© Algebra and Discrete Mathematics
Volume **30** (2020). Number 1, pp. 97–117
DOI:10.12958/adm1451

RESEARCH ARTICLE

On growth of generalized Grigorchuk's overgroups

S. T. Samarakoon

Communicated by V. Nekrashevych

Abstract. Grigorchuk's Overgroup $\widetilde{\mathcal{G}}$, is a branch group of intermediate growth. It contains the first Grigorchuk's torsion group $\mathcal{G}$ of intermediate growth constructed in 1980, but also has elements of infinite order. Its growth is substantially greater than the growth of $\mathcal{G}$. The group $\mathcal{G}$, corresponding to the sequence $(012)^\infty = 012012\ldots$, is a member of the family $\{G_\omega | \omega \in \Omega = \{0,1,2\}^{\mathbb{N}}\}$ consisting of groups of intermediate growth when sequence ω is not eventually constant. Following this construction, we define the family $\{\widetilde{G}_\omega, \omega \in \Omega\}$ of generalized overgroups. Then $\widetilde{\mathcal{G}} = \widetilde{G}_{(012)^\infty}$ and G_ω is a subgroup of $\widetilde{G}_\omega$ for each $\omega \in \Omega$. We prove, if ω is eventually constant, then $\widetilde{G}_\omega$ is of polynomial growth and if ω is not eventually constant, then $\widetilde{G}_\omega$ is of intermediate growth.

Introduction

The growth rate of groups is a long studied area [15, 17, 20] and it was known that growth rates of groups can vary from polynomial growth through intermediate growth to exponential growth. First group of intermediate growth (the growth which is neither polynomial nor exponential), known as the first Grigorchuk's torsion group $\mathcal{G}$, was constructed by Rostislav Grigorchuk in 1980 [10] as a finitely generated infinite torsion group and later [11] it was shown that it has intermediate growth. The growth rate

2010 MSC: 20E08.
Key words and phrases: growth of groups, intermediate growth, Grigorchuk group, growth bounds.

$\gamma_{\mathcal{G}}(n)$ of $\mathcal{G}$ was first shown to be bounded below by $e^{\sqrt{n}}$ and bounded above by e^{n^β}, where $\beta = \log_{32} 31 \approx 0.991$ [11,12]. In 1998, Laurent Bartholdi [1] and in 2001, Roman Muchnik and Igor Pak [18] independently refined the upper bound to $\gamma_{\mathcal{G}}(n) \preceq e^{n^\alpha}$, where $\alpha = \log(2)/\log(2/\eta) \approx 0.767$ and η is the real root of the polynomial $x^3 + x^2 + x - 2$. Recent work of Anna Erschler and Tianyi Zheng [8] showed $\gamma_{\mathcal{G}}(n) \succeq e^{n^{(\alpha-\epsilon)}}$ for any positive ϵ.

At the same time, in [11, 12] (also see [14]) an uncountable family of groups $\{G_\omega \colon \omega \in \Omega = \{0,1,2\}^{\mathbb{N}}\}$, known as generalized Grigorchuk's groups were constructed. They consist of groups of intermediate growth when sequence ω is not virtually constant and of polynomial growth when sequence ω is virtually constant [12].

Since the construction of the first Grigorchuk group, there was an expansion of the area of study and new groups of intermediate growth were introduced [3,4,13,16,19]. The group $\widetilde{\mathcal{G}}$ known as the Grigorchuk's overgroup [5] is an infinite finitely generated group of intermediate growth which shares many properties with first Grigorchuk's group [6]. In contrast, the Grigorchuk's overgroup has an element of infinite order [5]. As a corollary to Proposition 4 and Theorem 2″ of present article, the growth rate $\gamma_{\widetilde{\mathcal{G}}}(n)$ of $\widetilde{\mathcal{G}}$ satisfies,

$$\exp\left(\frac{n}{\log^{2+\epsilon} n}\right) \preceq \gamma_{\widetilde{\mathcal{G}}}(n) \preceq \exp\left(\frac{n\log(\log n)}{\log n}\right)$$

for any $\epsilon > 0$.

First introduced technique for getting an upper bound for $\mathcal{G}$ uses the strong contraction property [12] (also known as sum contraction property), which says that there is a finite index subgroup H of $\mathcal{G}$ such that any element $g \in H$ can be uniquely decomposed into some elements, whose sum of lengths in not larger than $C|g| + D$, where $0 < C < 1$ and D are constants independent of g [12]. Later this technique was developed and many variants were introduced [2,7,9]. In 2004, to get a lower bound for a certain class of groups of intermediate growth, Anna Erschler introduced a method for partial description of the Poisson boundary [7]. This idea was used to get the current known best lower bound for the growth of $\mathcal{G}$ [8]. We will be using a version of strong contraction property in this text.

Following the construction in [12], we introduce an uncountable family $\{\widetilde{G}_\omega, \omega \in \Omega\}$ called generalized Grigorchuk's overgroups, where $\Omega = \{0,1,2\}^{\mathbb{N}}$ (see Section 1.1).

Theorem 1. *Let $\omega \in \Omega$. Then $\widetilde{G}_\omega$ is of polynomial growth if ω is virtually constant and $\widetilde{G}_\omega$ is of intermediate growth if ω is not virtually constant.*

Let Ω_0, Ω_1 be subsets of Ω, where Ω_0 is the set consisting of all sequences containing $0, 1$ and 2 infinitely often, Ω_1 is the set consisting of sequences containing exactly two symbols infinitely often. Define Ω_0^* to be the subset of Ω_0 containing sequences $\omega = \{\omega_n\}$, such that there is an integer $M = M(\omega)$ with the property that for all $k \geqslant 1$, the set $\{\omega_k, \omega_{k+1}, \ldots, \omega_{k+M-1}\}$ contains all three symbols 0,1 and 2. Similarly, define Ω_1^* to be the subset of Ω_1 containing sequences $\omega = \{\omega_n\}$, such that there is an integer $M = M(\omega)$ with the property that for all $k \geqslant 1$, the set $\{\omega_k, \omega_{k+1}, \ldots, \omega_{k+M-1}\}$ contains at least two symbols. Let $\Omega^* = \Omega_0^* \cup \Omega_1^*$. Sequences in Ω^* are called homogeneous sequences.

Theorem 2. *Let $\omega \in \Omega^*$. Then*

$$\gamma_{\widetilde{G}_\omega}(n) \preceq \exp\left(\frac{n \log(\log n)}{\log n}\right).$$

Theorem 2 provides an upper bound for growth of $\widetilde{G}_\omega$ only for homogeneous sequences. In fact, it is impossible to give a unifying upper bound for growth of $\widetilde{G}_\omega$, for all $\omega \in \Omega_0 \cup \Omega_1$. This follows from Theorem 7.1 of [12], together with the fact that $G_\omega \subset \widetilde{G}_\omega$. However, it is possible to provide a unifying lower bound for the growth of groups $\widetilde{G}_\omega$ for all $\omega \in \Omega_0 \cup \Omega_1$ by a function of type $\exp\left(\dfrac{n}{\log^{2+\epsilon}(n)}\right)$ for arbitrary $\epsilon > 0$ (see Proposition 4).

We prove Theorem 1 in Section 2 and Theorem 2 in Section 3.

1. Preliminaries

First recall $\Omega = \{0, 1, 2\}^{\mathbb{N}}$ and $\Omega_0, \Omega_1 \subset \Omega$ are the set of all sequences containing $0, 1$ and 2 infinitely often and the set of all sequences containing exactly two symbols infinitely often, respectively. Now let $\Omega_2, \Omega_{1,2}$ be subsets of Ω, where Ω_2 is the set consisting of all eventually constant sequences and $\Omega_{1,2}$ is the set consisting of sequences containing at most two symbols. Let $\sigma \colon \Omega \to \Omega$ be the left shift. i.e. $(\sigma\omega)_n = \omega_{n+1}$.

1.1. Generalized Grigorchuk's groups G_ω and generalized Grigorchuk's overgroups $\widetilde{G}_\omega$

Let T_2 be the labeled binary rooted tree with root $\varnothing$, where vertices are labeled by finite strings of 0s and 1s (see Figure 1). Let $\mathrm{Aut}(T_2)$ be

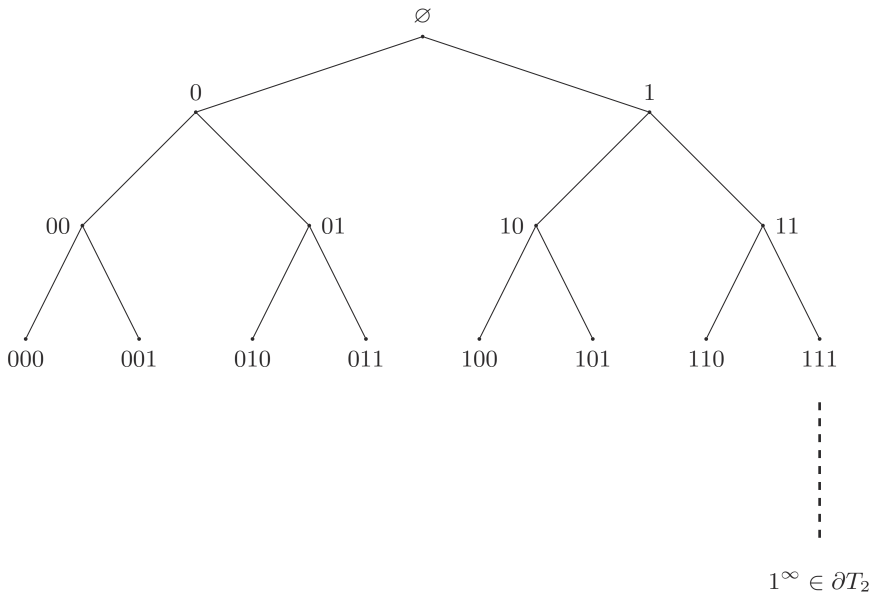

FIGURE 1. Labeled binary rooted tree T_2

the group of automorphisms of T_2, fixing the root and preserving the tree structure. That is, if $g \in \operatorname{Aut} T_2$, then g is a bijection from set of vertices of T_2 onto itself such that $g(\varnothing) = \varnothing$ and $g(u), g(v)$ are adjacent if and only if u, v are adjacent. For two vertices u, v of T_2, let uv be the vertex of T_2 labeled by the concatenation of labels of u and v.

