Conjugacy in finite state wreath powers of finite permutation groups

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Abstract. It is proved that conjugated periodic elements of the infinite wreath power of a finite permutation group are conjugated in the finite state wreath power of this group. Counter-examples for non-periodic elements are given.

1. Introduction

The conjugacy classes in the full automorphism group of a regular rooted tree are described in [1]. But for its subgroup of finite state automorphisms the corresponding description is a challenging task. In particular, there exist finite state level-transitive automorphisms (and therefore, conjugated in the full automorphism group) which are not conjugates in the finite state subgroup [2]. Deep results about conjugation of some special finite state automorphisms were obtained in [3] and [4].

The most natural way to introduce finite state automorphisms uses automata theory. But regarding our purposes we choose a language of infinite wreath products. The full automorphism group of $m$-regular rooted tree is the infinite wreath power of the symmetric group of degree $m$. If we restrict ourselves to some subgroup (or even subsemigroup) $G$ of this symmetric group we naturally obtain the infinite and finite state wreath powers of $G$ (cf. [5,6]).

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The work is organized as follows. In Section 2 we recall the finite state wreath power of a finite permutation group. In Section 3 we prove the main result of the paper. Namely, if periodic finite state elements of the infinite wreath power of a finite permutation group are conjugated in this wreath power then they are conjugated in the finite state wreath power of a given permutation group. In Section 4 we show how to construct non-periodic finite state elements conjugated in the wreath power but not conjugated in the finite state wreath power of a given permutation group.

For all other definitions used in the paper one can refer to [2].

2. Finite state wreath power

Let $A$ be a finite set of cardinality $m \geq 2$. Consider a finite group $G$ acting faithfully on the set $A$. In other words, the permutation group $(G, A)$ is a subgroup of the symmetric group $\text{Sym}(A)$. In the sequel we assume that the groups act on the sets from the right and denote by $a^g$ the result of the action of a group element $g$ on a point $a$.

Denote by $W^\infty(G, A)$ the infinitely iterated wreath product of $(G, A)$. The group $W^\infty(G, A)$ consists of permutations of the infinite cartesian product $X^\infty$ given by infinite sequences of the form

$$
\mathbf{g} = [g_1, g_2(x_1), \ldots, g_n(x_1, \ldots, x_{n-1}), \ldots],
$$

where $g_1 \in G$, $g_2(x_1) : A \to G$, $\ldots$, $g_n(x_1, \ldots, x_{n-1}) : A^{n-1} \to G$, $\ldots$ For each $n \geq 1$ we call $g_n$ the $n$-th term of $\mathbf{g}$ and denote it by $[\mathbf{g}]_n$. An element $\mathbf{g}$ acts on a point

$$
\bar{a} = (a_1, a_2, \ldots, a_n, \ldots) \in A^\infty
$$

by the rule

$$
\bar{a}^\mathbf{g} = (a_1^{g_1}, a_2^{g_2(a_1)}, \ldots, a_n^{g_n(a_1, \ldots, a_{n-1})}, \ldots).
$$

Let $\mathbf{g} = [g_1, g_2(x_1), g_3(x_1, x_2), \ldots] \in W^\infty(G, A)$ and $\bar{a} = (a_1, \ldots, a_n) \in A^n$ for some $n \geq 1$. Define an element $\text{rest}(\mathbf{g}, \bar{a}) \in W^\infty(G, A)$ as

$$
\text{rest}(\mathbf{g}, \bar{a}) = [h_1, h_2(x_1), h_3(x_1, x_2), \ldots],
$$

where

$$
\begin{align*}
    h_1 &= g_{n+1}(a_1, \ldots, a_n), \\
    h_2(x_1) &= g_{n+2}(a_1, \ldots, a_n, x_1), \\
    h_3(x_1, x_2) &= g_{n+3}(a_1, \ldots, a_n, x_1, x_2), \ldots
\end{align*}
$$
The tuple $\text{rest}(g, \bar{a})$ has the form (1) as well and therefore belongs to the group $W^\infty(G, A)$. The element $\text{rest}(g, \bar{a})$ is called the state of $g$ at $\bar{a}$. Also we consider $g$ as a state of itself.

