General formal local cohomology modules

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Abstract. Let $(R, \mathfrak{m})$ be a local ring, $\Phi$ a system of ideals of $R$ and $M$ a finitely generated $R$-module. In this paper, we define and study general formal local cohomology modules. We denote the $i$-th general formal local cohomology module $M$ with respect to $\Phi$ by $\mathcal{F}_\Phi^i(M)$ and we investigate some finiteness and Artinianness properties of general formal local cohomology modules.

Introduction

Throughout this paper, $R$ is a commutative Noetherian ring with identity, $\mathfrak{a}$ is an ideal of $R$, $\Phi$ a system of ideals of $R$ and $M$ is an $R$-module. Recall that the $i$-th local cohomology module of $M$ with respect to $\mathfrak{a}$ is denoted by $H_i^\mathfrak{a}(M)$. There are some generalizations of local cohomology theory. The following one is given in [2]. A system of ideals of $R$ is a non-empty set $\Phi$ of ideals of $R$ such that, whenever $a, b \in \Phi$, there exists $c \in \Phi$ with $c \subseteq a \cdot b$. For such a system, there is a $\Phi$-torsion functor $\Gamma_\Phi : \mathcal{C}(R) \to \mathcal{C}(R)$ (where $\mathcal{C}(R)$ denotes the category of $R$-modules and $R$-homomorphisms) such that for every $R$-module $M$,

$$
\Gamma_\Phi(M) := \{ x \in M : ax = 0 \text{ for some } a \text{ in } \Phi \}.
$$

In [2], $\Gamma_\Phi(-)$ is called the "general local cohomology functor with respect to $\Phi". For each $i \geq 0$, the $i$-th right derived functor of $\Gamma_\Phi(-)$ is denoted by $H_i^\Phi(-)$.


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For more details about general local cohomology modules see [2], [3].

Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and $M$ a finitely generated $R$-module. For each $i \geq 0$, $\mathcal{F}_\mathfrak{a}^i(M) := \lim_{\leftarrow n} H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ is called the $i$-th formal local cohomology of $M$ with respect to $\mathfrak{a}$.

The formal local cohomology modules have been studied by several authors; see for example [1], [4], [6], [9] and [10]. The purpose of this paper is to make a generalization of formal local cohomology theory as above. There are some generalization of formal local cohomology theory (see [7] and [11]). Here, we give a new generalization in terms of a system of ideals.

Let $(R,\mathfrak{m})$ be a local ring, $\Phi$ a system of ideals of $R$ and $M$ a finitely generated $R$-module. For each $i \geq 0$, we define $i$-th general formal local cohomology of $M$ with respect to $\Phi$ by

$$\mathcal{F}_\Phi^i(M) := \lim_{\leftarrow \mathfrak{a} \in \Phi} H^i_{\mathfrak{m}}(M/\mathfrak{a}M).$$

Clearly, for an ideal $\mathfrak{a}$ of $R$, if we put $\Phi := \{\mathfrak{a}^i \mid i \in \mathbb{N}\}$ then the above definition coincides with the original definition $\mathcal{F}_\mathfrak{a}^i(M)$.

In this paper, we get some results on Artinianness, vanishing and other properties of general formal local cohomology modules. Among other things, we will prove that, for any finitely generated $R$-module $M$ we have:

$$\inf\{i \in \mathbb{N} : \mathcal{F}_\Phi^i(M) \text{ is not representable}\} = \inf\{i \in \mathbb{N} : \mathcal{F}_\Phi^i(M) \text{ is not Artinian}\}$$

and

$$\sup\{i \in \mathbb{N} : \mathcal{F}_\Phi^i(M) \text{ is not representable}\} = \sup\{i \in \mathbb{N} : \mathcal{F}_\Phi^i(M) \text{ is not Artinian}\}.$$

Also, we study the structure of 0-th general formal local cohomology module and we will prove that for a complete local ring $(R,\mathfrak{m})$,

$$\operatorname{Ass}_R \mathcal{F}_\Phi^0(M) = \{p \in \operatorname{Ass}_R(M) : \dim R/(\mathfrak{a} + p) = 0 \text{ for all } \mathfrak{a} \in \Phi\}.$$
1. Results

Assume that \((R, m)\) is a local ring and that \(M\) is a finitely generated \(R\)-module. We investigate a generalization of formal local cohomology theory in terms of a system of ideals. A system of ideals of \(R\) is a non-empty set \(\Phi\) of ideals of \(R\) such that, whenever \(a, b \in \Phi\), there exists \(c \in \Phi\) with \(c \subseteq ab\). We define the relation \(\leq\) on \(\Phi\) by: \(a \leq b\) if and only if \(b \subseteq a\). It is easy to see that \(\Phi\) is a direct set by this relation. Now, let \(a, b \in \Phi\) such that \(a \leq b\), \(M\) be an \(R\)-module. Then for each integer \(n \geq 0\), the \(R\)-homomorphism \(M/b^n M \to M/a^n M\) induces the \(R\)-homomorphism \(\psi_{a}^{b} : H^{i}_{m}(M/b^n M) \to H^{i}_{m}(M/a^n M)\). Thus \(\{H^{i}_{m}(M/a^n M), \psi\}\) forms an inverse system of \(R\)-modules and \(R\)-homomorphisms over \(\Phi\). Now we are ready to give the following definition.

**Definition 1.** Let \((R, m)\) be a local ring, \(\Phi\) a system of ideals of \(R\) and \(M\) a finitely generated \(R\)-module. For each \(i \geq 0\);

\[ F^i_{\Phi}(M) := \lim_{\leftarrow a \in \Phi} H^{i}_{m}(M/a^n M) \]

is called the \(i\)-th general formal local cohomology of \(M\) with respect to \(\Phi\).