For $g \in \operatorname{Aut}(T_2), v \in T_2$, there is a unique element $g|_v \in \operatorname{Aut}(T_2)$, such that $g(vu) = g(v)g|_v(u)$, for all $u \in T_2$. $g|_v$ is called the section of g at v. In this article, we will only consider the sections for which $g(v) = v$.

Denote the identity in $\operatorname{Aut}(T_2)$ by 1. Let $a \in \operatorname{Aut}(T_2)$ be such that $a(0v) = 1v$ and $a(1v) = 0v$, for all $v \in \operatorname{Aut}(T_2)$. Note that any $g \in \operatorname{Aut}(T_2)$, that fixes vertices $1^{n-1}0$ for all $n \geqslant 1$, can be uniquely identified by its sections at $1^{n-1}0$, for all $n \geqslant 1$. So we identify an infinite sequence of automorphisms $\{A_n\}$ with the element $g \in \operatorname{Aut}(T_2)$ where $g(1^{n-1}0) = 1^{n-1}0$ and $g|_{1^{n-1}0} = A_n$ for all $n \geqslant 1$. Identify $x \in \operatorname{Aut}(T_2)$ with the sequence $(a, a, \ldots)$.

For $\omega \in \Omega$, define $b_\omega, c_\omega, d_\omega, \widetilde{b}_\omega, \widetilde{c}_\omega, \widetilde{d}_\omega \in \operatorname{Aut}(T_2)$ to be the elements identified with sequences $\{B_n\}$, $\{C_n\}$, $\{D_n\}$, $\{\widetilde{B}_n\}$, $\{\widetilde{C}_n\}$, $\{\widetilde{D}_n\}$, respectively, where

$$B_n = \begin{cases} a & \omega_n = 0 \text{ or } 1 \\ 1 & \omega_n = 2 \end{cases}, \qquad \widetilde{B}_n = \begin{cases} 1 & \omega_n = 0 \text{ or } 1 \\ a & \omega_n = 2 \end{cases},$$
$$C_n = \begin{cases} a & \omega_n = 0 \text{ or } 2 \\ 1 & \omega_n = 1 \end{cases}, \qquad \widetilde{C}_n = \begin{cases} 1 & \omega_n = 0 \text{ or } 2 \\ a & \omega_n = 1 \end{cases},$$
$$D_n = \begin{cases} a & \omega_n = 1 \text{ or } 2 \\ 1 & \omega_n = 0 \end{cases}, \qquad \widetilde{D}_n = \begin{cases} 1 & \omega_n = 1 \text{ or } 2 \\ a & \omega_n = 0 \end{cases}. \tag{1}$$

We will denote a, x by a_ω, x_ω, respectively, if it is convenient. Note that all these elements are involutions and all except a commute with each other. The generalized Grigorchuk group G_ω is the group generated by elements $a, b_\omega, c_\omega, d_\omega$ and the generalized overgroup $\widetilde{G}_\omega$ is the group generated by $a, b_\omega, c_\omega, d_\omega, x$. As follows from definition, $G_\omega \subset \widetilde{G}_\omega$ and it is useful to view $\widetilde{G}_\omega$ as the group generated by the set $\widetilde{S}_\omega = \{a, b_\omega, c_\omega, d_\omega, x, \widetilde{b}_\omega, \widetilde{c}_\omega, \widetilde{d}_\omega\}$.

For any $\omega \in \Omega$, elements in $\widetilde{S}_\omega$ satisfy the following relations called simple contractions;

$$\begin{gathered}
a^2 = x^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = \widetilde{b}_\omega^2 = \widetilde{c}_\omega^2 = \widetilde{d}_\omega^2 = 1, \\
b_\omega c_\omega = c_\omega b_\omega = d_\omega, \quad c_\omega d_\omega = d_\omega c_\omega = b_\omega, \quad d_\omega b_\omega = b_\omega d_\omega = c_\omega, \\
\widetilde{b}_\omega \widetilde{c}_\omega = \widetilde{c}_\omega \widetilde{b}_\omega = d_\omega, \quad \widetilde{c}_\omega \widetilde{d}_\omega = \widetilde{d}_\omega \widetilde{c}_\omega = b_\omega, \quad \widetilde{d}_\omega \widetilde{b}_\omega = \widetilde{b}_\omega \widetilde{d}_\omega = c_\omega, \\
b_\omega \widetilde{c}_\omega = \widetilde{c}_\omega b_\omega = \widetilde{d}_\omega, \quad c_\omega \widetilde{d}_\omega = \widetilde{d}_\omega c_\omega = \widetilde{b}_\omega, \quad d_\omega \widetilde{b}_\omega = \widetilde{b}_\omega d_\omega = \widetilde{c}_\omega, \\
\widetilde{b}_\omega c_\omega = c_\omega \widetilde{b}_\omega = \widetilde{d}_\omega, \quad \widetilde{c}_\omega d_\omega = d_\omega \widetilde{c}_\omega = \widetilde{b}_\omega, \quad \widetilde{d}_\omega b_\omega = b_\omega \widetilde{d}_\omega = \widetilde{c}_\omega, \\
b_\omega \widetilde{b}_\omega = \widetilde{b}_\omega b_\omega = c_\omega \widetilde{c}_\omega = \widetilde{c}_\omega c_\omega = d_\omega \widetilde{d}_\omega = \widetilde{d}_\omega d_\omega = x, \\
b_\omega x = x b_\omega = \widetilde{b}_\omega, \quad c_\omega x = x c_\omega = \widetilde{c}_\omega, \quad d_\omega x = x d_\omega = \widetilde{d}_\omega, \\
\widetilde{b}_\omega x = x \widetilde{b}_\omega = b_\omega, \quad \widetilde{c}_\omega x = x \widetilde{c}_\omega = c_\omega, \quad \widetilde{d}_\omega x = x \widetilde{d}_\omega = d_\omega.
\end{gathered} \tag{2}$$

Any word over the alphabet $\widetilde{S}_\omega$ can be reduced using simple contractions (2) to a word of the form,

$$(a) * a * a * \ldots * a * a * (a). \tag{3}$$

Here the first and the last a can be omitted and '$*$'s represent generators in $\widetilde{S}_\omega \setminus \{a\}$. A word of the form (3) is called a reduced word. Thus, each element $g \in \widetilde{G}_\omega$ can be represented using a reduced word.

Denote $\widetilde{H}_\omega := \widetilde{H}_\omega^{(1)} := \{g \in \widetilde{G}_\omega \colon g(v) = v \text{ for } v = 0, 1\}$. Then $g \in \widetilde{H}_\omega$ if and only if every word representing g has even number of 'a's. It is easy to see that $\widetilde{H}_\omega$ is generated by $\{s, s^a : s \in \widetilde{S}_\omega \setminus \{a\}\}$, using (3). There is a natural embedding $\widetilde{\psi}_\omega$ from $\{s, s^a : s \in \widetilde{S}_\omega \setminus \{a\}\}$ to $\left(\widetilde{S}_{\sigma\omega} \cup \{1\}\right)^2$ given by

$$\begin{aligned}
b_\omega &\mapsto (B_1, b_{\sigma\omega}), & c_\omega &\mapsto (C_1, c_{\sigma\omega}), & d_\omega &\mapsto (D_1, d_{\sigma\omega}),\\
b_\omega^a &\mapsto (b_{\sigma\omega}, B_1), & c_\omega^a &\mapsto (c_{\sigma\omega}, C_1), & d_\omega^a &\mapsto (d_{\sigma\omega}, D_1),\\
\widetilde{b}_\omega &\mapsto (\widetilde{B}_1, \widetilde{b}_{\sigma\omega}), & \widetilde{c}_\omega &\mapsto (\widetilde{C}_1, \widetilde{c}_{\sigma\omega}), & \widetilde{d}_\omega &\mapsto (\widetilde{D}_1, \widetilde{d}_{\sigma\omega}),\\
\widetilde{b}_\omega^a &\mapsto (\widetilde{b}_{\sigma\omega}, \widetilde{B}_1), & \widetilde{c}_\omega^a &\mapsto (\widetilde{c}_{\sigma\omega}, \widetilde{C}_1), & \widetilde{d}_\omega^a &\mapsto (\widetilde{d}_{\sigma\omega}, \widetilde{D}_1),\\
x &\mapsto (a, x), & x^a &\mapsto (x, a).
\end{aligned} \tag{4}$$