Define the set
\[ Q(g) = \{\text{rest}(g, \bar{a}) : \bar{a} \in A^n, n \geq 1\} \cup \{g\} \]
of all states of $g$. In particular, for the identity element $e \in W^\infty(G, A)$ we get the equality $Q(e) = \{e\}$. The following lemma is directly verified.

Lemma 1. Let $g, h \in W^\infty(G, A)$. Then
\[ Q(gh) \subseteq Q(g) \cdot Q(h), \quad Q(g^{-1}) = Q(g)^{-1}. \]
If $h \in Q(g)$ then $Q(h) \subseteq Q(g)$.

Let
\[ FW^\infty(G, A) = \{g \in W^\infty(G, A) : |Q(g)| < \infty\}. \]
Lemma 1 implies that this set form a subgroup of $W^\infty(G, A)$.

Definition 1. The group $FW^\infty(G, A)$ is called the finite state wreath power of the permutation group $(G, A)$.

The group $FW^\infty(G, A)$ is countable while $W^\infty(G, A)$ is not.

Another useful remark is that both permutation groups $(W^\infty(G, A), A^\infty)$ and $(FW^\infty(G, A), A^\infty)$ split into the wreath product of $(G, A)$ and itself, i.e.
\[ (W^\infty(G, A), A^\infty) = (G, A) \wr (W^\infty(G, A), A^\infty) \]
and
\[ (FW^\infty(G, A), A^\infty) = (G, A) \wr (FW^\infty(G, A), A^\infty). \]
This allows us to present an element $g = [g_1, g_2(x_1), g_3(x_1, x_2), \ldots] \in W^\infty(G, A)$ in the form
\[ g = [g_1; \text{rest}(g, a), a \in A]. \]

3. Conjugation of periodic elements

It is convenient for us to identify the set $A$ with the set $\{1, \ldots, m\}$. We need some additional notation here. Let elements $g \in G$ and $a \in A$ be fixed. Consider the cyclic decomposition of $g$ as a permutation on $A$. The length of the cycle containing $a$ will be denoted by $l(g, a)$. Then $a^{g^{l(g, a)}} = a$.
and $l(g, a)$ is the smallest integer satisfying this equality. The minimal element of this cycle we denote by $s(g, a)$. Note, that $l(g, s(g, a)) = l(g, a)$. The smallest integer $d \geq 0$ such that $a^{g^d} = s(g, a)$ will be denoted by $d(g, a)$. Then $0 \leq d(g, a) \leq l(g, a) - 1$.

We have the following

**Lemma 2.** Let $u$, $v$ and $h$ be elements of $G$ such that $u = h^{-1}vh$. Then for arbitrary $a \in A$ the equality $l(u, a^h) = l(v, a)$ holds.

**Proof.** From the definition of $l(v, a)$ we have $a^{l(v, a)} = a$. Thus

$$(a^h)^{l(v, a)} = (a^h)^{(h^{-1}vh)^{l(v, a)}} = (a^h)^{h^{-1}v^{l(v, a)}h} = (a^{v^{l(v, a)}})^h = a^h.$$ 

Hence, $l(u, a^h) \leq l(v, a)$. The inequality $l(u, a^h) \geq l(v, a)$ is proved analogously. 

The rules of multiplication and taking inverses in wreath products imply that for arbitrary $u = [u_1, \ldots], v = [v_1, \ldots] \in W^\infty(G, A)$ and $a \in A$ one have the equalities:

$$\text{rest}(uv, a) = \text{rest}(u, a)\text{rest}(v, a^{u_1}) \quad \text{and} \quad \text{rest}(u^{-1}, a) = \text{rest}(u, a^{v_1^{-1}})^{-1}.$$ 

Then we can prove

**Lemma 3.** Let $u = [u_1, \ldots], v = [v_1, \ldots]$ and $h = [h_1, \ldots]$ be elements of the group $W^\infty(G, A)$ such that $u = h^{-1}vh$. Then for arbitrary $a \in A$ the equality

$$\text{rest}(u^{l(u_1, a^{h_1})}, a^{h_1}) = (\text{rest}(h, a))^{-1}\text{rest}(v^{l(v_1, a)}, a)\text{rest}(h, a)$$

holds.

