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Poisson brackets on some skew PBW extensions

B. A. Zambrano

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This article is dedicated to my family

Abstract. In [1] the author gives a description of Poisson brackets on some algebras of quantum polynomials $\mathcal{O}_q$, which is called the *general algebra of quantum polynomials*. The main of this paper is to present a generalization of [1] through a description of Poisson brackets on some skew PBW extensions of a ring A by the extensions $\mathcal{O}^{r,n}_{q,\delta}$, which are generalization of $\mathcal{O}_q$, and show some examples of skew PBW extension where we can apply this description.

1. Introduction

Skew PBW extensions were introduced in [2] and some of its properties have been studied in [4], [5], [6], [7], among others. Let A and R be rings, we say that R is a skew PBW extension of A, if the following conditions hold:

1) $A \subset R$
2) There exist finitely many elements $x_1, \ldots, x_n \in R$ such that R is a left A-free module with basis

$$\mathrm{Mon}\{x_1, \ldots, x_n\} = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} | (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}\}.$$

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3) For every $i = 1, \dots, n$ there exist an injective ring endomorphism $\sigma_i : A \to A$ and a σ_i-derivation $\delta_i : A \to A$ such that

$$x_i a = \sigma_i(a) x_i + \delta_i(a)$$

for all $a \in A$.

4) For every $1 \leqslant i < j \leqslant n$ there exists $q_{ij} \in A$ left invertible and $a_{ij}^{(t)} \in A$ such that

$$x_j x_i = q_{ij} x_i x_j + a_{ij}^{(0)} + a_{ij}^{(1)} x_1 + \cdots + a_{ij}^{(n)} x_n$$

Under these conditions we will denote $R = \sigma(A)\langle x_1, \dots, x_n\rangle$.

Fix $1 \leqslant r \leqslant n$, we will denote $\mathcal{O}_{q,\delta}^{r,n}$ an extension of A such that

1) $A \subset \mathcal{O}_{q,\delta}^{r,n}$.
2) There exist finitely many elements $x_1, \dots, x_n, x_1^{-1}, \dots, x_r^{-1} \in \mathcal{O}_{q,\delta}^{r,n}$ such that $\mathcal{O}_{q,\delta}^{r,n}$ is a left A-free module with basis

$$\mathrm{Mon}\{x_1^{\pm}, \dots, x_r^{\pm}, x_{r+1}, \dots, x_n\} = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} | (\alpha_1, \dots, \alpha) \in Z\},$$

where $Z = \mathbb{Z}^r \times \mathbb{N}^{n-r}$.

3) For every $i = 1, \dots, n$ there exist a derivation $\delta_i : A \to A$ such that

$$x_i a = a x_i + \delta_i(a)$$

for all $a \in A$.

4) For every $1 \leqslant i < j \leqslant n$ there exists $q_{ij} \in Z(A)$ invertible and $a_{ij}^{(t)} \in Z(A)$ such that

$$x_j x_i = q_{ij} x_i x_j + a_{ij}^{(0)} + a_{ij}^{(1)} x_1 + \cdots + a_{ij}^{(n)} x_n$$

5) $x_i x_i^{-1} = x_i^{-1} x_i = 1$ for $i = 1, \dots, r$.

In the present paper we shall assume that $\mathcal{O}_{q,\delta}^{r,n}$ satisfies the following conditions

1) $\delta_1 = 0$.
2) For every i fixed, $i = 1, \dots, n$ and $(m_1, \dots, m_n) \in Z \setminus \{(0, \dots, 0)\}$, $\left(1 - \prod_{j=1, j\neq i}^{n} q_{ij}^{m_j}\right) \in A^*$.
3) $\delta_t(q_{ij}) = \delta_t(a_{ij}^{(m)}) = 0$ for all $m = 0, \cdots, n$ and $i, j, t = 1, \cdots, n$.
4) $a_{1j}^{(0)} = 0$ for all $j = 1, \dots, n$. If $1 \leqslant i \leqslant r$ and $i < j$ then $a_{ij}^{(m)} = 0$ for all $m = i, \cdots, n$.

We will denote $\bar{p}_{ij} = p_{ij} - a_{ij}^{(0)}$.

Remark 1.1. We will consider the deglex order over $\mathrm{Mon}(\mathcal{O}_{q,\delta}^{r,n}) := \mathrm{Mon}\{x_1^{\pm}, \ldots, x_r^{\pm}, x_{r+1}, \ldots, x_n\}$ that is defined by [2] as

$$x^\alpha \succeq x^\beta = \begin{cases} x^\alpha = x^\beta & \text{or} \\ x^\alpha \neq x^\beta \text{but} |\alpha| > |\beta| & \text{or} \\ x^\alpha \neq x^\beta, \ |\alpha| = |\beta| \text{but} \exists i & \\ \quad \text{with} \alpha_1 = \beta_i, \ldots, \alpha_{i-1} = \beta_{i-1} \text{and} \alpha_i > \beta_i & \end{cases}$$

where $x^\alpha, x^\beta \in \mathrm{Mon}(\mathcal{O}_{q,\delta}^{r,n})$. Each element $f \in \mathcal{O}_{q,\delta}^{r,n} \setminus \{0\}$ can be represented in a unique way as $f = \eta_{v_0} x^{v_0} + \cdots + \eta_{v_k} x^{v_k}$, with $\eta_{v_l} \in A \setminus \{0\}$, $1 \leqslant l \leqslant k$, and $x^{v_k} \succ \cdots \succ x^{v_0}$; we take $\deg(f) = \deg(x^{v_k}) := |v_k|$[1]. We will denote $it(f) = \eta_{v_k} x^{v_k}$ the *leader term of* f.