Let W be a word with even number of 'a's, over alphabet $\widetilde{S}_\omega$, in the reduced form (3). Then W can be written in the form,

$$(*^a) * *^a * *^a \ldots * *^a * (*^a), \tag{5}$$

where '$*$'s represent elements in $\widetilde{S}_\omega \setminus \{a\}$. Here the first and last $*^a$ can be omitted depending on first and last letter of W. Now using (5) and (4), we can extend $\widetilde{\psi}_\omega$ to the set of words W, of the reduced form (3), over alphabet $\widetilde{S}_\omega$, with even number of a's. Then $W \mapsto (\widetilde{W_0}, \widetilde{W_1})$, where $\widetilde{W_0}, \widetilde{W_1}$ are words over alphabet $\widetilde{S}_{\sigma\omega}$, which need not to be in reduced form (3). Note that each $*, *^a$ in (5) contributes either a letter or no letters (if the corresponding coordinate is 1) to each of $\widetilde{W_0}$ and $\widetilde{W_1}$. Write W in the form (5), and suppose there are n number of '$*$'s in it. Then $\left|\widetilde{W_0}\right|, \left|\widetilde{W_1}\right| \leqslant n$. If $W = **^a**^a \ldots **^a*$, then $|W| = 2n-1$. If $W = (*^a)**^a**^a \ldots **^a*$ or $W = **^a**^a \ldots **^a*(*^a)$, then $|W| = 2n$. If $W = (*^a)**^a**^a \ldots **^a*(*^a)$, then $|W| = 2n+1$. In either case, we get

$$\left|\widetilde{W_0}\right|, \left|\widetilde{W_1}\right| \leqslant \frac{|W|+1}{2} \quad \text{and} \quad \left|\widetilde{W_0}\right| + \left|\widetilde{W_1}\right| \leqslant |W| + 1. \tag{6}$$

Here $|W|, \left|\widetilde{W_0}\right|, \left|\widetilde{W_1}\right|$, represents the number of letters in $W, \widetilde{W_0}, \widetilde{W_1}$, respectively. In fact, we can give a better upper bound,

$$\left|\widetilde{W_0}\right| + \left|\widetilde{W_1}\right| \leqslant |W| + 1 - \alpha, \tag{7}$$

where α is the number of letters in W, whose first coordinate of the natural embedding is 1. $\widetilde{\psi}_\omega$ can also be extended to a map from $\widetilde{H}_\omega$ into $\widetilde{G}_{\sigma\omega} \times \widetilde{G}_{\sigma\omega}$ given by,

$$\widetilde{\psi}_\omega(g) = (g|_0, g|_1). \tag{8}$$

We will denote both of theses extensions also by $\widetilde{\psi}_\omega$ and we may drop the subscript ω if there is no ambiguity.

Let G be a group with a finite generating set S. For a word W over the alphabet S, the number of letters in W is denoted by $|W|$ and for $s \in S$, the number of occurrences of s in W is denoted by $|W|_s$. For any element $g \in G$, the length of g, denoted by $|g|$, is defined by

$$|g| = \min\{|W| : g = W \text{ in } G\}.$$

It is easy to see from (8) and (6), that for $g \in \widetilde{H}_\omega$,

$$|g|_0| , |g|_1| \leqslant \frac{|g|+1}{2} \qquad \text{and} \qquad |g|_0| + |g|_1| \leqslant |g| + 1. \tag{9}$$

Now define $\gamma_{G,S}$, a positive integer valued function on nonnegative integers by

$$\gamma_{G,S}(n) = |B_{G,S}(n)| ,$$

where $B_{G,S}(n) = \{g \in G \colon\ |g| \leqslant n\}$ is the ball of radius n in the Cayley graph of G with respect to the generating set S. $\gamma_{G,S}$ is called the volume growth function of G with respect to the finite generating set S.

There is a partial order relation $\preceq$ for growth functions defined by $f \preceq g$ if and only if there are constants A and B such that $f(n) \leqslant Ag(Bn)$ for all n. We define an equivalence relation $\simeq$ by, $f \simeq g$ if and only if $f \preceq g$ and $g \preceq f$. The equivalence class of $\gamma_{G,S}(n)$ is known as the growth rate of the group G. The growth rate of a group does not depend on the generating set. So we denote the growth rate of a group G, by $\gamma_G(n)$. Growth rate can be polynomial, exponential, or intermediate if $\gamma_{G,S}(n) \simeq n^d$ for some positive integer d, $\gamma_{G,S}(n) \simeq e^n$, or $n^d \precneqq \gamma_{G,S}(n) \precneqq e^n$ for all positive integers d, respectively. Growth above polynomial is called super polynomial and growth below exponential is called subexponential.

The growth exponent $\lambda_{G,S}$ of a group G generated by S, is given by $\lambda_{G,S} = \lim_n (\gamma_{G,S}(n))^{1/n}$, and $\lambda_{G,S} \geqslant 1$ for any finitely generated group G. Note that $1/\lambda_{G,S}$ is the radius of convergence of the generating function of $\{\gamma_{G,S}(n)\}$. An easy exercise shows that, for finitely generated, infinite group G with generating set S

$$\lim_n (\gamma_{G,S}(n))^{1/n} = \lim_n \left(\gamma'_{G,S}(n)\right)^{1/n}, \tag{10}$$

by looking at the radii of convergence of generating functions of $\{\gamma_{G,S}(n)\}$ and $\{\gamma'_{G,S}(n)\}$, where $\gamma'_{G,S}(n) = |B_{G,S}(n) \setminus B_{G,S}(n-1)| = \gamma_{G,S}(n) -$

$\gamma_{G,S}(n-1)$ is the spherical growth function of G with respect to the generating set S. For finite indexed subgroup H of G,

$$\gamma_{H,S}(n) \leqslant \gamma_{G,S}(n) \leqslant \gamma_{H,S}(n+N),$$

where $\gamma_{H,S}(n) = |B_{G,S}(n) \cap H|$ and N is the maximum of lengths of right coset representatives of H in G. Thus for infinite group G, we get

$$\lim_n \left(\gamma_{H,S}(n)\right)^{1/n} = \lim_n \left(\gamma'_{H,S}(n)\right)^{1/n} = \lim_n \left(\gamma_{G,S}(n)\right)^{1/n}. \tag{11}$$

Here $\gamma'_{H,S}(n) = |(B_{G,S}(n) \setminus B_{G,S}(n-1)) \cap H|$. It is known that $\lambda_{G,S} > 1$ if and only if G has exponential growth [12]. We will be using $\widetilde{\gamma}_\omega, \widetilde{\lambda}_\omega$ in this text to denote $\gamma_{\widetilde{G}_\omega, \widetilde{S}_\omega}, \lambda_{\widetilde{G}_\omega, \widetilde{S}_\omega}$, where $\widetilde{S}_\omega = \{a, b_\omega, c_\omega, d_\omega, x, \widetilde{b}_\omega, \widetilde{c}_\omega, \widetilde{d}_\omega\}$.

2. Growth of $\widetilde{G}_\omega$

Proposition 1. *$\widetilde{G}_\omega$ has subexponential growth for each $\omega \in \Omega_1 \cup \Omega_2$.*

Before proceeding to the proof, we start with three lemmas.

Lemma 1. *A non-decreasing semi-multiplicative function $\gamma(n)$ with argument a natural number, can be extended to a non-decreasing semi-multiplicative function $\gamma(x)$, with argument a non-negative real number.*

Proof. See Lemma 3.1 of [12]. □

Lemma 2. *For any $\omega \in \Omega$, $\widetilde{\lambda}_\omega \leqslant \widetilde{\lambda}_{\sigma\omega}$.*

Proof. Denote $\widetilde{B}_\omega(n) = B_{\widetilde{G}_\omega, \widetilde{S}_\omega}(n)$ and $\widetilde{H}_\omega(n) = \widetilde{H}_\omega \cap \widetilde{B}_\omega(n)$. Any element $g \in \widetilde{B}_\omega(n)$ is either in $\widetilde{H}_\omega$ or is of the form $g = ag'$, where $g' \in \widetilde{H}_\omega$ and $|g'| \leqslant |g| + 1 \leqslant n+1$. Thus,

$$\widetilde{\gamma}_\omega(n) = |\widetilde{B}_\omega(n)| \leqslant |\widetilde{H}_\omega(n)| + |\widetilde{H}_\omega(n+1)| \leqslant 2|\widetilde{H}_\omega(n+1)|.$$

For each $g \in \widetilde{H}_\omega$, $g|_0, g|_1 \in \widetilde{G}_{\sigma\omega}$ from (8) satisfy (9) and so,

$$|\widetilde{H}_\omega(n)| \leqslant |\widetilde{B}_{\sigma\omega}(\frac{n+1}{2})|^2 = \left(\widetilde{\gamma}_{\sigma\omega}(\frac{n+1}{2})\right)^2.$$