**Proof.** Note, that $u_1 = h_1^{-1}v_1h_1$. By Lemma 2 we have equalities

$$u^{l(u_1, a^{h_1})} = u^{l(v_1, a)} = h^{-1}v^{l(v_1, a)}h.$$

Hence,

$$\text{rest}(u^{l(u_1, a^{h_1})}, a^{h_1}) = \text{rest}(h^{-1}v^{l(v_1, a)}h, a^{h_1})$$

$$= \text{rest}(h^{-1}, a^{h_1})\text{rest}(v^{l(v_1, a)}, a)\text{rest}(h, a^{l(v_1, a)})$$

$$= (\text{rest}(h, a))^{-1}\text{rest}(v^{l(v_1, a)}, a)\text{rest}(h, a).$$
**Theorem 1.** Arbitrary elements of finite order of the group \(FW^\infty(G, A)\), conjugated in the group \(W^\infty(G, A)\), are conjugated in the group \(FW^\infty(G, A)\) as well.

*Proof.* Denote by \(M\) the set of all pairs \((u, v) \in FW^\infty(G, A) \times FW^\infty(G, A)\) such that elements \(u\) and \(v\) have finite order and are conjugated in the group \(W^\infty(G, A)\). For each pair \(\theta = (u, v) \in M\) let us fix an element \(\Psi(\theta) \in W^\infty(G, A)\) such that the equality \(u = (\Psi(\theta))^{-1}v\Psi(\theta)\) holds. The correspondence \(\theta \mapsto \Psi(\theta)\) may be regarded as a mapping \(\Psi : M \rightarrow W^\infty(G, A)\).

Now we proceed as follows. Using \(\Psi\) we construct new mappings

\[
\Psi_* : M \rightarrow W^\infty(G, A) \quad \text{and} \quad \Phi : M \times A \rightarrow W^\infty(G, A).
\]

Then we show that for each pair \(\theta = (u, v) \in M\) the equality

\[u = (\Psi_*(\theta))^{-1}v\Psi_*(\theta)\]

holds. Finally, we prove that indeed \(\Psi_*(M) \subset FW^\infty(G, A)\). Then the statement of the theorem follows.

For a pair \(\theta = (u, v) \in M\), where \(u = [u_1, u_2(x_1), \ldots]\) and \(u = [v_1, v_2(x_1), \ldots]\), we will use the notation \(\Psi(\theta) = [h_1, h_2(x_1), \ldots]\).

*Step 1.* Let us define mappings \(\Psi_*\) and \(\Phi\).

It is sufficient to check that recursive equalities

\[
\Psi_*(\theta) = [h_1; \text{rest}(v^{d(v_1, a)}, a)\Phi(\theta, a)(\text{rest}(u^{d(v_1, a)}, a^{h_1}))^{-1}, a \in A],
\]

\[
\Phi(\theta, a) = \Psi_*(\text{rest}(u^{l(u_1, a^{h_1})}, (s(u_1, a))^{h_1}), \text{rest}(v^{l(v_1, a)}, s(v_1, a)))\]

\[a \in A\]

(5)

(6)

correctly define required mappings \(\Psi_*\) and \(\Phi\). First of all, from (5) we have \([\Psi_*(\theta)]_1 = [\Psi(\theta)]_1\) and hence the term \([\Psi_*(\theta)]_1\) is well-defined. To define other terms we need \(\Phi(\theta, a), a \in A\). Then, by Lemma 3, for each \(a \in A\) the pair

\[
(\text{rest}(u^{l(u_1, a^{h_1})}, (s(u_1, a))^{h_1}), \text{rest}(v^{l(v_1, a)}, s(v_1, a)))
\]

belongs to \(M\). Hence, equality (6) defines the first term of \(\Phi(\theta, a), a \in A\). Again looking at (5), we obtain the second term of \(\Psi_*(\theta)\) and so on. Inductively, for arbitrary \(k \geq 1\), having defined the \(k\)th term of \(\Psi_*(\theta)\) by (5), we define the \(k\)th term of \(\Phi(\theta)\) by (6) and this gives us a possibility to define the \((k + 1)\)th term of \(\Psi_*(\theta)\) by (5).
Note that for every $a \in A$ the equality $\Phi(\theta, a^{v_1}) = \Phi(\theta, a)$ holds.

