, x_1, \ldots, x_n, y\}|.\)

Assume now that \(C_{P_1}(Q) = 1\) and \(C_{P_1}(P_2) \neq 1.\) So \(P_1 = [P_1, Q]\) and hence there exists \(c \in C_{P_1}(P_2),\) where \(c = [a, x_1^{-1}],\) for some \(a \in P_1\) and \(x_1 \in Q \setminus \Phi(Q).\) Thus there exist \(x_2, \ldots, x_n \in Q\) such that \(\langle x_1, \ldots, x_n \rangle^{P_2} = Q.\) Let \(X = \{x_1^a, \ldots, x_n^a\}.\) Since \(x_1^a = [a, x_1^{-1}]x_1 = cx_1\) for some \(1 \neq c \in P_1,\) we have \((x_1^a)^{-1}(x_1^a)^y = x_1^{-1}x_1^y.\) Moreover \(1 \neq x_1^{-1}x_1^y \in \langle X \rangle.\) Hence \(\langle x_1 \rangle^{P_2} \subseteq \langle X \rangle\) and \(x_1 \in \langle X \rangle.\) So \(1 \neq x_1^{-1}x_1^a \in X \cap P_1\) and \(G = \langle X \rangle.\) It follows that \(\{a, x_1, \ldots, x_n, y\}\) and \(\{y, x_1^a, \ldots, x_n^a\}\) are a pp-base of \(G.\)

2. Now there exists \(a \in P_1\) and at least one \(x_i,\) say \(x_i\) such that \([a, y] \neq 1 \neq [x_1, y].\) Then \(y^a(y^{x_1})^{-1} = a^{-1}a^{y^{-1}}x_1^{-1} \neq 1.\) It follows that \(\{y^a, y^{x_1}, x_2, \ldots, x_n\}\) is a pp-base of \(G.\)

3. Since \([P_1, Q] \neq 1,\) there exists \(a \in P_1\) such that \(Q \neq Q^a.\) Moreover \(P_1 = [P_1, Q].\) Otherwise \(C_{P_1}(Q)\) is a normal subgroup of \(G,\) contradicting the minimality of \(P_1.\) Hence there exist \(c_1, \ldots, c_n \in P_1\) such that \(x_1^{a_1} = c_1x_1, \ldots, x_n^{a_n} = c_nx_n.\) So we obtain \((x_1^{a})^{-1}(x_1^a)^y = x_1^{-1}x_1^y \neq 1.\) It follows that \(\{x_1^a, \ldots, x_n^a, y\}\) is a pp-base of \(G.\)

4. In this case \(\{g, x_1, x_2, \ldots, x_n, y\}\) and \(\{gx_1y, x_2, \ldots, x_n, y\}\) are pp-bases of \(P \times H.\)

Hence \(G\) is not a \(B_{pp}\)-group in all the cases. So the proof is complete. \(\square\)
Remark 3.5. Let $G$ be a solvable group with a minimal normal $p$-subgroup $P$, where $p$ is a prime and let $H$ be a complement to $P$ in $G$. Assume that $H/\Phi(H)$ is a scalar extension. It follows, by Theorem 2.1, that $H$ is a $\mathcal{B}_{pp}$-group and we may assume that $d$ is the size of every pp-bases of $H$, for some positive integer $d$. Then from proofs of Lemmas 3.2, 3.4 we immediately deduce that there exist pp-bases $B_1$ and $B_2$ of $G$ such that $|B_1| = n + 1$ and $|B_2| < n + 1$.

Proof of Theorem 1.2. Let $G$ be a Frattini-free solvable group with property $\mathcal{B}_{pp}$. We use induction on $|G|$. Let $P = O_p(G)$ be a maximal normal $p$-group of $G$, for some prime $p$. Hence $\Phi(P) \leq \Phi(G) = 1$ and $P$ is an elementary abelian $p$-group. By [3, Theorem 10.6], there exists a subgroup $H$ of $G$ such that $G = P \rtimes H$. From Theorem 2.3, $H$ is a $\mathcal{B}_{pp}$-group. So by the induction assumption $H = H_1 \times \ldots \times H_k$, where $H_i/\Phi(H_i)$ is an elementary abelian $q$-group or a scalar extension for $i = 1, \ldots, k$. By [3, Theorem 10.6], $P = P_1 \times \ldots \times P_n$, where $P_i$ is a minimal normal subgroup of $G$, for $i \in \{1, \ldots, n\}$.

Let $a_i \in P_i$ be a non-trivial element for $i \in \{1, \ldots, n\}$ and $\{h_1, \ldots, h_r\}$ be a pp-base of $H$. Then $B = \{a_1, \ldots, a_n, h_1, \ldots, h_r\}$ is a pp-base of $G$. Assume that $P_i$ and $P_j$ are not isomorphic as $\mathbb{F}_p[H]$-modules for some $i \neq j$. Then $B' = (B \setminus \{a_i, a_j\}) \cup \{a_ia_j\}$ is a pp-base of $G$. Since $|B'| = |B| - 1$, $G$ is not a $\mathcal{B}_{pp}$-group, a contradiction. So all the $P_i$ are isomorphic to each another as $\mathbb{F}_p[H]$-modules. In particular, this implies that $C_H(P_i) = 1$ for each $P_i$.