Therefore,

$$\widetilde{\gamma}_\omega(n) \leqslant 2\left(\widetilde{\gamma}_{\sigma\omega}(\frac{n+2}{2})\right)^2.$$

Consequently,

$$\widetilde{\lambda}_\omega = \lim_n (\widetilde{\gamma}_\omega(n))^{1/n} \leqslant \lim_n \left(2\left(\widetilde{\gamma}_{\sigma\omega}(\frac{n+2}{2})\right)^2\right)^{1/n}$$
$$= \lim_n \left(\widetilde{\gamma}_{\sigma\omega}(\frac{n+2}{2})\right)^{2/n} = \widetilde{\lambda}_{\sigma\omega}. \qquad \square$$

Lemma 3. *For any $\omega \in \Omega_{1,2}$, $\widetilde{G}_\omega = G_\omega$.*

Proof. First note that $x \in G_\omega \implies xb_\omega, xc_\omega, xd_\omega \in G_\omega \implies \widetilde{b}_\omega, \widetilde{c}_\omega, \widetilde{d}_\omega \in G_\omega \implies \widetilde{G}_\omega \subset G_\omega \implies \widetilde{G}_\omega = G_\omega$. To prove Lemma 3, we only need to show that $x \in G_\omega$. For definiteness we may assume ω consists only of symbols $0, 1$. Then by (1), $b_\omega = (a, a, a, ...) = x$. Therefore $x \in G_\omega$ and thus the result is true. $\square$

Proof of Proposition 1. Let $\omega \in \Omega_1 \cup \Omega_2$. Then there exists $N \in \mathbb{N}$ such that $\sigma^N\omega \in \Omega_{1,2}$. Then by Lemma 3, $\widetilde{G}_{\sigma^N\omega} = G_{\sigma^N\omega}$. Therefore $\widetilde{\lambda}_{\sigma^N\omega} = \lambda_{\sigma^N\omega}$. For any ω, G_ω is of intermediate growth if $\omega \in \Omega_1$ and of polynomial growth if $\omega \in \Omega_2$ [12]. Thus $\lambda_{\sigma^N\omega} = 1$. So by Lemma 2, $\widetilde{\lambda}_\omega \leqslant \widetilde{\lambda}_{\sigma^N\omega} = 1$. Thus $\widetilde{G}_\omega$ is of subexponential growth. $\square$

Proposition 2. *$\widetilde{G}_\omega$ has intermediate growth for $\omega \in \Omega_1$.*

Proof. By Proposition 1, $\widetilde{G}_\omega$ is of subexponential growth. Since $G_\omega \subset \widetilde{G}_\omega$ and G_ω is of super-polynomial growth [12], $\widetilde{G}_\omega$ is of super-polynomial growth. Hence $\widetilde{G}_\omega$ is of intermediate growth. $\square$

Proposition 3. *$\widetilde{G}_\omega$ has polynomial growth for $\omega \in \Omega_2$.*

Proof. Since $\omega \in \Omega_2$, there is a natural number N such that $\omega_n = \omega_N$ for all $n \geqslant N$, where $\omega = \{\omega_n\}$. Then $\widetilde{G}_{\sigma^{N-1}\omega} = \langle a, x\rangle \cong \mathbb{D}_\infty$, the infinite Dihedral group. Let $\mathbb{G}$ be the subgroup of $\mathrm{Aut}(T_2)$ containing elements g such that $g|_v \in \langle a, x\rangle$ for all v in level $N-1$ of T_2. Then $\widetilde{G}_\omega \subset \mathbb{G}$. Let $\mathbb{G}_0$ be the subgroup of $\mathbb{G}$ containing automorphisms fixing all vertices in the first $N-1$ levels of T_2. Note that $\mathbb{G}_0 \lhd \mathbb{G}$ and $[\mathbb{G} : \mathbb{G}_0] \leqslant 2^{2^N-1}$. But $\mathbb{G}_0 \cong \langle a, x\rangle^{2^{N-1}} \cong \mathbb{D}_\infty^{2^{N-1}}$. Thus $\mathbb{G}_0$ is virtually abelian and of polynomial growth. Since $[\mathbb{G} : \mathbb{G}_0] < \infty, \mathbb{G}$ is of polynomial growth. $\widetilde{G}_\omega \subset \mathbb{G}$ implies that $\widetilde{G}_\omega$ is of polynomial growth. $\square$

Theorem 3. *$\widetilde{G}_\omega$ has intermediate growth for $\omega \in \Omega_0$.*

We will, from now on, consider the generating set of $\widetilde{G}_\omega$ to be $\widetilde{S}_\omega = \{a, b_\omega, c_\omega, d_\omega, \widetilde{b}_\omega, \widetilde{c}_\omega, \widetilde{d}_\omega, x\}$. A reduced word W satisfying $g = W$ in $\widetilde{G}_\omega$ and $|g| = |W|$ is called a minimal representation of g. For any $\epsilon > 0$ define $\mathcal{F}^\epsilon(n) = \mathcal{F}^\epsilon_\omega(n)$ to be the set of length n elements g in $\widetilde{G}_\omega$ such that for any minimal representation W of g over alphabet $\widetilde{S}_\omega$,

$$|W|_* > (1/2 - \epsilon)n, \quad \text{for some } * \in \widetilde{S}_\omega \setminus \{a\}. \tag{12}$$

So for any minimal representation of elements in $\mathcal{F}^\epsilon(n)$, its reduced form (3) has most of $*$s as the same letter. Now define $\mathcal{D}^\epsilon(n) = \mathcal{D}^\epsilon_\omega(n)$ to be the complement of $\mathcal{F}^\epsilon(n)$ in $\widetilde{B}_\omega(n) \setminus \widetilde{B}_\omega(n-1)$, the sphere of radius n. Thus if $g \in \mathcal{D}^\epsilon(n)$, then g has a minimal representation W satisfying

$$|W|_* \leqslant (1/2 - \epsilon)n, \quad \text{for all } * \in \widetilde{S}_\omega \setminus \{a\}. \tag{13}$$

For any $\delta > 0$ define $\widetilde{\mathcal{F}}^\delta(n')$ to be the set of words W' over the alphabet $\widetilde{S}_\omega \setminus \{a\}$ of length n' such that

$$\left|W'\right|_* > (1 - \delta)n', \quad \text{for some } * \in \widetilde{S}_\omega \setminus \{a\}. \tag{14}$$

Therefore, each word in $\widetilde{\mathcal{F}}^\delta(n')$ has mostly equal letters.

Lemma 4. *Let $0 < \epsilon < 1/2$ and let W be a minimal representation of an element in $\mathcal{F}^\epsilon(n)$. Let W' be the word obtained by deleting all letters a from W. Then $W' \in \widetilde{\mathcal{F}}^\delta(n')$ where*

$$\frac{n-1}{2} \leqslant n' \leqslant \frac{n+1}{2}, \tag{15}$$

$$\delta = 2\epsilon + \frac{(1-2\epsilon)}{n-1}. \tag{16}$$

Proof. Since W is a reduced word, by (3), we observe that, $2\left|W\right|_a - 1 \leqslant |W| \leqslant 2\left|W\right|_a + 1$. Thus $\frac{|W|-1}{2} \leqslant \left|W\right|_a \leqslant \frac{|W|+1}{2}$, and so $\frac{|W|-1}{2} \leqslant |W| - \left|W\right|_a \leqslant \frac{|W|+1}{2}$. So we get (15).

By (12),

$$\begin{aligned}\left|W'\right|_* &= |W|_* > (1/2 - \epsilon)n \geqslant (1/2-\epsilon)(2n'-1) \\ &= \left(1 - 2\epsilon - \frac{(1-2\epsilon)}{2n'}\right) n' \geqslant \left(1 - 2\epsilon - \frac{(1-2\epsilon)}{n-1}\right) n' = (1-\delta)n',\end{aligned}$$

from (16). $\square$

Lemma 5. *If* $\delta < 1$, *then* $\overline{\lim_k} |\widetilde{\mathcal{F}}^\delta(k)|^{1/k} \leqslant (1-\delta)^{-1}(\delta/6)^{-\delta}$.