Step 2. Let us prove the equality $u = (\Psi_*(\theta))^{-1}v\Psi_*(\theta)$, where $\theta = (u, v) \in M$.

We will prove by induction on $k$ the equality

$$[u]_k = [(\Psi_*(\theta))^{-1}v\Psi_*(\theta)]_k.$$  

Since $[\Psi_*(\theta)]_1 = [\Psi(\theta)]_1$ and $$[u]_1 = [(\Psi(\theta))^{-1}v\Psi(\theta)]_1$$ we obtain the required statement for $k = 1$.

Assume that for the $(k - 1)$th terms the equality is proved. Proceed with the $k$th ones. Fix an element $a \in A$. Denote by $g$ the state of $(\Psi_*(\theta))^{-1}v\Psi_*(\theta)$ at $a^{h_1}$. It is sufficient to check the equality $[\text{rest}(u, a^{h_1})]_k = [g]_k$. For $g$ we have the equalities:

$$g = \text{rest}((\Psi_*(\theta))^{-1}v\Psi_*(\theta), a^{h_1})$$  

$$= \text{rest}((\Psi_*(\theta))^{-1}, a^{h_1})\text{rest}(v, a)\text{rest}(\Psi_*(\theta), a^{v_1})$$  

$$= (\text{rest}(\Psi_*(\theta), a))^{-1}\text{rest}(v, a)\text{rest}(\Psi_*(\theta), a^{v_1})$$  

$$= \text{rest}(u^{d(v_1, a), a^{h_1}})(\Phi(\theta, a))^{-1}(\text{rest}(v^{d(v_1, a), a}))^{-1}\text{rest}(v, a)$$  

$$\cdot \text{rest}(v^{d(v_1, a^{v_1}), a^{v_1}})\Phi(\theta, a^{v_1})(\text{rest}(u^{d(v_1, a^{v_1}), a^{v_1}h_1}))^{-1}$$  

$$= \text{rest}(u^{d(v_1, a), a^{h_1}})(\Phi(\theta, a))^{-1}(\text{rest}(v^{d(v_1, a), a}))^{-1}$$  

$$\cdot \text{rest}(v^{d(v_1, a^{v_1})+1, a})\Phi(\theta, a^{v_1})(\text{rest}(u^{d(v_1, a^{v_1}), a^{v_1}h_1}))^{-1}.$$  

There are two possibilities: $s(v_1, a) = a$ or $s(v_1, a) \neq a$. Consider these cases.

1) Let $s(v_1, a) = a$. Then $d(v_1, a) = 0$ and $d(v_1, a^{v_1}) = l(v_1, a) - 1$. This implies $\text{rest}(u^{d(v_1, a), a^{h_1}}) = e$ and $\text{rest}(v^{d(v_1, a), a}) = e$. Lemma 2 and equality (6) then implies

$$\Phi(\theta, a) = \Psi_*(\text{rest}(u^{l(v_1, a), a^{h_1}}), \text{rest}(v^{l(v_1, a), a})).$$  

Then, in view of the inductive hypothesis, the equalities follow:

$$[g]_k = [(\Phi(\theta, a))^{-1}\text{rest}(v^{l(v_1, a), a})\Phi(\theta, a^{v_1})(\text{rest}(u^{l(v_1, a)-1, a^{v_1}h_1}))^{-1}]_k$$  

$$= [\text{rest}(u^{l(v_1, a), a^{h_1}})(\text{rest}(u^{l(v_1, a)-1, a^{h_1}u_1}))^{-1}]_k$$  

$$= [\text{rest}(u, a^{h_1})\text{rest}(u^{l(v_1, a)-1, a^{h_1}u_1})(\text{rest}(u^{l(v_1, a)-1, a^{h_1}u_1}))^{-1}]_k$$  

$$= [\text{rest}(u, a^{h_1})]_k.$$
2) Let $s(v_1, a) \neq a$. Then $d(v_1, a^{v_1}) = d(v_1, a) - 1$. For $g$ now we have:

$$
g = \text{rest}(u^{d(v_1, a)}, a^{h_1})(\Phi(\theta, a))^{-1}(\text{rest}(v^{d(v_1, a)}, a))^{-1}$$