**Theorem 1.** Let \((R, m)\) be a local ring, \(\Phi\) a system of ideals of \(R\) and \(M\) a finitely generated \(R\)-module. For each \(i \geq 0\);

\[ F^i_{\Phi}(M) \cong \lim_{\leftarrow a \in \Phi} F^i_{a}(M). \]

**Proof.** Let \(a, b \in \Phi\) such that \(a \leq b\). If \(n\) is an integer then the natural homomorphism \(M/b^n M \to M/a^n M\) induces the homomorphism \(H^{i}_{m}(M/b^n M) \to H^{i}_{m}(M/a^n M)\) for any integer \(i \geq 0\). On the other hand, if \(n \leq m\) we have the following commutative diagram:

\[
\begin{array}{c}
H^{i}_{m}(M/b^n M) \quad \longrightarrow \quad H^{i}_{m}(M/a^n M) \\
\uparrow \quad \uparrow \\
H^{i}_{m}(M/b^m M) \quad \longrightarrow \quad H^{i}_{m}(M/a^m M)
\end{array}
\]

From the above diagram we get a homomorphism

\[ \lambda^{b}_{a} : \lim_{\leftarrow n} H^{i}_{m}(M/b^n M) \to \lim_{\leftarrow n} H^{i}_{m}(M/b^n M) \]

and so, we have

\[ \lambda^{b}_{a} : F^{i}_{b}(M) \to F^{i}_{a}(M). \]

This shows that \(\{F^{i}_{a}(M), \lambda\}_{a \in \Phi}\) is an inverse system of \(R\)-modules and \(R\)-homomorphisms over the directed set \(\Phi\). Thus we may form \(\lim_{\leftarrow a \in \Phi} F^{i}_{a}(M)\).

But, for each integer \(k \in \mathbb{N}\) and any ideal \(a \in \Phi\) there exists an ideal \(b \in \Phi\) such that \(b \subseteq a^k\). Thus, by using a proof similar to the proof
of [12, Lemma 3.8] for each integer \( k \) we have
\[
\lim_{\overset{\leftarrow}{a \in \Phi}} \lim_{m} H_{m}^{i}(M/\alpha M) \simeq \lim_{\overset{\leftarrow}{a \in \Phi}} \lim_{k} H_{m}^{i}(M/\alpha^{k}M)
\]
and so
\[
\lim_{\overset{\leftarrow}{a \in \Phi}} \Phi^{i}(M) \simeq \lim_{\overset{\leftarrow}{a \in \Phi}} \lim_{k} H_{m}^{i}(M/\alpha^{k}M) \simeq \lim_{k} \lim_{\overset{\leftarrow}{a \in \Phi}} H_{m}^{i}(M/\alpha^{k}M)
\]
\[
\simeq \lim_{\overset{\leftarrow}{a \in \Phi}} \lim_{m} H_{m}^{i}(M/\alpha M) \simeq \Phi^{i}(M).
\]

Let \((R, \mathfrak{m})\) be a local ring, \(\Phi\) a system of ideals of \(R\) and \(M\) a finitely generated \(R\)-module. Let \(x\) denotes a system of elements of \(R\) such that \(m = \text{Rad}(xR)\). Let \(\check{C}_{\overline{x}}\) denotes the Čech complex of \(R\) with respect to \(x\). For an \(R\)-module \(M\) and an ideal \(a\), it is easy to see that there exists an inverse system of \(R\)-complexes \(\{\check{C}_{\overline{x}} \otimes M/\alpha M\}_{a \in \Phi}\). Hence, we may form the inverse limit \(\lim_{\overset{\leftarrow}{a \in \Phi}} (\check{C}_{\overline{x}} \otimes M/\alpha M)\). By a proof similar to the proof of [12, proposition 3.2] we obtain the next result.

**Theorem 2.** With the previous notation, there is an isomorphism
\[
\Phi^{i}(M) \simeq H^{i}(\lim_{\overset{\leftarrow}{a \in \Phi}} (\check{C}_{\overline{x}} \otimes M/\alpha M))
\]
for all \(i \in \mathbb{Z}\).

*Proof.* It follows by a straightforward modification of the proof of [12, proposition 3.2]. \(\square\)

**Theorem 3.** Let \((R, \mathfrak{m})\) be a local ring, \(\Phi\) a system of ideals of \(R\) and \(M\) a finitely generated \(R\)-module. Then \(\Phi^{i}(M) = 0\) for all \(i > \dim(M)\).

*Proof.* Let \(i > \dim(M)\). By [12, Theorem 4.5] \(\Phi^{i}(M) = 0\) for all \(a \in \Phi\). Thus \(\Phi^{i}(M) = \lim_{\overset{\leftarrow}{a \in \Phi}} \Phi^{i}(M) = 0\), as required. \(\square\)

Let \(f : R \to R'\) be a homomorphism of Noetherian commutative rings. Set \(\Phi R' := \{aR' : a \in \Phi\}\). Then \(\Phi R'\) is a system of ideals of \(R'\). Now by using this notation we give the following result:

**Theorem 4.** Let \((R, \mathfrak{m})\) be a local ring, \(\Phi\) a system of ideals of \(R\) and \(M\) a finitely generated \(R\)-module. Then \(\Phi^{i}(M) \simeq \Phi^{i}R(\hat{M})\) for all \(i \in \mathbb{Z}\).