Proof. Any word $W \in \widetilde{\mathcal{F}}^\delta(k)$ can be constructed by choosing a letter $*$ out of $\{b, c, d, \widetilde{b}, \widetilde{c}, \widetilde{d}, x\}$, which satisfies (14). So, W contains the letter $*$ at least $k - \lfloor \delta k \rfloor$ times and possibly t times more, where $0 \leqslant t \leqslant \lfloor \delta k \rfloor$. The rest of the positions of W can be filled by the other six letters with frequencies $i_1, \dots, i_6$, where $\sum i_j = \lfloor \delta k \rfloor - t$. Therefore, we have

$$\left|\widetilde{\mathcal{F}}^\delta(k)\right| \leqslant 7 + 7 \sum_{t=0}^{\lfloor \delta k \rfloor} \sum_{\sum i_j = \lfloor \delta k \rfloor - t} \frac{k!}{(k - \lfloor \delta k \rfloor + t)! i_1! \dots i_6!}.$$

Let $(\delta k - t)_* := 6 \left\lfloor \dfrac{\lfloor \delta k - t \rfloor}{6} \right\rfloor$ be the largest integer not greater than $\lfloor \delta k - t \rfloor$, that is divisible by 6. Since $i_1, \dots, i_6$ are non negative integers, we have

$$i_1! \dots i_6! \geqslant \left\lfloor \frac{\sum i_j}{6} \right\rfloor !^6 = \left\lfloor \frac{\lfloor \delta k \rfloor - t}{6} \right\rfloor !^6 = \left\lfloor \frac{\lfloor \delta k - t \rfloor}{6} \right\rfloor !^6 = \left(\frac{(\delta k - t)_*}{6} \right) !^6.$$

Since the number of ways to choose non negative integers $i_1, \dots, i_6$ such that $\sum i_j = \lfloor \delta k \rfloor - t$ is $\binom{\lfloor \delta k \rfloor - t + 5}{5}$, we get

$$\begin{aligned}
\left|\widetilde{\mathcal{F}}^\delta(k)\right| &\leqslant 7 + 7 \sum_{t=0}^{\lfloor \delta k \rfloor} \binom{\lfloor \delta k \rfloor - t + 5}{5} \frac{k!}{(k - \lfloor \delta k \rfloor + t)! \left(\frac{(\delta k - t)_*}{6}\right)!^6} \\
&\leqslant 7 + 7 \binom{\lfloor \delta k \rfloor + 5}{5} \sum_{t=0}^{\lfloor \delta k \rfloor} \frac{k!}{(k - \lfloor \delta k \rfloor + t)! \left(\frac{(\delta k - t)_*}{6}\right)!^6} \\
&\leqslant (\lfloor \delta k \rfloor + 5)^5 \sum_{t=0}^{\lfloor \delta k \rfloor} \frac{k!}{(k - \lfloor \delta k \rfloor + t)! \left(\frac{(\delta k - t)_*}{6}\right)!^6} \\
&\leqslant (\lfloor \delta k \rfloor + 5)^5 \sum_{t=0}^{\lfloor \delta k \rfloor} \frac{e\sqrt{k} k^k e^{-k} e^{(k - \lfloor \delta k \rfloor + t)} e^{(\delta k - t)_*}}{(k - \lfloor \delta k \rfloor + t)^{(k - \lfloor \delta k \rfloor + t)} \left(\frac{(\delta k - t)_*}{6}\right)^{(\delta k - t)_*}}.
\end{aligned}$$

Here we used the Stirling's formula $\dfrac{n^n}{e^n} \leqslant n! \leqslant e\sqrt{n}\dfrac{n^n}{e^n}$. Since $0 \leqslant (\lfloor \delta k \rfloor - t) - (\delta k - t)_* \leqslant 6$,

$$\left|\widetilde{\mathcal{F}}^\delta(k)\right| \leqslant e(\lfloor \delta k \rfloor + 5)^5 \sum_{t=0}^{\lfloor \delta k \rfloor} \frac{\sqrt{k} k^{(\lfloor \delta k \rfloor - t) - (\delta k - t)_*} e^{(\delta k - t)_* - (\lfloor \delta k \rfloor - t)}}{\left(1 - \frac{\lfloor \delta k \rfloor}{k} + \frac{t}{k}\right)^{(k - \lfloor \delta k \rfloor + t)} \left(\frac{(\delta k - t)_*}{6k}\right)^{(\delta k - t)_*}}$$

$$\leqslant e(\lfloor \delta k\rfloor + 5)^5 \sum_{t=0}^{\lfloor \delta k\rfloor} \frac{\sqrt{k}k^6}{\left(1-\frac{\lfloor \delta k\rfloor}{k}+\frac{t}{k}\right)^{(k-\lfloor \delta k\rfloor+t)} \left(\frac{(\delta k-t)_*}{6k}\right)^{(\delta k-t)_*}}$$

$$\leqslant ek^6(\lfloor \delta k\rfloor + 5)^5\sqrt{k}(1-\delta)^{-k} \sum_{t=0}^{\lfloor \delta k\rfloor} \left(\frac{(\delta k-t)_*}{6k}\right)^{-(\delta k-t)_*}.$$

Note that the real valued function, $\xi \mapsto \xi^{-\xi}$ for $\xi > 0$, is an increasing function on the interval $(0, e^{-1})$. Since $\delta/6 < 1/6 < e^{-1}$, we get

$$\left(\frac{(\delta k-x)_*}{6k}\right)^{-\left(\frac{(\delta k-x)_*}{6k}\right)} \leqslant \left(\frac{\delta}{6}\right)^{-\left(\frac{\delta}{6}\right)}.$$

Therefore,

$$\left|\widetilde{\mathcal{F}}^\delta(k)\right| \leqslant ek^6(\lfloor \delta k\rfloor + 5)^5\sqrt{k}(1-\delta)^{-k}(\lfloor \delta k\rfloor + 1)\left(\frac{\delta}{6}\right)^{-\left(\frac{\delta}{6}\right)6k}.$$

Hence,

$$\overline{\lim_k}\left|\widetilde{\mathcal{F}}^\delta(k)\right|^{1/k} \leqslant (1-\delta)^{-1}(\delta/6)^{-\delta}. \qquad \square$$

Corollary 1. *Let* $\epsilon < 1/2$. *Then,* $\overline{\lim_n}|\mathcal{F}^\epsilon(n)|^{1/n} \leqslant (1-2\epsilon)^{-1/2}(\epsilon/3)^{-\epsilon}$.

Proof. If n is even, then minimal representations of at most two elements in $\mathcal{F}^\epsilon(n)$ give the same word in $\widetilde{\mathcal{F}}^\delta(n/2)$. So,

$$|\mathcal{F}^\epsilon(n)| \leqslant 2|\widetilde{\mathcal{F}}^\delta(n/2)|.$$

If n is odd, then for each element in $\mathcal{F}^\epsilon(n)$, we can assign a unique word in $\widetilde{\mathcal{F}}^\delta((n-1)/2)$ or $\widetilde{\mathcal{F}}^\delta((n+1)/2)$, and so,

$$|\mathcal{F}^\epsilon(n)| \leqslant |\widetilde{\mathcal{F}}^\delta((n-1)/2)| + |\widetilde{\mathcal{F}}^\delta((n+1)/2)|.$$

Note that,

$$\overline{\lim_n}|\widetilde{\mathcal{F}}^\delta(n/2)|^{1/n} \leqslant \lim_n \left((1-\delta)^{-1}(\delta/6)^{-\delta}\right)^{1/2},$$

$$\overline{\lim_n}|\widetilde{\mathcal{F}}^\delta((n-1)/2)|^{1/n} \leqslant \lim_n \left((1-\delta)^{-1}(\delta/6)^{-\delta}\right)^{1/2},$$

$$\overline{\lim_n}|\widetilde{\mathcal{F}}^\delta((n+1)/2)|^{1/n} \leqslant \lim_n \left((1-\delta)^{-1}(\delta/6)^{-\delta}\right)^{1/2},$$

and thus,

$$\overline{\lim_n}|\mathcal{F}^\epsilon(n)|^{1/n} \leqslant \lim_n \left((1-\delta)^{-1}(\delta/6)^{-\delta}\right)^{1/2}.$$

Since $\delta = 2\epsilon + \dfrac{(1-2\epsilon)}{n-1}$, $\lim_n \delta = 2\epsilon$ and therefore,

$$\lim_n \left((1-\delta)^{-1}(\delta/6)^{-\delta}\right)^{-1/2} = (1-2\epsilon)^{-1/2}(\epsilon/3)^{-\epsilon}.$$

Hence we get the desired result. □

For each $s \geqslant 1$, let $\widetilde{H}_\omega^{(s)} := \{g \in \widetilde{G}_\omega \colon g(v) = v \text{ for } v \text{ in level } s\}$ and denote the canonical generators of $\widetilde{G}_{\sigma^s\omega}$ by $a, b_s, c_s, d_s, \widetilde{b}_s, \widetilde{c}_s, \widetilde{d}_s, x$. We assign above symbols, when $s = 0$, to the generators of $\widetilde{G}_\omega$. Using the map $\widetilde{\psi}$, we get the following:

$$\begin{aligned}
\omega_s = 0 \implies & b_{s-1} = (a, b_s) \quad c_{s-1} = (a, c_s) \quad d_{s-1} = (1, d_s) \quad x = (a, x)\\
& \widetilde{b}_{s-1} = (1, \widetilde{b}_s) \quad \widetilde{c}_{s-1} = (1, \widetilde{c}_s) \quad \widetilde{d}_{s-1} = (a, \widetilde{d}_s),\\[1em]
\omega_s = 1 \implies & b_{s-1} = (a, b_s) \quad c_{s-1} = (1, c_s) \quad d_{s-1} = (a, d_s) \quad x = (a, x)\\
& \widetilde{b}_{s-1} = (1, \widetilde{b}_s) \quad \widetilde{c}_{s-1} = (a, \widetilde{c}_s) \quad \widetilde{d}_{s-1} = (1, \widetilde{d}_s),\\[1em]
\omega_s = 2 \implies & b_{s-1} = (1, b_s) \quad c_{s-1} = (a, c_s) \quad d_{s-1} = (a, d_s) \quad x = (a, x)\\
& \widetilde{b}_{s-1} = (a, \widetilde{b}_s) \quad \widetilde{c}_{s-1} = (1, \widetilde{c}_s) \quad \widetilde{d}_{s-1} = (1, \widetilde{d}_s).
\end{aligned} \tag{17}$$