$$
\cdot \text{rest}(v^{d(v_1, a^{v_1})+1}, a)\Phi(\theta, a^{v_1})(\text{rest}(u^{d(v_1, a^{v_1})}, a^{v_1h_1}))^{-1}
$$

$$
= \text{rest}(u^{d(v_1, a)}, a^{h_1})(\text{rest}(u^{d(v_1, a)-1}, a^{h_1u_1}))^{-1}
$$

$$
= \text{rest}(u, a^{h_1})\text{rest}(u^{d(v_1, a)-1}, a^{h_1u_1})(\text{rest}(u^{d(v_1, a)-1}, a^{h_1u_1}))^{-1}
$$

$$
= \text{rest}(u, a^{h_1}).
$$

In both cases we obtained the equality $[\text{rest}(u, a^{h_1})]_k = [g]_k$. Hence, our statement is true for the $k$th terms.

**Step 3.** Let us check the inclusion $\Psi_*(M) \subset FW^\infty(G, A)$.

Denote by $M_k$ the subset of all pairs $(u, v) \in M$ such that the orders of $u$ and $v$ equal $k$. These subsets are pairwise disjoint and

$$
M = \bigcup_{k=1}^{\infty} M_k.
$$

Let us prove by induction on $k$ that $\Psi_*(M_k) \subset FW^\infty(G, A)$.

In case $k = 1$ we have $M_1 = \{ (e, e) \}$. Since

$$
\text{rest}(e, a) = e, \quad a \in A,
$$

equalities (5) and (6) imply

$$
\text{rest}(\Psi_*(e, e), a) = \Psi_*(e, e), \quad a \in A.
$$

Hence, $g \in FW^\infty(G, A)$.

Suppose that $\Psi_*(M_i) \subset FW^\infty(G, A)$ for all $i < k$. We are going to prove the inclusion $\Psi_*(M_k) \subset FW^\infty(G, A)$. If the set $M_k$ is empty then the statement is true. Let $\theta = (u, v)$ be a pair belonging to the set $M_k$. We have to show that $|Q(\Psi_*(\theta))| < \infty$.

For an element $g \in W^\infty(G, A)$ its set of stable states is defined as

$$
SQ(g) = \{ \text{rest}(g, a) : \text{rest}(g^2, a) = (\text{rest}(g, a))^2, a \in A^n, n \geq 1 \} \cup \{ g \}.
$$

Since $u, v \in FW^\infty(G, A)$ the set

$$
Q_1 = \Psi_*\left( (SQ(u) \times SQ(v)) \cap M_k \right)
$$

is finite. In particular, $\Psi_*((u, v)) \in Q_1$. We are going to show that $Q_1 \subset FW^\infty(G, A)$. 

Fix arbitrary \( g \in Q_1 \). Then

\[
g = \Psi_*((\hat{u}, \hat{v}))
\]

for some \( \hat{u} \in SQ(u) \), \( \hat{v} \in SQ(v) \) such that \( (\hat{u}, \hat{v}) \in M_k \).

Let \( a \in A \). Denote \( l([\hat{v}]_1, a) \) by \( \ell \). Two possible cases arise.

Case 1: \( \ell = 1 \). Then \( s([\hat{v}]_1, a) = a \) and \( d([\hat{v}]_1, a) = 0 \). By equalities (5) and (6) we now obtain

\[
\text{rest}(g, a) = \text{rest}(\Psi_*((\hat{u}, \hat{v})), a) = \Phi((\hat{u}, \hat{v}), a)
\]

\[
= \Psi_*\left(\text{rest}(\hat{u}, a^{\Psi((\hat{u}, \hat{v}))}), \text{rest}(\hat{v}, a)\right).
\]

The latter element belongs to \( Q_1 \). Hence, in this case

\[
\text{rest}(g, a) \in Q_1.
\]

Case 2: \( \ell > 1 \). By Lemma 2 the equality \( l([\hat{u}]_1, a^{\Psi((\hat{u}, \hat{v}))}) = l([\hat{v}]_1, a) = \ell \) holds. By (6) we have

\[
\Phi((\hat{u}, \hat{v}), a) = \Psi_*\left(\text{rest}(\hat{u}^\ell, (s([\hat{v}]_1, a))^{\Psi((\hat{u}, \hat{v}))}), \text{rest}(\hat{v}^\ell, s([\hat{v}]_1, a))\right).
\]  