*Proof.* By [12, Proposition 3.3], \(\Phi^{i}(M) \simeq \Phi^{i}R(\hat{M})\). Thus \(\lim_{\overset{\leftarrow}{a \in \Phi}} \Phi^{i}(M) \simeq \lim_{\overset{\leftarrow}{a \in \Phi}} \Phi^{i}R(\hat{M})\). Now Theorem 1 completes the proof. \(\square\)
Recall that a dualizing complex $D_R$ for a local ring $(R, \mathfrak{m})$ is a bounded complex of injective $R$-modules whose cohomology modules $H^i(D_R)$ are finitely generated $R$-modules for all $i \in \mathbb{Z}$. For more details see [13]. It is well known that $R$ possesses a dualizing complex if and only if $R$ is the factor ring of a Gorenstein ring. The next result is an expression of the general formal local cohomology in terms of a certain general local cohomology of the dualizing complex.

**Theorem 5.** Let $(R, \mathfrak{m})$ be a local ring possessing a dualizing complex $D_R$, $\Phi$ a system of ideals of $R$ and $M$ a finitely generated $R$-module. Then

$$\mathfrak{F}_\Phi^i(M) \simeq \text{Hom}_R(H^i_{\Phi}(\text{Hom}_R(M, D_R)), E(R/\mathfrak{m})),$$

for all $i \in \mathbb{Z}$.

**Proof.** By Local Duality Theorem there are the isomorphisms

$$H^i_m(M/\alpha M) \simeq \text{Hom}_R(H^{-i}(\text{Hom}_R(M/\alpha M, D_R)), E(R/\mathfrak{m})),$$

for all $i \in \mathbb{Z}$ and $\alpha \in \Phi$. Thus we have

$$\lim_{\leftarrow \alpha \in \Phi} H^i_m(M/\alpha M) \simeq \text{Hom}_R(H^{-i}(\lim_{\rightarrow \alpha \in \Phi} \text{Hom}_R(M/\alpha M, D_R)), E(R/\mathfrak{m})),$$

for all $i \in \mathbb{Z}$. But $\lim_{\rightarrow \alpha \in \Phi} \text{Hom}_R(M/\alpha M, D_R)) \simeq \Gamma_\Phi(\text{Hom}_R(M, D_R))$ and so

$$\lim_{\leftarrow \alpha \in \Phi} H^i_m(M/\alpha M) \simeq \text{Hom}_R(H^{-i}(\Gamma_\Phi(\text{Hom}_R(M, D_R)), E(R/\mathfrak{m})),$$

for all $i \in \mathbb{Z}$. Therefore

$$\mathfrak{F}_\Phi^i(M) \simeq \text{Hom}_R(H^i_{\Phi}(\text{Hom}_R(M, D_R)), E(R/\mathfrak{m})),$$

for all $i \in \mathbb{Z}$, as required. \qed

**Theorem 6.** Let $(R, \mathfrak{m})$ be a local ring, $\Phi$ a system of ideals of $R$ and $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ a short exact sequence of finitely generated $R$-modules. Then there is a long exact sequence

$$\cdots \rightarrow \mathfrak{F}_\Phi^i(A) \rightarrow \mathfrak{F}_\Phi^i(B) \rightarrow \mathfrak{F}_\Phi^i(C) \rightarrow \mathfrak{F}_\Phi^{i+1}(A) \rightarrow \cdots.$$

**Proof.** It follows by an argument similar to the proof of [12, Theorem 3.11]. \qed
Theorem 7. Let \((R, \mathfrak{m})\) be a local ring, \(\Phi\) a system of ideals of \(R\) and \(M\) a finitely generated \(R\)-module. If \(w := \max \{\dim (M/aM) | a \in \Phi\}\) is finite then \(\mathfrak{f}_\Phi^w(M) \neq 0\) and \(\mathfrak{f}_\Phi^i(M) = 0\) for all \(i > w\).

Proof. Let \(i > w\). Since \(i > \dim (M/\mathfrak{a}M)\) for all \(\mathfrak{a} \in \Phi\), [12, Theorem 4.5] implies that \(\mathfrak{f}_\mathfrak{a}^i(M) = 0\) for all \(\mathfrak{a} \in \Phi\). Thus \(\mathfrak{f}_\Phi^i(M) = \lim_{\mathfrak{a} \to \Phi} \mathfrak{f}_\mathfrak{a}^i(M) = 0\). On the other hand, since \(w\) is finite there exists an ideal \(\mathfrak{b} \in \Phi\) such that \(\dim (M/\mathfrak{b}M) = w\). Now, put \(\Psi = \{\mathfrak{c} \in \Phi \mid \mathfrak{c} \subseteq \mathfrak{b}\}\). Then \(\Psi\) is cofinal in \(\Phi\). Thus we may assume that \(\mathfrak{a} \subseteq \mathfrak{b}\) for all \(\mathfrak{a} \in \Phi\). Let \(\mathfrak{c} \in \Phi\). It is easy to see that \(\dim (\mathfrak{b}M/\mathfrak{c}M) \leq \dim M/\mathfrak{c}M \leq w\) and so the exact sequence \(0 \to \mathfrak{b}M/\mathfrak{c}M \to M/\mathfrak{c}M \to M/\mathfrak{b}M \to 0\) induces \(H^w_m(M/\mathfrak{c}M) \to H^w_m(M/\mathfrak{b}M) \to 0\). Now for each \(\mathfrak{d} \in \Phi\) with \(\mathfrak{d} \leq \mathfrak{c}\), i.e., \(\mathfrak{c} \subseteq \mathfrak{d}\), we have the following commutative diagram:

\[
\begin{array}{ccc}
H^w_m(M/\mathfrak{d}M) & \xrightarrow{f_\mathfrak{d}} & H^w_m(M/\mathfrak{b}M) \\
\uparrow & & \uparrow \\
H^w_m(M/\mathfrak{c}M) & \xrightarrow{f_\mathfrak{c}} & H^w_m(M/\mathfrak{b}M) \to 0
\end{array}
\]