Let W be a minimal representation of an element in $\widetilde{H}_\omega^{(s)}$. Then there are $\widetilde{W_0}, \widetilde{W_1}$ such that $W = (\widetilde{W_0}, \widetilde{W_1})$ using substitutions in (17). Let W_0, W_1 be obtained by doing simple contractions on $\widetilde{W_0}, \widetilde{W_1}$. Let α_1 denote the number of such simple contractions. So W_0, W_1 are minimal representations of words in $\widetilde{H}_{\sigma^1\omega}^{(s-1)}$ and by (6),

$$|W_0| + |W_1| \leqslant \left|\widetilde{W_0}\right| + \left|\widetilde{W_1}\right| - \alpha_1 \leqslant |W| + 1 - \alpha_1. \tag{18}$$

Now there are $\widetilde{W_{00}}$, $\widetilde{W_{01}}$, $\widetilde{W_{10}}$, $\widetilde{W_{11}}$ such that $W_0 = (\widetilde{W_{00}}, \widetilde{W_{01}})$, $W_1 = (\widetilde{W_{10}}, \widetilde{W_{11}})$ using substitutions in (17). Let $W_{00}, W_{01}, W_{10}, W_{11}$ be obtained by doing simple contractions on $\widetilde{W_{00}}, \widetilde{W_{01}}, \widetilde{W_{10}}, \widetilde{W_{11}}$. Let α_2 denote the number of such simple contractions. So $W_{00}, W_{01}, W_{10}, W_{11}$

are minimal representations of elements in $\widetilde{H}^{(s-2)}_{\sigma^2\omega}$ and applying (18), we get

$$\begin{aligned}|W_{00}|+|W_{01}|+|W_{10}|+|W_{11}| &\leqslant |W_0|+1+|W_1|+1-\alpha_2\\ &\leqslant |W|+2^2-1-(\alpha_1+\alpha_2).\end{aligned}$$

Proceeding this manner we get $\{W_{i_1i_2\ldots i_s}\}_{i_j\in\{0,1\}}$ minimal representations of elements in $\widetilde{H}^{(s-s)}_{\sigma^s\omega}=\widetilde{G}_{\sigma^s\omega}$. Denote by α_s the number of simple contractions done to obtain $\{W_{i_1i_2\ldots i_s}\}_{i_j\in\{0,1\}}$ from $\{\widetilde{W}_{i_1i_2\ldots i_s}\}_{i_j\in\{0,1\}}$. Then by applying (18) repeatedly, we get

$$\sum_{i_1,i_2,\ldots,i_s}|W_{i_1i_2\ldots i_s}|\leqslant |W|+2^s-1-\sum_1^{s-1}\alpha_i. \tag{19}$$

Let $X_0:=|W|_{d_0}+|W|_{\widetilde{b}_0}+|W|_{\widetilde{c}_0}, Y_0:=|W|_{c_0}+|W|_{\widetilde{b}_0}+|W|_{\widetilde{d}_0}$ and $Z_0:=|W|_{b_0}+|W|_{\widetilde{c}_0}+|W|_{\widetilde{d}_0}$. Also for $j=1,2,\ldots s$, let

$$\begin{aligned}X_j&=\sum\left(|W_{i_1i_2\ldots i_j}|_{d_j}+|W_{i_1i_2\ldots i_j}|_{\widetilde{b}_j}+|W_{i_1i_2\ldots i_j}|_{\widetilde{c}_j}\right),\\ Y_j&=\sum\left(|W_{i_1i_2\ldots i_j}|_{c_j}+|W_{i_1i_2\ldots i_j}|_{\widetilde{b}_j}+|W_{i_1i_2\ldots i_j}|_{\widetilde{d}_j}\right),\\ Z_j&=\sum\left(|W_{i_1i_2\ldots i_j}|_{b_j}+|W_{i_1i_2\ldots i_j}|_{\widetilde{c}_j}+|W_{i_1i_2\ldots i_j}|_{\widetilde{d}_j}\right).\end{aligned}$$

Lemma 6. *Let $\epsilon>0, n_\epsilon\in\mathbb{N}$ such that $n_\epsilon\epsilon>5/2$. Let $n\geqslant n_\epsilon$. Let $s\in\mathbb{N}$ such that ω_s is the first time that the third symbol appears in ω. Let W be a minimal representation of an element in $\mathcal{D}^\epsilon(n)\cap\widetilde{H}^{(s)}_\omega$. Then,*

$$\sum_{i_1,i_2,\ldots,i_s}|W_{i_1i_2\ldots i_s}|\leqslant\left(1-\frac{\epsilon}{5}\right)n+2^s-1.$$

Proof. For definiteness, suppose the sequence ω begins with the symbol 0, first 1 appears in the t-th position, and first 2 appears in the s-th position. That is, $\omega_1=\ldots=\omega_{t-1}=0$, $\omega_t=1$, $\omega_m\neq 2$ for every $m<s$, and $\omega_s=2$. First note that each simple contraction decreases Y_i,Z_i by at most 2. Thus,

$$Y_{t-1}\geqslant Y_0-2\sum_1^{t-1}\alpha_i\geqslant Y_0-2\sum_1^{s-1}\alpha_i\quad\text{ and }\quad Z_{s-1}\geqslant Z_0-2\sum_1^{s-1}\alpha_i. \tag{20}$$

Since $\omega_1=0$ there are X_0 of letters in W, with 1 in their first coordinate when written using (17). Thus we modify (19), as done in (7) to be

$$\sum_{i_1,i_2,\ldots,i_s}|W_{i_1i_2\ldots i_s}|\leqslant n+2^s-1-\sum_1^{s-1}\alpha_i-X_0.$$

Similarly, since $\omega_t = 1$ and $\omega_s = 2$, we get

$$\sum_{i_1,i_2,\ldots,i_s} |W_{i_1 i_2 \ldots i_s}| \leqslant n + 2^s - 1 - \sum_1^{s-1} \alpha_i - X_0 - Y_{t-1} - Z_{s-1}. \tag{21}$$

Now let us show that $X_0 + Y_{t-1} + Z_{s-1} + \sum_1^{s-1} \alpha_i > n\epsilon/5$. To the contrary assume $X_0+Y_{t-1}+Z_{s-1}+\sum_1^{s-1} \alpha_i \leqslant n\epsilon/5$. Therefore, $\sum_1^{s-1} \alpha_i \leqslant n\epsilon/5$ and by (20) and (21),we get

$$\begin{aligned} X_0 + Y_0 + Z_0 &\leqslant X_0 + \left(Y_{t-1} + 2\sum_1^{s-1} \alpha_i \right) + \left(Z_{s-1} + 2\sum_1^{s-1} \alpha_i \right) \\ &\leqslant \left(X_0 + Y_{t-1} + Z_{s-1} + \sum_1^{s-1} \alpha_i \right) + 3\left(\sum_1^{s-1} \alpha_i \right) \leqslant \frac{4}{5} n\epsilon. \end{aligned}$$

But $n = |W| \leqslant |W|_a + |W|_x + X_0 + Y_0 + Z_0 \leqslant \frac{n+1}{2} + \frac{n}{2} - n\epsilon + \frac{4}{5} n\epsilon$, since $|W|_x \leqslant (1/2 - \epsilon)n$ by (13). Thus $n\epsilon \leqslant 5/2$, which is a contradiction. So $X_0 + Y_{t-1} + Z_{s-1} + \sum_1^{s-1} \alpha_i > n\epsilon/5$. Therefore,

$$\sum_{i_1,i_2,\ldots,i_s} |W_{i_1 i_2 \ldots i_s}| \leqslant \left(1 - \frac{\epsilon}{5}\right) n + 2^s - 1. \qquad \square$$

Proof of Theorem 3. Take a fixed $0 < \epsilon < 1/2$. Suppose first that there are positive integers k, s, such that there exists an infinite set $N_0 \subset \mathbb{N}$ where

$$\left| \widetilde{H}^{(s)}_{\sigma^k \omega} \cap \mathcal{F}^{\epsilon}_{\sigma^k \omega}(n) \right| \geqslant \left| \widetilde{H}^{(s)}_{\sigma^k \omega} \cap \mathcal{D}^{\epsilon}_{\sigma^k \omega}(n) \right|, \tag{22}$$

for all $n \in N_0$. By Lemma 2 and (11),

$$\begin{aligned} \widetilde{\lambda}_\omega \leqslant \widetilde{\lambda}_{\sigma^k \omega} &= \lim_n |\widetilde{\gamma}_{\sigma^k \omega}(n)|^{1/n} \\ &= \lim_n |\gamma'_{\widetilde{G}_{\sigma^k \omega}, \widetilde{S}_{\sigma^k \omega}}(n)|^{1/n} = \lim_{n \in N_0} |\gamma'_{\widetilde{G}_{\sigma^k \omega}, \widetilde{S}_{\sigma^k \omega}}(n)|^{1/n} \\ &= \lim_{n \in N_0} \left(\left| \widetilde{H}^{(s)}_{\sigma^k \omega} \cap \mathcal{F}^{\epsilon}_{\sigma^k \omega}(n) \right| + \left| \widetilde{H}^{(s)}_{\sigma^k \omega} \cap \mathcal{D}^{\epsilon}_{\sigma^k \omega}(n) \right| \right)^{1/n}. \end{aligned}$$