(7)

Since cyclic decompositions of both \([\hat{u}]_1\) and \([\hat{v}]_1\) contain a cycle of length \( \ell \), the number \( \ell \) divides the orders of both \( \hat{u} \) and \( \hat{v} \). It implies that the orders of arguments of \( \Psi_* \) in (7) are strictly less than \( k \). Indeed, they are states of the \( \ell \)th powers of elements \( \hat{u} \) and \( \hat{v} \) correspondingly at elements belonging to cycles of length \( \ell \). Applying the inductive hypothesis we get \( \Phi((\hat{u}, \hat{v}), a) \in FW^\infty(G, A) \). Now from the definition of \( \Psi_* \) we obtain that the state

\[
\text{rest}(g, a) = \text{rest}(\Psi_*((\hat{u}, \hat{v})), a)
\]

belongs to \( FW^\infty(G, A) \) as the product of elements from \( FW^\infty(G, A) \).

Thus, the state \( \text{rest}(g, a) \) belongs to the finite set \( Q_1 \) or the set \( Q(\text{rest}(g, a)) \) is finite.

Let

\[
Q_2 = \{ \text{rest}(g, a) : g \in Q_1, a \in A \} \cap FW^\infty(G, A)
\]

and

\[
Q_3 = \bigcup_{h \in Q_2} Q(h).
\]

Since sets \( Q_1 \) and \( A \) are finite, the set \( Q_2 \) is finite. Being a union of finite number of finite sets, the set \( Q_3 \) is finite as well. Then, using the definition of the state we obtain

\[
Q(g) \subseteq Q_1 \cup Q_3.
\]

Therefore, \( g \in FW^\infty(G, A) \). The proof is complete. \( \Box \)
Observe that rewriting mapping $\Psi_*$ constructed in the proof of theorem 1 may be defined on the set of all pairs of conjugated elements of $W_\infty(G, A)$. Additional conditions on such elements were used only to prove that the image of $\Psi_*$ belongs to the finite state wreath power of $(G, A)$. It would be interesting to examine this image in general case.

4. Non-conjugated elements of infinite order

Let us show how to construct two elements of the group $FW_\infty(G, A)$ which are conjugated in the group $W_\infty(G, A)$ but are not conjugated in the group $FW_\infty(G, A)$.

Let $g$ be a non-identity element of the group $G$. If $n$ is the order of the element $g$ and $p$ is a prime divisor of $n$ then the element $g^* = g^{n/p}$ has order $p$ and as a permutation on $A$ is a product of independent cycles of length $p$. Without loss of generality we assume that $g^*$ has no fixed points. We will identify the set $A$ with the set $\{0, \ldots, m-1\}$ in such a way that for some $k \geq 1$ the permutation $g^*$ will be expressed in the form

$$g^* = (0, \ldots, p-1)(p, \ldots, 2p-1) \ldots ((k-1)p, \ldots, kp-1).$$

Let us consider the set $A_p = \{0, \ldots, p-1\}$, the cyclic group $G_p = \langle \sigma \rangle$ generated by the permutation $\sigma = (0, \ldots, p-1)$ and the mapping $c : G_p \to G$ that maps an element $h \in G_p$ to the permutation acting on the set $\{0, \ldots, kp-1\}$ by the rule $x \mapsto (x \mod p)^h + [x/p] \cdot p$ and trivially on other elements of the set $A$. In other words the mapping $c$ duplicates action on the set $A_p$ onto the sets $\{p, \ldots, 2p-1\}, \ldots, \{(k-1)p, \ldots, kp-1\}$.

Using mapping $c$ one can transform any element $u \in W_\infty(G_p, A_p)$ into an element $u^{(k)} \in W_\infty(G, A)$ by the equality

$$[u^{(k)}]_n(x_1, \ldots, x_{n-1}) = \begin{cases} c([u]_n(x_1 \mod p, \ldots, x_{n-1} \mod p)), & 0 \leq x_1, \ldots, x_{n-1} < kp, \\ e, & \text{otherwise}. \end{cases}$$

Denote by $f$ the function that for any $u \in W_\infty(G_p, A_p)$ computes $u^{(k)} \in W_\infty(G, A)$. The function $f$ is well-defined.