The family of \(R\)-modules \(\{\ker f_\mathfrak{c}\}_{\mathfrak{c} \in \Phi}\), as a family of Artinian \(R\)-modules, satisfies the Mittag-Leffler condition. Hence the above diagram induces an exact sequence \(\lim_{\mathfrak{c} \to \Phi} H^w_m(M/\mathfrak{c}M) \to H^w_m(M/\mathfrak{b}M) \to 0\). By Theorem 1 we get \(\mathfrak{f}_\Phi^w(M) \to H^w_m(M/\mathfrak{b}M) \to 0\). By Grothendieck’s non-vanishing Theorem \(H^w_m(M/\mathfrak{b}M) \neq 0\). Therefore \(\mathfrak{f}_\Phi^w(M) \neq 0\), as required.

Theorem 8. Let \((R, \mathfrak{m})\) be a local ring, \(\Phi\) a system of ideals of \(R\) and \(M\) a finitely generated \(R\)-module of dimension \(d\). Then \(\mathfrak{f}_\Phi^d(M)\) is homomorphic image of \(H^d_m(M)\), and so \(\mathfrak{f}_\Phi^d(M)\) is Artinian.

Proof. Let \(\mathfrak{a} \in \Phi\). We have \(\dim \mathfrak{a}M \leq \dim M\), so that, by the Grothendieck’s Vanishing Theorem, the short exact sequence

\[
0 \to \mathfrak{a}M \to M \to M/\mathfrak{a}M \to 0
\]

induces an exact sequence

\[
H^d_m(M) \xrightarrow{\phi_\mathfrak{a}} H^d_m(M/\mathfrak{a}M) \to 0.
\]

The family of \(R\)-modules \(\{\ker \phi_\mathfrak{a}\}_{\mathfrak{a} \in \Phi}\), as a family of Artinian \(R\)-modules, satisfies the Mittag-Leffler condition. Therefore, the above exact sequence induces an exact sequence \(\lim_{\mathfrak{a} \to \Phi} H^d_m(M) \to \lim_{\mathfrak{a} \to \Phi} H^d_m(M/\mathfrak{a}M) \to 0\) and
so we have the exact sequence $H^d_m(M) \rightarrow \mathcal{F}^d_{\Phi}(M) \rightarrow 0$, and the proof is complete.

In the next result, we investigate the 0-th general formal local cohomology module. Let $\mathfrak{a}$ be an ideal of $R$ and $M$ a finitely generated $R$-module. For a submodule $N$ of $M$ we denote the ultimate constant value of the increasing sequence

$$N \subseteq N :_M \mathfrak{a} \subseteq N :_M \mathfrak{a}^2 \subseteq \cdots \subseteq N :_M \mathfrak{a}^i \subseteq \cdots$$

by $N :_M (\mathfrak{a})$. Let $0 = \bigcap_{j=1}^n Q_j$ denotes a reduced primary decomposition of the zero submodule 0 in $M$ and $Q_j$ is a $p_j$-primary submodule of $M$, for all $j = 1, \cdots, n$. Put $T(\mathfrak{a}, M) := \{ \mathfrak{p} \in \text{Ass}_R M : \dim R/(\mathfrak{a} + \mathfrak{p}) > 0 \}$ and $u_M(\mathfrak{a}) := \bigcap_{\mathfrak{p} \in T(\mathfrak{a}, M)} Q_i$ also $T(\Phi, M) := \{ \mathfrak{p} \in \text{Ass}_R M : \text{there exists } \mathfrak{a} \in \Phi \text{ such that } \dim R/(\mathfrak{a} + \mathfrak{p}) > 0 \}$ and $u_M(\Phi) := \bigcap_{\mathfrak{p} \in T(\Phi, M)} Q_i$. With these notations we have:

**Theorem 9.** Let $(R, \mathfrak{m})$ be a local ring, $\Phi$ a system of ideals of $R$ and $M$ a finitely generated $R$-module. Then

i) $\bigcap_{\mathfrak{a} \in \Phi} u_M(\mathfrak{a}) = u_M(\Phi)$,

ii) $u_M(\Phi) = \bigcap_{\mathfrak{a} \in \Phi} (\mathfrak{a}M :_M (\mathfrak{m}))$,

iii) $\mathcal{F}^0_{\Phi}(M) \simeq u_M(\hat{\Phi} \hat{R})$.

**Proof.** i) It is easy to see that

$$\bigcap_{\mathfrak{a} \in \Phi} u_M(\mathfrak{a}) = \bigcap_{\mathfrak{a} \in \Phi} \bigcap_{\mathfrak{p} \in T(\mathfrak{a}, M)} Q_i = \bigcap_{\mathfrak{p} \in T(\Phi, M)} Q_i = u_M(\Phi).$$

ii) By [12, Lemma 4.1(a)], $u_M(\mathfrak{a}) = \bigcap_{n \geq 1} (\mathfrak{a}^nM :_M (\mathfrak{m}))$. Thus

$$u_M(\Phi) = \bigcap_{\mathfrak{a} \in \Phi} u_M(\mathfrak{a}) = \bigcap_{\mathfrak{a} \in \Phi} \bigcap_{n \geq 1} (\mathfrak{a}^nM :_M (\mathfrak{m})) \subseteq \bigcap_{\mathfrak{a} \in \Phi} (\mathfrak{a}M :_M (\mathfrak{m})).$$