Using (22), we get

$$\begin{aligned} \widetilde{\lambda}_\omega &\leqslant \overline{\lim_{n \in N_0}} \left(2 \left| \widetilde{H}^{(s)}_{\sigma^k \omega} \cap \mathcal{F}^{\epsilon}_{\sigma^k \omega}(n) \right| \right)^{1/n} = \overline{\lim_{n \in N_0}} \left(\left| \widetilde{H}^{(s)}_{\sigma^k \omega} \cap \mathcal{F}^{\epsilon}_{\sigma^k \omega}(n) \right| \right)^{1/n} \\ &\leqslant \overline{\lim_{n \in N_0}} \left| \mathcal{F}^{\epsilon}_{\sigma^k \omega}(n) \right|^{1/n} \leqslant \overline{\lim_n} \left| \mathcal{F}^{\epsilon}_{\sigma^k \omega}(n) \right|^{1/n}. \end{aligned}$$

Using Corollary 1 we get

$$\widetilde{\lambda}_\omega \leqslant (1-2\epsilon)^{-1/2}(\epsilon/3)^{-\epsilon}. \tag{23}$$

Now suppose that for every $k, s \in \mathbb{N}$, there exists an $N(k,s)$ such that for all $n \geqslant N(k,s)$,

$$\left|\widetilde{H}^{(s)}_{\sigma^k\omega} \cap \mathcal{F}^{\epsilon}_{\sigma^k\omega}(n)\right| < \left|\widetilde{H}^{(s)}_{\sigma^k\omega} \cap \mathcal{D}^{\epsilon}_{\sigma^k\omega}(n)\right|. \tag{24}$$

As before, let $\widetilde{H}^{(s)}_\omega(n) := \widetilde{B}_\omega(n) \cap \widetilde{H}^{(s)}_\omega$ and $\widetilde{H}^{(s)}_{\sigma^k\omega}(n) := \widetilde{B}_{\sigma^k\omega}(n) \cap \widetilde{H}^{(s)}_{\sigma^k\omega}$. Let $\omega = \omega_1 \ldots \omega_{s_1}\omega_{s_1+1} \ldots \omega_{s_1+s_2}\omega_{s_1+s_2+1} \ldots \omega_{s_1+s_2+s_3} \ldots$ where s_1 is the first time third symbol appears in ω, s_2 is the first time third symbol appears in $\sigma^{s_1}\omega$, and so on.

Since $[\widetilde{G}_\omega : \widetilde{H}^{(s_1)}_\omega] \leqslant 2^{2^{s_1}-1} =: K_1$, there is a fixed Schreier system of representatives of the right cosets of $\widetilde{G}_\omega$ modulo $\widetilde{H}^{(s_1)}_\omega$ with Schreier representatives of length less than K_1. So for any $g \in \widetilde{B}_\omega(n)$, there are $h \in \widetilde{H}^{(s_1)}_\omega$, l a Schreier representative such that $g = hl$ and since $|l| \leqslant K_1$, we have $|h| \leqslant n + K_1$. Therefore,

$$\left|\widetilde{B}_\omega(n)\right| \leqslant K_1 \left|\widetilde{H}^{(s_1)}_\omega(n+K_1)\right|. \tag{25}$$

Let $N_1 = \max\{n_\epsilon, N(0,s_1)\}$, where n_ϵ is defined in Lemma 6 and $N(0,s_1)$ is defined in (24). Note that

$$\begin{aligned}\left|\widetilde{H}^{(s_1)}_\omega(n+K_1)\right| &= 1 + \sum_{k=1}^{n+K_1} \left|\widetilde{H}^{(s_1)}_\omega(n+K_1) \cap \left(\widetilde{B}_\omega(k) \setminus \widetilde{B}_\omega(k-1)\right)\right| \\ &\leqslant N_1 \left|\widetilde{B}_\omega(N_1)\right| + \sum_{k=N_1}^{n+K_1} \left|\widetilde{H}^{(s_1)}_\omega(n+K_1) \cap \left(\widetilde{B}_\omega(k) \setminus \widetilde{B}_\omega(k-1)\right)\right|.\end{aligned}$$

From (24), for $k \geqslant N_1$,

$$\begin{aligned}&\left|\widetilde{H}^{(s_1)}_\omega(n+K_1) \cap \left(\widetilde{B}_\omega(k) \setminus \widetilde{B}_\omega(k-1)\right)\right| \\ &\qquad = \left|\widetilde{H}^{(s_1)}_\omega(n+K_1) \cap \mathcal{F}^{\epsilon}(k)\right| + \left|\widetilde{H}^{(s_1)}_\omega(n+K_1) \cap \mathcal{D}^{\epsilon}(k)\right| \\ &\qquad \leqslant 2\left|\widetilde{H}^{(s_1)}_\omega(n+K_1) \cap \mathcal{D}^{\epsilon}(k)\right|.\end{aligned}$$

Therefore,

$$\left|\widetilde{H}^{(s_1)}_\omega(n+K_1)\right| \leqslant N_1 \left|\widetilde{B}_\omega(N_1)\right| + 2\sum_{k=N_1}^{n+K_1} \left|\widetilde{H}^{(s_1)}_\omega(n+K_1) \cap \mathcal{D}^{\epsilon}(k)\right|.$$

Now using Lemma 6,

$$\left|\widetilde{H}_\omega^{(s_1)}(n+K_1)\right| \leqslant N_1 \left|\widetilde{B}_\omega(N_1)\right| + 2 \sum_{j_1,\dots,j_{2^{s_1}}} \left|\widetilde{B}_{\sigma^{s_1}\omega}(j_1)\right| \dots \left|\widetilde{B}_{\sigma^{s_1}\omega}(j_{2^{s_1}})\right|, \tag{26}$$

where $\sum_{i=1}^{2^{s_1}} j_i \leqslant \left(1-\frac{\epsilon}{5}\right)(n+K_1)+2^{s_1}-1.$

Note that

$$\widetilde{\lambda}_{\sigma^{s_1}\omega} = \lim_j |\widetilde{B}_{\sigma^{s_1}\omega}(j)|^{1/j},$$

and therefore, for each $\delta > 0$, there exists an $J = J(\delta)$ such that for $j \geqslant J$,

$$|\widetilde{B}_{\sigma^{s_1-1}\omega}(j)| \leqslant (\widetilde{\lambda}_{\sigma^{s_1}\omega}+\delta)^j.$$

Thus for all j

$$|\widetilde{B}_{\sigma^{s_1-1}\omega}(j)| \leqslant |\widetilde{B}_{\sigma^{s_1-1}\omega}(J)|(\widetilde{\lambda}_{\sigma^{s_1}\omega}+\delta)^j,$$

which implies

$$\begin{aligned}\left|\widetilde{B}_{\sigma^{s_1}\omega}(j_1)\right| \dots \left|\widetilde{B}_{\sigma^{s_1}\omega}(j_{2^{s_1}})\right| &\leqslant \left|\widetilde{B}_{\sigma^{s_1-1}\omega}(J)\right|^{2^{s_1}} (\widetilde{\lambda}_{\sigma^{s_1}\omega}+\delta)^{\sum_{i=1}^{2^{s_1}} j_i} \\ &\leqslant \left|\widetilde{B}_{\sigma^{s_1-1}\omega}(J)\right|^{2^{s_1}} (\widetilde{\lambda}_{\sigma^{s_1}\omega}+\delta)^{\left(1-\frac{\epsilon}{5}\right)(n+K_1)+2^{s_1}-1}. \end{aligned} \tag{27}$$

The number of summands in the right hand side of (26) is

$$\begin{aligned}\binom{\left(1-\frac{\epsilon}{5}\right)(n+K_1)+2^{s_1}-1+2^{s_1}}{2^{s_1}} &\leqslant \binom{n+K_1+2^{s_1+1}-1}{2^{s_1}} \\ &\leqslant (n+K_1+2^{s_1+1}-1)^{2^{s_1}}. \end{aligned} \tag{28}$$

Now by (25), (26), (27) and (28) we get

$$\begin{aligned}\left|\widetilde{B}_\omega(n)\right| \leqslant K_1 N_1 \left|\widetilde{B}_\omega(N_1)\right| \\ &+ \Big(2K_1(n+K_1+2^{s_1+1}-1)^{2^{s_1}} \left|\widetilde{B}_{\sigma^{s_1-1}\omega}(J)\right|^{2^{s_1}} \\ &\times (\widetilde{\lambda}_{\sigma^{s_1}\omega}+\delta)^{\left(1-\frac{\epsilon}{5}\right)(n+K_1)+2^{s_1}-1}\Big). \end{aligned}$$