**Lemma 4.** If $u \in FW_\infty(G_p, A_p)$ then $u^{(k)} \in FW_\infty(G, A)$.

**Proof.** If $u \in W_\infty(G_p, A_p)$ then the value of $[u]_n$ equals to some power of $\sigma$. By definition of the transformation the value of $[u^{(k)}]_n$ equals to the same power of $g^*$ or $e$ depending on the arguments. Thus $u^{(k)} \in W_\infty(G, A)$.
Denote by $A_{kp}$ the set $\{0, \ldots, kp - 1\}$ and denote by $\bar{a} \mod p$ the element 

$$(a_1 \mod p, a_2 \mod p, \ldots, a_n \mod p) \in A_p^n$$

for $\bar{a} = (a_1, a_2, \ldots, a_n) \in A^n$. We are going to prove for $u \in W^\infty(G_p, A_p)$ that 

$$\text{rest}(f(u), \bar{a}) = \begin{cases} f(\text{rest}(u, \bar{a} \mod p)), & \bar{a} \in A_{kp}^n, \ n \geq 1, \\ e, & \text{otherwise.} \end{cases}$$ (8)

For $\bar{a} = (a_1, \ldots, a_n) \in A^n$ and $n \geq 1$ we have 

$$[\text{rest}(f(u), \bar{a})]_m(x_1, \ldots, x_{m-1}) = [f(u)]_{n+m}(a_1, \ldots, a_n, x_1, \ldots, x_{m-1}).$$

If $\bar{a} \notin A_{kp}^n$ then $[\text{rest}(f(u), \bar{a})]_m(x_1, \ldots, x_{m-1}) = e$ for all $x_1, \ldots, x_{m-1}$ that implies $\text{rest}(f(u), \bar{a}) = e$. In case $\bar{a} \in A_{kp}^n$ the equality 

$$[\text{rest}(f(u), \bar{a})]_m(x_1, \ldots, x_{m-1}) =$$

$$= \begin{cases} c([u]_{n+m}(a_1 \mod p, \ldots, x_{m-1} \mod p)), & 0 \leq x_1, \ldots, x_{m-1} < kp, \\ e, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} c(\text{rest}(u, \bar{a} \mod p)]_m(x_1 \mod p, \ldots, x_{m-1} \mod p)), & 0 \leq x_1, \ldots, x_{m-1} < kp, \\ e, & \text{otherwise,} \end{cases}$$

$$= [f(\text{rest}(u, \bar{a} \mod p))]_m(x_1, \ldots, x_{m-1})$$

holds. Thus $\text{rest}(f(u), \bar{a}) = f(\text{rest}(u, \bar{a} \mod p))$.

From equality (8) for $u \in W^\infty(G_p, A_p)$ we get 

$$Q(u^{(k)}) = Q(f(u)) = \{\text{rest}(f(u), \bar{a}) : \bar{a} \in A^n, \ n \geq 1\} \cup \{f(u)\} \subset$$

$$\subset \{f(\text{rest}(u, \bar{a} \mod p)) : \bar{a} \in A_{kp}^n, \ n \geq 1\} \cup \{e\} \cup \{f(u)\} \subset$$

$$\subset f(Q(u)) \cup \{e\} \cup \{f(u)\} = f(Q(u)) \cup \{e\}.$$

This implies that if $u \in FW^\infty(G_p, A_p)$ then $u^{(k)} \in FW^\infty(G, A)$. 

Suppose that we have two elements $u, v \in FW^\infty(G_p, A_p)$ that satisfy conditions: 1) $u$ and $v$ are conjugated in the group $W^\infty(G_p, A_p)$; 2) growth of $u$ is logarithmic; 3) growth of $v$ is exponential. Using elements $u$ and $v$ we construct elements $u^{(k)}$ and $v^{(k)}$. By lemma 4 these new elements belongs to the group $FW^\infty(G, A)$. Since 

$$f(Q(u)) \subset Q(f(u)) \subset f(Q(u)) \cup \{e\}.$$
u and \( f(u) \) have equivalent growth. If \( gu = vg \) then \( f(g)f(u) = f(v)f(g) \). Therefore \( u \) and \( f(u) \) satisfy the following conditions: 1') \( u^{(k)} \) and \( v^{(k)} \) are conjugated in the group \( W^\infty(G, A) \); 2') growth of \( u^{(k)} \) is logarithmic; 3') growth of \( v^{(k)} \) is exponential. Since the growth of an element is invariant under conjugation in \( FW^\infty(G, A) \) (see [2, subsection 4.3]) This implies that elements \( u^{(k)} \) and \( v^{(k)} \) are non-conjugated in the group \( FW^\infty(G, A) \).