Conversely, let $x \in \bigcap_{\mathfrak{a} \in \Phi} (\mathfrak{a}M :_M (\mathfrak{m}))$. Let $\mathfrak{a} \in \Phi$ be an ideal. Then there exists an integer $u$ such that $xm^u \subseteq \mathfrak{a}M$. For any integer $k$, there exists an ideal $\mathfrak{b} \in \Phi$ such that $\mathfrak{b} \subseteq \mathfrak{a}^k$. Since $x \in (\mathfrak{b}M :_M (\mathfrak{m}))$ there exists an integer $t$ such that $xm^t \subseteq \mathfrak{b}M \subseteq \mathfrak{a}^kM$. Hence $x \in (\mathfrak{a}^kM :_M (\mathfrak{m}))$ and so $x \in \bigcap_{n \geq 1} (\mathfrak{a}^nM :_M (\mathfrak{m}))$ for each ideal $\mathfrak{a} \in \Phi$. Therefore $x \in \bigcap_{\mathfrak{a} \in \Phi} \bigcap_{n \geq 1} (\mathfrak{a}^nM :_M (\mathfrak{m})) = u_M(\Phi)$.

iii) By Theorem 4 we may assume that $M = \hat{M}$ and $R = \hat{R}$. Let $\mathfrak{b}$ be a proper ideal of $R$ such that $\mathfrak{b} \in \Phi$. It is easy to see that $\bigcap_{\mathfrak{a} \in \Phi} \mathfrak{a}M \subseteq \bigcap_{n \geq 0} \mathfrak{b}^nM$. Thus Krull’s intersection theorem implies that $\bigcap_{\mathfrak{a} \in \Phi} \mathfrak{a}M = 0$. Now the proof is a straightforward modification of the proof of [12, Lemma 4.1(c)].
Corollary 1. Let \((R, \mathfrak{m})\) be a complete local ring, \(\Phi\) a system of ideals of \(R\) and \(M\) a finitely generated \(R\)-module. Then

\[\text{Ass}_R \mathfrak{S}_\Phi^0(M) = \{p \in \text{Ass}_R M : \dim R/(a + p) = 0 \text{ for all } a \in \Phi\}.\]

Proof. By [12, Lemma 2.7] \(\text{Ass}_R u_M(\Phi) = \text{Ass}_R M \setminus T(\Phi, M)\). But

\[\text{Ass}_R M \setminus T(\Phi, M) = \{p \in \text{Ass}_R M : \dim R/(a + p) = 0 \text{ for all } a \in \Phi\}\]

and \(\mathfrak{S}_\Phi^0(M) = u_M(\Phi)\) by Theorem 9(iii) and this finishes the proof. 

Corollary 2. Let \((R, \mathfrak{m})\) be a local ring, \(\Phi\) a system of ideals of \(R\) and \(M\) a finitely generated \(R\)-module. Then \(\mathfrak{S}_\Phi^0(M) = 0\) if and only if \(\text{Ass}_R \hat{M} = T(\Phi \hat{R}, \hat{M})\).

Proof. By Theorem 3(iii) \(\mathfrak{S}_\Phi^0(M) = 0\) if and only if \(u_\hat{M}(\Phi \hat{R}) = 0\). But

\[\text{Ass}_R u_M(\Phi \hat{R}) = \text{Ass}_R \hat{M} \setminus T(\Phi \hat{R}, \hat{M})\]

by [12, Lemma 2.7]. Thus \(u_\hat{M}(\Phi \hat{R}) = 0\) if and only if \(\text{Ass}_R \hat{M} = T(\Phi \hat{R}, \hat{M})\) and the proof is complete.

The next theorem gives a result for representable general formal local cohomology modules.

Theorem 10. Let \((R, \mathfrak{m})\) be a local ring, \(\Phi\) a system of ideals of \(R\) and \(M\) a finitely generated \(R\)-module. Let \(i\) be an integer such that \(\mathfrak{S}_\Phi^i(M)\) is nonzero and representable. Then there exists an ideal \(a \subseteq p\) for all \(p \in \text{Att}_R \mathfrak{S}_\Phi^i(M)\).

Proof. Let \(\mathfrak{S}_\Phi^i(M) = S_1 + S_2 + \ldots + S_n\) be a minimal secondary representation of \(\mathfrak{S}_\Phi^i(M)\) where \(S_j\) is non-zero and \(p_j\)-Secondary for \(j = 1, 2, \ldots, n\).

Let \(1 \leq j \leq n\). Since \(S_j \neq 0\), there exists \(0 \neq a = (a_i) \in S_j \subseteq \mathfrak{S}_\Phi^i(M) = \lim \longrightarrow_{a \in \Phi} H^i_m(M/\mathfrak{m}M)\).

Let \(a_k\) be the first nonzero component of \(a\). Thus there exists an ideal \(a_k \in \Phi\) such that \(a_k \in H^i_m(M/a_kM)\). We claim \(a_k \subseteq p_j\). If not, then there exists \(u \in a_k \setminus p_j\). Since \(u \notin p_j\), we have \(uS_j = S_j\). Thus \(a \in S_j = uS_j \subseteq u\mathfrak{S}_\Phi^i(M)\) But \(uH^i_m(M/a_kM) = 0\) and so the \(k\)-th component of each element of \(u\mathfrak{S}_\Phi^i(M)\) is zero. But \(a \in u\mathfrak{S}_\Phi^i(M)\) and the \(k\)-th component of \(a\) is not zero. It follows that \(a_k \subseteq p_j\) where \(a_k \in \Phi\). Hence, we proved that for each integer \(j \in \{1, \ldots, n\}\) there exists an ideal \(b_j \in \Phi\) such that \(b_j \subseteq p_j\). Since \(\Phi\) is a system of ideals there exists an ideal \(a \in \Phi\) such that \(a \subseteq b_1b_2\cdots b_n \subseteq p_j\) for all \(j \in \{1, \ldots, n\}\), this completes the proof. 