Again, by Theorem 2.3, we may suppose that $P$ is a minimal normal subgroup of $G$. Thus $G = P \rtimes (H_1 \times \ldots \times H_k)$ and $P = \langle a \rangle^H$, for every $1 \neq a \in P$. Assume that $B_i$ is a pp-base of $H_i$ for $i = 1, \ldots, k$. Then $\{a\} \cup B_1 \cup \ldots \cup B_k$ is a pp-base of $P \rtimes H$, as $|\langle H_i \rangle|, |\langle H_j \rangle| = 1$, for $i \neq j$.

If for some $i \in \{1, \ldots, k\}$, $H_i$ is a $q$-group, then $q \neq p$, by the choice of $P$. So suppose that $H_1/\Phi(H_1)$ is a scalar extension and $|P, H_1| \neq 1$. Let $P_1 = \langle a \rangle^{H_1} \leq P$, for some $a \in P_1$. Then $\{a\} \cup B_1$ is a pp-base of $P_1 \rtimes H_1$. Moreover, by Remark 3.5, there exists a pp-base, say $B$ of $P_1 \rtimes H_1$, such that $|B| < |B_1| + 1$. Observe that $B \cup B_2 \cup \ldots \cup B_k$ is a generating set of pp-elements of $G$. So there exists a pp-base $C \subseteq B \cup B_2 \cup \ldots \cup B_k$ of $G$ such that $|C| < |\{a\} \cup B_1 \cup \ldots \cup B_k|$, a contradiction. Hence either $|P, H_1| = 1$ or $H_1$ is a $p$-group. If $|P, H_1| = 1$ and $P$ and $H$ have not coprime orders, then by Case 4. of Lemma 3.4 and analogous consideration as the above, we obtain that $G$ is not a $\mathcal{B}_{pp}$-group.

It follows that if $H_i$ is not a $q$-group, then $H_i$ centralises $P$. It implies that $G = [P \rtimes (H_1 \times \ldots \times H_k)] \rtimes H_{r+1} \times \ldots \times H_k$. Moreover $H_i$ is a $q_i$-group,
where \([H_i, P] \neq 1\) and \((q_i, p) = 1\) for \(i = 1, \ldots, r\) while \(|H_i| |p| = 1\), for \(i = r + 1, \ldots, k\).

Further, by Theorem 2.3, \(P \rtimes (H_1 \times \cdots \times H_r)\) is a \(B_{pp}\)-group. Then, by Lemma 3.3, only one \(H_i\) acts non-trivially on \(P\) and such \(H_i\) is cyclic. It follows that \(G = (P \rtimes H_1) \times H_2 \times \cdots \times H_k\), where \(H_1\) is a cyclic \(q\)-group, and \(|P \rtimes H_1||H_j| = 1\) for \(j = 2, \ldots, n\). Moreover, by Theorems 2.1, 2.3, \(P \rtimes H_1\) is a scalar extension or is abelian.

Conversely, let \(G = G_1 \times \cdots \times G_n\), where \(G_i\) is either an elementary abelian \(p\)-group or a scalar extension and \(|G_i| |G_j| = 1\) for \(i \neq j\). Then, by Theorem 2.1, every direct factor of \(G\) is a \(B_{pp}\)-group. Hence, by Theorem 2.4, \(G\) is a \(B_{pp}\)-group. So the proof is complete. \(\square\)

4. Solvable groups with the pp-basis exchange property

In this section we investigate the structure of finite solvable groups which have the pp-basis exchange property. We start from the statement analogous to Theorem 2.4.

**Proposition 4.1.** Let \(G_1\) and \(G_2\) be groups with coprime orders. Then \(G_1\) and \(G_2\) have the pp-basis exchange property if and only if \(G_1 \times G_2\) has the pp-basis exchange property.

**Proof.** Since \(G_1\) and \(G_2\) have coprime orders, an element \(g = g_1g_2 \in G_1 \times G_2\), where \(g_1 \in G_1, g_2 \in G_2\), is a pp-element if and only if \(g = g_1\) or \(g = g_2\). Hence \(B\) is a pp-base of \(G_1 \times G_2\) if and only if \(B = B_1 \cup B_2\), where \(B_1, B_2\) are pp-bases of \(G_1, G_2\), respectively. From here the result follows immediately. \(\square\)

**Proposition 4.2.** Let \(G\) be a Frattini-free solvable group. If \(G\) has the pp-basis exchange property, then \(G\) has property \(B_{pp}\) and all pp-elements of \(G\) have prime orders.