Let us consider the following elements of the group \( FW^\infty(G_p, A_p) \)

\[
e = [e; e, \ldots, e], \quad a_i = [\sigma^i; a_0, a_1, \ldots, a_{p-1}], \quad 0 \leq i < p,
\]

\[
s = [\sigma; e, \ldots, e, s], \quad b_i = [\sigma^i; a_i, a_i, \ldots, a_i, b_{i+1}], \quad 0 \leq i < p.
\]

To simplify notations we will identify \( b_p \) and \( b_0 \). Let us show that the elements \( s \) and \( b_1 \) satisfy conditions 1)–3).

**Lemma 5.** An element \( g \in W^\infty(G_p, A_p) \) is level transitive (acts transitively on the sets \( A_p^k \), \( k \geq 1 \)) if and only if \( g^*_k = \prod_{v \in A_p^{k-1}}[g]_k(v) \neq e \) for all \( k \geq 1 \).

**Proof.** The proof is similar to the proof of lemma 4.4 in [2] and we use two additional facts that the group \( G_p \) is abelian and every non-unity element generates a transitive subgroup. \( \square \)

**Lemma 6.** Let \( p \) be an odd prime number. Then the element \( b_1 \) satisfies equalities \( (b_1)^*_k = \sigma \) for all \( k \geq 1 \) which implies that \( b_1 \) is level transitive.

**Proof.** Equalities \( (a_i)^*_1 = (b_i)^*_1 = \sigma^i \) are obvious and the recurrent formulas

\[
(a_i)^*_{k+1} = (a_0)^*_k(a_1)^*_k \cdots (a_{p-1})^*_k,
\]

\[
(b_i)^*_{k+1} = ((a_i)^*_k)^{p-1}(b_{i+1})^*_k
\]

follow from definitions. The first of the recurrent formulas implies

\[
(a_i)^*_{2} = (a_0)^*_1(a_1)^*_1 \cdots (a_{p-1})^*_1 = \sigma^{0+1+\ldots+(p-1)} = \sigma^{\frac{p(p-1)}{2}} = e
\]

and by induction we get \( (a_i)^*_k = 1 \) for all \( k \geq 2 \). The second of the recurrent formulas implies

\[
(b_i)^*_{2} = ((a_i)^*_1)^{p-1}(b_{i+1})^*_1 = \sigma^{-i}\sigma^{i+1} = \sigma,
\]

\[
(b_i)^*_{k} = ((a_i)^*_k-1)^{p-1}(b_{i+1})_{k-1}^* = (b_{i+1})_{k-1}^* = \sigma, \quad k \geq 3. \quad \square
\]

**Lemma 7.** The elements \( s \) and \( b_1 \) are conjugated in the group \( W^\infty(G_p, A_p) \).
Proof. The adding machine $s$ is level transitive. The element $b_1$ is level transitive by the lemma 6. Thus the elements $s$ and $b_1$ are conjugated in the group $W^\infty(S_p, A_p)$.

Suppose that equality $b_1 = g^{-1}sg$ holds for some $g \in W^\infty(S_p, A_p)$. Let us prove that $g \in W^\infty(G_p, A_p)$. The element $g$ for every $k \geq 0$ satisfies equality $b_1^k = g^{-1}sg^k$ which implies $[g]_k(\bar{a}) = (s)^*_k[g]_k(\bar{a})$ for $\bar{a} \in A_p^k$. From the last equality it follows by lemma 6 that $[g]_k(\bar{a})\sigma = \sigma[g]_k(\bar{a})$ and finally we get $[g]_k(\bar{a}) \in G_p$. □

Lemma 8. The element $b_1$ has exponential growth.

Proof. The proof is analogous to the proof of the proposition 4.2 in [2]. □

References

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