\(\Box\)
Corollary 3. Let \((R, \mathfrak{m})\) be a local ring, \(\Phi\) a system of ideals of \(R\) and \(M\) a finitely generated \(R\)-module. Let \(i\) be an integer such that \(\mathfrak{F}_\Phi^i(M)\) is nonzero and representable. Then there exists an ideal \(a \in \Phi\) such that \(a\mathfrak{F}_\Phi^i(M) = 0\).

Proof. By \([5, 7.2.11]\) \(\bigcap_{p \in \text{Att } \mathfrak{F}_\Phi^i(M)} \mathfrak{p} = \sqrt{(0 : \mathfrak{F}_\Phi^i(M))}\). Thus by Theorem 10 we conclude that there exists an ideal \(b\) in \(\Phi\) and an integer \(n\) such that, \(b^n \mathfrak{F}_\Phi^i(M) = 0\). Since \(\Phi\) is a system of ideals, there exists an ideal \(a\) in \(\Phi\) such that \(a \subseteq b^n\). Therefore \(a\mathfrak{F}_\Phi^i(M) = 0\), as desired. \(\square\)

Let \(R\) be a ring, \(\Phi\) a system of ideals of \(R\) and \(M\) an \(R\)-module. Recall that

\[ \Gamma_\Phi(M) := \{ x \in M : ax = 0 \text{ for some } a \in \Phi \}. \]

We say that \(M\) is \(\Phi\)-torsion if \(M = \Gamma_\Phi(M)\) and that \(M\) is \(\Phi\)-torsion-free if \(\Gamma_\Phi(M) = 0\). For a finitely generated \(R\)-module \(M\), it is easy to see that \(M\) is \(\Phi\)-torsion-free if and only if, for each \(a \in \Phi\), \(a\) contains a non-zero-divisor on \(M\).

In order to state the next result we recall the concept of Matlis dual.

Let \(M\) be an \(R\)-module and \(E(R/\mathfrak{m})\) the injective envelope of \(R/\mathfrak{m}\). The module \(D(M) = \text{Hom}_R(M, E(R/\mathfrak{m}))\) is called the Matlis dual of \(M\).

Lemma 1. Let \((R, \mathfrak{m})\) be a complete local ring, \(\Phi\) a system of ideals of \(R\) and \(M\) a finitely generated \(R\)-module. Then

(i) \(M\) is \(\Phi\)-adically complete (i.e \(M \simeq \lim_{\leftarrow a \in \Phi} (M/aM)\)),

(ii) \(\mathfrak{F}_\Phi^0(M)\) is finitely generated \(R\)-module.

Proof. i) Since \(M\) is finitely generated, \(D(M)\) is Artinian and so \(D(M)\) is \(\mathfrak{m}\)-torsion. For each \(i \in \mathbb{N}\), there exists \(a \in \Phi\) such that \(a \subseteq \mathfrak{m}^i\). Hence \(D(M)\) is \(\Phi\)-torsion and we have

\[ D(M) = \bigcup_{a \in \Phi} (0 :_{D(M)} a) \simeq \lim_{\leftarrow a \in \Phi} \text{Hom}_R(R/a, D(M)). \]

Thus

\[ M \simeq D D(M) \simeq D(\lim_{\leftarrow a \in \Phi} \text{Hom}_R(R/a, D(M))) \simeq \lim_{\leftarrow a \in \Phi} R/a \otimes R D D(M) \simeq \lim_{\leftarrow a \in \Phi} M/aM. \]

ii) By definition \(\mathfrak{F}_\Phi^0(M) = \lim_{\leftarrow a \in \Phi} H^0_m(M/aM)\). Since \(H^0_m(M/aM) \subseteq M/aM\) for all \(a \in \Phi\), by (i) we get

\[ \mathfrak{F}_\Phi^0(M) \subseteq \lim_{\leftarrow a \in \Phi} (M/aM) \simeq M. \]
Since $M$ is finitely generated we conclude that $\mathfrak{F}_\Phi^0(M)$ is finitely generated, as required. $\square$

**Lemma 2.** Let $\Phi$ be a system of ideals of $R$ and $L \xrightarrow{f} M \xrightarrow{g} N$ be a exact sequence of $R$-modules and $R$-homomorphisms. Suppose that there exist two ideal $a$ and $b$ in $\Phi$ such that $a L = 0$ and $b N = 0$. Then there exists an ideal $c \in \Phi$ such that $c M = 0$.

**Proof.** Since $b g(M) = 0$, we have $b M \subseteq \ker g = \text{im} f$. But $a L = 0$, and so $a(\text{im} f) = 0$. Thus $a b M = 0$. But, there exists an ideal $c \in \Phi$ such that $c \subseteq a b$. Therefore $c M = 0$ and the proof is complete. $\square$

For the following proof we need the next Lemma.

**Lemma 3.** Let $(R, m)$ be a local ring, $\Phi$ a system of ideals of $R$ and $M$ a finitely generated $R$-module. Let $M$ be an $\Phi$-torsion $R$-module. Then $\mathfrak{F}_\Phi^i(M) \cong H_m^i(M)$. Therefore $\mathfrak{F}_\Phi^i(M)$ is Artinian for all $i \geq 0$.