Therefore,

$$\widetilde{\lambda}_\omega = \lim_n \left|\widetilde{B}_\omega(n)\right|^{1/n} \leqslant \left(\widetilde{\lambda}_{\sigma^{s_1}\omega}+\delta\right)^{\left(1-\frac{\epsilon}{5}\right)}.$$

Since δ is arbitrary,

$$\widetilde{\lambda}_\omega \leqslant \left(\widetilde{\lambda}_{\sigma^{s_1}\omega}\right)^{\left(1-\frac{\epsilon}{5}\right)}.$$

In the same way, still under the assumption (24), and replacing ω by $\omega, \sigma^{s_1}\omega, \sigma^{s_1+s_2}\omega, \sigma^{s_1+s_2+s_3}\omega, \ldots$, we get

$$\begin{aligned}
\widetilde{\lambda}_\omega &\leqslant \left(\widetilde{\lambda}_{\sigma^{s_1}\omega}\right)^{\left(1-\frac{\epsilon}{5}\right)}\\
\widetilde{\lambda}_{\sigma^{s_1}\omega} &\leqslant \left(\widetilde{\lambda}_{\sigma^{s_1+s_2}\omega}\right)^{\left(1-\frac{\epsilon}{5}\right)}\\
\widetilde{\lambda}_{\sigma^{s_1+s_2}\omega} &\leqslant \left(\widetilde{\lambda}_{\sigma^{s_1+s_2+s_3}\omega}\right)^{\left(1-\frac{\epsilon}{5}\right)}\\
&\vdots
\end{aligned}$$

Thus for each $k \in \mathbb{N}$,

$$\widetilde{\lambda}_\omega \leqslant \left(\widetilde{\lambda}_{\sigma^{s_1+\ldots+s_k}\omega}\right)^{\left(1-\frac{\epsilon}{5}\right)^k}. \tag{29}$$

But the growth index λ of a group with 8 generators of order 2 cannot exceed 9. Since k may be chosen arbitrarily large, it follows from (29) that $\widetilde{\lambda}_\omega = 1$. If there exists an $\epsilon > 0$ satisfying (24), then $\widetilde{\lambda}_\omega = 1$. If not, then for all $\epsilon > 0$ we have (22). Thus by (23) and

$$\lim_{\epsilon\to 0}(1-2\epsilon)^{-1/2}(\epsilon/3)^{-\epsilon} = 1,$$

we get $\widetilde{\lambda}_\omega = 1$ in all cases. Since $\widetilde{\lambda}_\omega = 1$, $\widetilde{G}_\omega$ has subexponential growth.

We know $G_\omega \subset \widetilde{G}_\omega$ and by [12], G_ω is of intermediate growth. Therefore $\widetilde{G}_\omega$ is of intermediate growth. □

Note that the Theorem 1 follows directly from Proposition 2, Proposition 3, and Theorem 3.

3. Growth bounds for $\widetilde{G}_\omega$

Proposition 4. *Let $\omega \in \Omega_0 \cup \Omega_1$. Then for each $\epsilon > 0$,*

$$\gamma_{\widetilde{G}_\omega}(n) \succeq \exp\left(\frac{n}{\log^{2+\epsilon}(n)}\right).$$

Proof. Let $\omega \in \Omega_0 \cup \Omega_1$. We may assume ω has infinitely many 0's and 2's. Then, by (1), b_ω as a sequence of P's and I's contains both symbols infinitely often. By Theorem 2 of [7] the group generated by elements a, b_ω, x has growth bounded below by $\exp\left(\frac{n}{\log^{2+\epsilon}(n)}\right)$. Since $\widetilde{G}_\omega$ contains the elements a, b_ω, x, we get the required result. $\square$

Theorem 2′. *Let $\omega \in \Omega_1^*$. Then,*

$$\gamma_{\widetilde{G}_\omega}(n) \preceq \exp\left(\frac{n \log(\log(n))}{\log(n)}\right).$$

Proof. Since $\omega \in \Omega_1^*$, there is an N such that $\sigma^N \omega$ contains exactly two symbols, say i, j. Then by Lemma 3, $\widetilde{G}_{\sigma^N \omega} = G_{\sigma^N \omega}$. By Theorem 3 of [7], we get

$$\gamma_{\widetilde{G}_{\sigma^n \omega}}(n) \preceq \exp\left(\frac{n \log(\log(n))}{\log(n)}\right),$$

and therefore,

$$\begin{aligned}\gamma_{\widetilde{G}_\omega}(n) &\approx \left(\gamma_{\widetilde{G}_{\sigma^n \omega}}(n)\right)^{2^N} \\ &\preceq \left(\exp\left(\frac{n \log(\log(n))}{\log(n)}\right)\right)^{2^N} \approx \exp\left(\frac{n \log(\log(n))}{\log(n)}\right). \quad \square\end{aligned}$$

While Theorem 3 states that $\widetilde{G}_\omega$ has intermediate growth for all $\omega \in \Omega_0$, for homogeneous sequences from Ω_0^*, we can actually provide an explicit upper bound on growth.

Theorem 2″. *Let $\omega \in \Omega_0^*$. Then,*

$$\gamma_{\widetilde{G}_\omega}(n) \leqslant \exp\left(\frac{n \log(\log(n))}{\log(n)}\right).$$

Proof. The proof follows similarly as of the proof of Theorem 3 of [7] by replacing Lemma 6.2 (1) of [7] by Lemma 6. $\square$

Theorem 2′ together with Theorem 2″ implies Theorem 2.

References

[1] L. Bartholdi, *The growth of Grigorchuk's torsion group*, Int. Math. Res. Not. IMRN, N.**20**, 1998, pp.1049-1054.

[2] L. Bartholdi, *A Wilson group of non-uniformly exponential growth*, C. R. Math. Acad. Sci. Paris, **336**, 2003, N.7, pp.549-554.

[3] M. G. Benli, R. I. Grigorchuk, T. Nagnibeda, *Universal groups of intermediate growth and their invariant random subgroups*, Funct. Anal. Appl., **49**, 2015, N.3, pp.159-174.

[4] L. Bartholdi, A. Erschler, *Groups of given intermediate word growth*, Ann. Inst. Fourier (Grenoble), **64**, 2014, N.5, pp.2003-2036.

[5] L. Bartholdi, R. I. Grigorchuk, *On the spectrum of Hecke type operators related to some fractal groups*, Tr. Mat. Inst. Steklova, N.**231**, 2000, Din. Sist., Avtom. i Beskon. Gruppy, pp.5-45; translation in Proc. Steklov Inst. Math., 2000, no. 4(231), pp.1-41.

[6] L. Bartholdi, R. I. Grigorchuk, *On parabolic subgroups and Hecke algebras of some fractal groups*, Serdica Math. J., **28**, 2002, N.1, pp.47-90.

[7] A. Erschler, *Boundary behavior for groups of subexponential growth*, Ann. of Math., **160**, 2004, N.3, pp.1183-1210.

[8] A. Erschler, T. Zheng, *Growth of periodic Grigorchuk groups*, Invent. math., **219**, 2020, pp.1069-1155.

[9] D. Francoeur, *On the subexponential growth of groups acting on rooted trees*, Groups Geom. Dyn., **14**, 2020, N.1, pp.1-24.

[10] R. I. Grigorchuk, *On Burnside's problem on periodic groups*, Funct. Anal. Appl., **14**, 1980, N.1, pp.41-43.

[11] R. I. Grigorchuk, *On the Milnor problem of group growth*, Soviet Math. Dokl., **28**, 1983, N.1, pp.23-26.

[12] R. I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat., **48**, 1984, N.5, pp.939-985.

[13] R. I. Grigorchuk, *Construction of p-groups of intermediate growth that have a continuum of factor-groups*, Algebra Logika, **23**, 1984, N.4, pp.383-394.

[14] R. I. Grigorchuk, *Degrees of growth of p-groups and torsion-free groups*, Math. USSR-Sb., **54**, 1986, N.1, pp.185-205.

[15] R. I. Grigorchuk, *On growth in group theory*, Proceedings of the International Congress of Mathematicians Vol **I** (August 21-29, 1990), Kyoto, Japan, Math. Soc. Japan, 1991, pp.325-338.

[16] M. Kassabov, I. Pak, *Groups of Oscillating Intermediate Growth*, Ann. of Math., **177**, 2013, N.3, pp.1113-1145.

[17] J. Milnor, *Problem 5603*, Amer. Math. Monthly, **75**, 1968, N.6, pp.685-686.

[18] R. Muchnik, I. Pak, *On growth of Grigorchuk groups*, Internat. J. Algebra Comput. **11**, 2001, N.1, pp.1-17.

[19] V. Nekrashevych, *Palindromic subshifts and simple periodic groups of intermediate growth*, Ann. of Math., **187**, 2018, N.3, pp.667-719.

[20] A. I. Schwarz, *A volume invariant of covering*, Dokl. Akad. Nauk SSSR, N.**105**, 1955, pp.32-34.

Contact information

Supun Thamara Samarakoon
Department of Mathematics
Mailstop 3368
Texas A&M University
College Station, TX 77843-3368
United States
E-Mail(s): sts@tamu.edu
Web-page(s): www.math.tamu.edu/~stsamster

Received by the editors: 06.09.2019
and in final form 30.06.2020.