**Proof.** Assume that \(B_1, B_2\) are two pp-bases of \(G\) such that \(|B_1| < |B_2|\). We choose \(B_1\) and \(B_2\) with the property that \(|B_2 \setminus B_1|\) is minimal. Let \(x \in B_2 \setminus B_1\). Since \(G\) has the pp-basis exchange property, there exists \(y \in B_1 \setminus B_2\) such that \((B_2 \setminus \{x\}) \cup \{y\}\) is a pp-base of \(G\). Moreover \(|(B_2 \setminus \{x\}) \cup \{y\} \setminus B_1| < |B_2 \setminus B_1|\). This contradicts the minimality of \(|B_1 \setminus B_2|\). So \(G\) is a \(B_{pp}\)-group.

Now, by Theorem 1.2, \(G = G_1 \times \cdots \times G_k\) where \(G_i\) is either an elementary abelian \(p\)-group or a scalar extension. If \(G_i\) is elementary abelian, then obviously all elements have prime orders. So assume that \(G_i = P \rtimes \langle x \rangle\) is
a scalar extension and \( x \) is a \( q \)-element. Moreover assume \( x^q \notin C_G(P) \).

Let \( a_1, \ldots, a_s \in P \) be a minimal set such that \( \langle a_1, \ldots, a_s \rangle^{(x)} = P \). Hence \( B_1 = \{ a_1x, a_2, \ldots, a_s, x^q \} \) and \( B_2 = \{ a_1, \ldots, a_s, x \} \) are pp-bases of \( G_i \).

Moreover \( \langle (B_1 \setminus \{ a_1x \}) \cup \{ y \} \rangle \neq G_i \) for every \( y \in B_2 \). Hence \( G_i \) has not the pp-basis exchange property. So, by Theorem 4.1, \( G \) also has not the pp-basis exchange property, a contradiction. Hence \( x^q \in C_G(P) \). Since \( G \) is Frattini-free, it follows, by Theorem 2.2, that all pp-elements of \( G_i \) have prime orders.

**Lemma 4.3.** Let \( G = P \times Q \) be a scalar extension. If all pp-elements of \( G \) have prime orders, then \( G \) has the pp-basis exchange property.

**Proof.** Assume that \( |Q| = q \), then all pp-elements of \( G \) have prime orders and all pp-basis have \( n \) elements. Let \( B_1 = \{ x_1, \ldots, x_n \} \) and \( B_2 = \{ y_1, \ldots, y_n \} \) be pp-bases of \( G \). Assume that \( x_1 \notin B_2 \) and \( H = \langle x_2, \ldots, x_n \rangle \).

We show that \( H \) is a maximal subgroup of \( G \). In this purpose we consider two cases:

1. \( x_1 \in P \). Then \( \langle x_1 \rangle^Q \) is a minimal normal subgroup of \( G \) and \( H = P/\langle x_1 \rangle^Q \times Q^a \), where \( a \in P \). So \( H \) is a maximal subgroup of \( G \).

2. \( x_1 \notin P \). Then \( x_1 \) is a \( q \)-element, where \( q \) is a prime and \( Q \) is a \( q \)-group. Since \( H \nsubseteq P \) there exists in \( H \) another \( q \)-element, say \( x_2 \). We may assume that \( x_1 = x^{a_1} \) and \( x_2 = x^{a_2} \), where \( a_1a_2^{-1} \notin C_P(Q) \). Hence \( x^{a_1}x^{-a_2} = c \in P \) and \( c \notin H \). Indeed, if \( c \in H \) and \( x^{a_2} \in H \), then \( x^{a_1} \in H \), a contradiction. It follows, by analogous as in (1), that \( H \) is a maximal subgroup in \( G \).

By assumption, there exists \( y_i \notin H \) for some \( i \in \{ 1, \ldots, n \} \). Since \( H \) is a maximal subgroup of \( G \), \( \langle H, y_i \rangle = G \). It follows that \( \langle (B_1 \setminus \{ x_1 \}) \cup \{ y_1 \} \rangle = G \) and \( |(B_1 \setminus \{ x_1 \}) \cup \{ y_1 \}| = n \). Since \( G \) is a \( B_{pp} \)-group, \( (B_1 \setminus \{ x_1 \}) \cup \{ y_1 \} \) is a pp-base of \( G \). The proof is complete.

**Proof of Theorem 1.3.** It follows immediately from Proposition 4.1, Proposition 4.2 and Lemma 4.3.

Using Theorem 1.2 we obtain

**Corollary 4.4.** Let \( G \) be a Frattini-free solvable group. Then \( G \) has the pp-basis exchange property if and only if it is one of the following groups:

1) an elementary abelian \( p \)-group;

2) a scalar extension \( P \times Q \), where \( P \) is an elementary abelian \( p \)-group, \( Q \) has order \( q \) for distinct primes \( p \neq q \);

3) a direct product of groups given in (1) and (2) with pairwise coprime orders.
Remark 4.5. By [5], we know that every simple group is generated by an involution and an element of prime order. So a simple group has a 2-element pp-base. On the other hand, by the Classification of Finite Simple Group, we know that every simple group is generated by at least three involutions, so every simple group has a pp-base which has at least 3 elements. It implies that all simple groups do not have property $B_{pp}$. By the first part of the proof of Proposition 4.2, we may deduced that if a simple group has not property $B_{pp}$, then it has not the pp-basis exchange property.

References


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