**Proof.** It is easy to see that, since $M$ is finitely generated and $\Phi$-torsion there exists an ideal $a$ in $\Phi$ such that $a M = 0$. We put $\Psi = \{ b \in \Phi \mid b \subseteq a \}$. Then $\Psi$ is cofinal in $\Phi$. Thus we may assume that $b \subseteq a$ for all $b \in \Phi$ and so $b M = 0$ for all $b \in \Phi$. Hence

$$\mathfrak{F}_\Phi^i(M) \cong \lim_{\leftarrow b \in \Phi} H_m^i(M/b M) \cong \lim_{\leftarrow b \in \Phi} H_m^i(M) \cong H_m^i(M)$$

for all $i \geq 0$, as desired. $\square$

**Theorem 11.** Let $(R, m)$ be a local ring, $\Phi$ a system of ideals of $R$ and $M$ a finitely generated $R$-module. Let $t \in \mathbb{N}$. Then the following statements are equivalent:

(i) $\mathfrak{F}_\Phi^i(M)$ is Artinian for all $i < t$,

(ii) $\mathfrak{F}_\Phi^i(M)$ is representable for all $i < t$,

(iii) there exists an ideal $a$ in $\Phi$ such that, $a\mathfrak{F}_\Phi^i(M) = 0$ for all $i < t$.

**Proof.** (i) $\Rightarrow$ (ii): Any Artinian $R$-module is representable.

(ii) $\Rightarrow$ (iii): By Corollary 3.

(iii) $\Rightarrow$ (i): We use induction on $t$. Since $\mathfrak{F}_\Phi^i(M) \cong \mathfrak{F}_\Phi^i(\hat{M})$ by Theorem 4, we may assume that $R$ is complete. Let $t = 1$. By Lemma 1(ii), $\mathfrak{F}_\Phi^0(M)$ is a finitely generated $R$-module. By assumption $\text{Supp}_R(\mathfrak{F}_\Phi^0(M)) \subseteq V(a)$ and so by Corollary 1 we conclude that $\text{Supp}_R(\mathfrak{F}_\Phi^0(M)) \subseteq V(m)$. Thus $\mathfrak{F}_\Phi^0(M)$ is Artinian.
Now suppose, inductively, that \( t > 0 \) and \( \mathfrak{F}_a^{t-1}(M) \) is Artinian for all \( i \leq t - 2 \). We show that \( \mathfrak{F}_a^{t-1}(M) \) is Artinian. By Theorem 6, the short exact sequence  

\[
0 \rightarrow \Gamma \Phi(M) \rightarrow M \rightarrow M/\Gamma \Phi(M) \rightarrow 0
\]

implies the long exact sequence  

\[
\cdots \rightarrow \mathfrak{F}^{i-1}_a(\Gamma \Phi(M)) \rightarrow \mathfrak{F}^{i-1}_a(M) \rightarrow \mathfrak{F}^{i-1}_a(M/\Gamma \Phi(M)) \rightarrow \mathfrak{F}^i_a(\Gamma \Phi(M)) \rightarrow \cdots.
\]

But \( \mathfrak{F}^i_a(\Gamma \Phi(M)) \) is Artinian for all \( i \) by Lemma 3. Thus by using the above long exact sequence it follows that \( \mathfrak{F}_a^{t}(M) \) is Artinian if and only if \( \mathfrak{F}^i_a(M/\Gamma \Phi(M)) \) is Artinian for all \( i \). On the other hand, since \( \Phi \) is a system of ideals, by Corollary 3 we can find an ideal \( b \in \Phi \) such that \( b\mathfrak{F}^i_a(\Gamma \Phi(M)) = 0 \) for all \( i \leq t \). By assumption and lemma 2 we conclude that there exists an ideal \( c \in \Phi \) such that \( c\mathfrak{F}^i_a(M/\Gamma \Phi(M)) = 0 \) for all \( i < t \). Therefore we can and do assume that \( M \) is an \( \Phi \)-torsion-free \( R \)-module.

Since \( a \in \Phi \), it is easy to see that \( a \) contains an element \( r \) which is a non-zerodivisor on \( M \). The short exact sequence  

\[
0 \rightarrow M \rightarrow M \rightarrow M/rM \rightarrow 0
\]

induces a long exact sequence  

\[
0 \rightarrow \mathfrak{F}^0_a(M) \rightarrow \mathfrak{F}^0_a(M) \rightarrow \cdots \rightarrow \mathfrak{F}^i_a(M) \rightarrow \mathfrak{F}^i_a(M) \rightarrow \mathfrak{F}^i_a(M/rM) \rightarrow \cdots.
\]

By assumption and the above long exact sequence and lemma 2, it follows that there exists an ideal \( b \in \Phi \) such that \( b\mathfrak{F}^i_a(M/rM) = 0 \) for all \( i < t - 1 \). Thus, by the inductive hypothesis, we conclude that \( \mathfrak{F}^{t-2}_a(M/rM) \) is Artinian. Since \( r\mathfrak{F}^{t-1}_a(M) \subseteq a\mathfrak{F}^{t-1}_a(M) = 0 \), the above long exact sequence implies that \( \mathfrak{F}^{t-2}_a(M/rM) \rightarrow \mathfrak{F}^{t-1}_a(M) \rightarrow 0 \) is exact. But \( \mathfrak{F}^{t-2}_a(M/rM) \) is Artinian and so \( \mathfrak{F}^{t-1}_a(M) \) is Artinian, as required.

\textbf{Theorem 12.} Let \((R, \mathfrak{m})\) be a local ring, \( \Phi \) a system of ideals of \( R \) and \( M \) a finitely generated \( R \)-module. Let \( t \in \mathbb{N} \). Then the following statements are equivalent:

\begin{itemize}
  \item [(i)] \( \mathfrak{F}_a^{t}(M) \) is Artinian for all \( i > t \),
  \item [(ii)] \( \mathfrak{F}_a^{t}(M) \) is representable for all \( i > t \),
  \item [(iii)] there exists an ideal \( a \) in \( \Phi \) such that, \( a\mathfrak{F}_a^{t}(M) = 0 \) for all \( i > t \).
\end{itemize}
Proof. (i) \( \Rightarrow \) (ii): It is clear.
(ii) \( \Rightarrow \) (iii): By Corollary 3.
(iii) \( \Rightarrow \) (i): The proof can be easily obtained by extending the proof of [4, Theorem 2.9] *mutatis mutandis* to this general case. \( \square \)

Let \( \mathfrak{a} \) be an ideal of a local ring \( (R, \mathfrak{m}) \) and \( M \) a finitely generated \( R \)-module of dimension \( d \). By Theorem 8 \( \mathfrak{g}^d_\Phi(M) \) is Artinian. In the next result we determine the set \( \text{Att}_R \mathfrak{g}^d_\Phi(M) \).

**Theorem 13.** Let \( (R, \mathfrak{m}) \) be a local ring, \( \Phi \) a system of ideals of \( R \) and \( M \) a finitely generated \( R \)-module of dimension \( d \). Then there exists an ideal \( \mathfrak{a} \) in \( \Phi \) such that \( \text{Att}_R \mathfrak{g}^d_\Phi(M) = \text{Assh}_R(M) \cap \text{V}(\mathfrak{a}) \).

**Proof.** Let \( w := \max \{ \dim(M/\mathfrak{a}M) : \mathfrak{a} \in \Phi \} \). If \( w < d \) then \( \mathfrak{g}^d_\Phi(M) = 0 \) by Theorem 7 and so there is nothing to prove. Thus we suppose that \( w = d \).

By Theorem 10 there exists an ideal \( \mathfrak{a} \in \Phi \) such that \( \text{Att}_R \mathfrak{g}^d_\Phi(M) \subseteq \text{V}(\mathfrak{a}) \). But by Theorem 8 and [5, 7.3.2] \( \text{Att}_R \mathfrak{g}^d_\Phi(M) \subseteq \text{Att}_R H^d_{\mathfrak{m}}(M) = \text{Assh}_R(M) \). Thus \( \text{Att}_R \mathfrak{g}^d_\Phi(M) \subseteq \text{Assh}_R(M) \cap \text{V}(\mathfrak{a}) \).

Conversely, assume that \( \mathfrak{a} \in \Phi \). We show that \( \text{Assh}_R(M) \cap \text{V}(\mathfrak{a}) \subseteq \text{Att}_R \mathfrak{g}^d_\Phi(M) \). Let \( \mathfrak{p} \in \text{Assh}_R(M) \cap \text{V}(\mathfrak{a}) \). By [8, 6.8], there exists a \( \mathfrak{p} \)-primary submodule \( N \) of \( M \) such that \( \text{Ass}(M/N) = \{ \mathfrak{p} \} \) and \( \mathfrak{p} = \sqrt{(0 : (M/N))} \). Thus \( \dim M/N = \dim R/\mathfrak{p} = \dim M \). Since \( \mathfrak{a} \subseteq \mathfrak{p} \) we have \( \sqrt{\mathfrak{a}} \subseteq \sqrt{(0 : (M/N))} \).

Thus we can see that \( \text{Supp}_R((M/N)/\mathfrak{a}(M/N)) = \text{Supp}_R(M/N) \) and \( \dim ((M/N)/\mathfrak{a}(M/N)) = \dim (M/N) \). Now by Theorem 7, \( \mathfrak{g}^d_\Phi(M/N) \neq 0 \). Hence

\[ \phi \neq \text{Att}_R \mathfrak{g}^d_\Phi(M/N) \subseteq \text{Att}_R H^d_{\mathfrak{m}}(M/N) \subseteq \text{Ass}(M/N) = \{ \mathfrak{p} \} \]

Therefore we have \( \text{Att}_R \mathfrak{g}^d_\Phi(M/N) = \{ \mathfrak{p} \} \). But the exact sequence

\[ 0 \to N \to M \to M/N \to 0 \]

induces \( \mathfrak{g}^d_\Phi(M) \to \mathfrak{g}^d_\Phi(M/N) \to 0 \). Thus \( \{ \mathfrak{p} \} = \text{Att}_R \mathfrak{g}^d_\Phi(M/N) \subseteq \text{Att}_R \mathfrak{g}^d_\Phi(M) \). Therefore \( \mathfrak{p} \in \text{Att}_R \mathfrak{g}^d_\Phi(M) \). This completes the proof. \( \square \)

**Corollary 4.** Let \( (R, \mathfrak{m}) \) be a local ring, \( \Phi \) a system of ideals of \( R \) and \( M \) and \( N \) be two finitely generated \( R \)-modules of dimension \( d \) such that \( \text{Supp}_R M = \text{Supp}_R N \). Then \( \text{Att}_R \mathfrak{g}^d_\Phi(M) = \text{Att}_R \mathfrak{g}^d_\Phi(N) \).
Proof. By Theorem 13 there exist two ideals \( a \) and \( b \) in \( \Phi \) such that \( \text{Att}_R \mathfrak{S}_d^d(M) = \text{Assh}_R M \cap V(a) \) and \( \text{Att}_R \mathfrak{S}_d^d(N) = \text{Assh}_R N \cap V(b) \). But by assumption we have \( \text{Assh}_R M = \text{Assh}_R N \). On the other hand, by using the proof of Theorem 13,

\[
\text{Att}_R \mathfrak{S}_d^d(M) = \text{Assh}_R M \cap V(a) = \text{Assh}_R N \cap V(a) \subseteq \text{Att}_R \mathfrak{S}_d^d(N).
\]

Similarly \( \text{Att}_R \mathfrak{S}_d^d(N) \subseteq \text{Att}_R \mathfrak{S}_d^d(M) \). This completes the proof. \( \square \)

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