Lemma 1.2. For all $z \in \mathcal{O}_{q,\delta}^{r,n}$, $i, j = 1, \ldots, n$, and $m = 0, \ldots, n$ we have that $q_{ij} z = z q_{ij}$ and $a_{ij}^{(m)} z = z a_{ij}^{(m)}$.

Proof. It is enough to see that $q_{ij} x_t = x_t q_{ij}$ and $a_{ij}^{(m)} x_t = x_t a_{ij}^{(m)}$ for all t. By the definition of $\mathcal{O}_{q,\delta}^{r,n}$ we have that $x_t q_{ij} = q_{ij} x_t + \delta_t(q_{ij}) = q_{ij} x_t$ and $x_t a_{ij}^{(m)} = a_{ij}^{(m)} x_t + \delta_t(a_{ij}^{(m)}) = a_{ij}^{(m)} x_t$. □

Lemma 1.3. Let $x_1^{a_1} \cdots x_n^{a_n} \in \mathrm{Mon}(\mathcal{O}_{q,\delta}^{r,n})$ then

1) For all $i = 1, \ldots, n$ there exists $p_{i,a} \in \mathcal{O}_{q,\delta}^{r,n}$ such that $x_1^{a_1} \cdots x_n^{a_n} x_i = \left(\prod_{j>i} q_{ij}^{a_j}\right) x_1^{a_1} \cdots x_i^{a_i+1} \cdots x_n^{a_n} + p_{i,a}$ with $\deg(p_{i,a}) < a_1 + \cdots + a_n + 1$ or $p_{i,a} = 0$ where $a = (a_1, \ldots, a_n)$.
2) For all $i = 1, \ldots, n$ there exists $p_{a,i} \in \mathcal{O}_{q,\delta}^{r,n}$ such that $x_i x_1^{a_1} \cdots x_n^{a_n} = \left(\prod_{j<i} q_{ji}^{a_j}\right) x_1^{a_1} \cdots x_i^{a_i+1} \cdots x_n^{a_n} + p_{a,i}$ with $\deg(p_{a,i}) < a_1 + \cdots + a_n + 1$ or $p_{a,i} = 0$ where $a = (a_1, \ldots, a_n)$.

Proof. Fix $x_i \in \mathcal{O}_{q,\delta}^{r,n}$ and take $x_{t_1}^{a_1} \cdots x_{t_j}^{a_j} \in \mathcal{O}_{q,\delta}^{r,n}$ such that $a_l \neq 0$. We will prove the first claimed by induction on j.

1) $(j = 1, x_{t_1}^{a_1} = x_t^b)$ We will show this claimed by induction on b, we can suppose that $i < t$, in other way, we have the claimed.

(a) $(b = 1)$ By definition, for all $i < t$ we have that $x_t x_i = q_{it} x_i x_t + p_{i,t}$, where $\deg(p_{i,t}) < 2$.

[1]If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ then we take $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

(b) $(b+1)$ Put $i<t$. By induction hypothesis there exist $p_{l,b}$ such that $x_t^b x_l = q_{lt}^b x_l x_t^b + p_{l,b}$ where $\deg(p_{l,b}) < b+1$ for all $l<t$, then

$$\begin{aligned} x_t^{b+1}x_i &= x_t^b\left(q_{it}x_ix_t + a_{it}^{(0)} + \sum_{l=1}^{n} a_{it}^{(l)}x_l\right) \\ &= q_{it}x_t^b x_i x_t + \left(a_{it}^{(0)}x_t^b + \sum_{l=1}^{n} a_{it}^{(l)}x_t^b x_l\right) \\ &= q_{it}q_{it}^b x_i x_t^{b+1} + q_{it}p_{i,b} + a_{it}^{(0)}x_t^b + \sum_{l=1}^{t-1} a_{it}^{(l)}q_{lt}^b x_l x_t^b \\ &\quad + \sum_{l=1}^{t-1} a_{it}^{(l)}p_{l,b} + \sum_{l=t}^{n} a_{it}^{(l)}x_t^b x_l \\ &= q_{it}^{b+1}x_ix_t^{b+1} + p_{i,b+1} \end{aligned}$$

where $p_{i,b+1} = a_{it}^{(0)}x_t^b + \sum_{l=1}^{t-1} a_{it}^{(l)}q_{lt}^b x_l x_t^b + \sum_{l=1}^{t-1} a_{it}^{(l)}p_{l,b} + \sum_{l=t}^{n} a_{it}^{(l)}x_t^b x_l$ and

$$\deg(a_{it}^{(0)}x_t^b),\ \deg(a_{it}^{(l)}q_{lt}^b x_l x_t^b),\ \deg(a_{it}^{(l)}p_{l,b}),\ \deg(a_{it}^{(l)}x_t^b x_l) < b+2$$

(c) ($b=-1$, if $t \leqslant r$) We will show that $p_{i,-1} = \sum_{h=1}^{i-1}\sum_{k=1}^{s(i,-1)} r_{h,k}x_hx_t^{-k}$ with $r_{h,k} \in \langle a_{ij}^{(m)}, q_{ij} | i,j = 1,\dots,n,\ m=0,\dots,n\rangle = G$, sub-ring of A, by induction on $1 \leqslant i \leqslant r$. If $i=1$ then $x_t^{-1}x_1 = x_t^{-1}(x_1x_t)x_t^{-1} = x_t^{-1}(q_{t1}x_tx_1)x_t^{-1} = q_{1t}^{-1}x_1x_t^{-1}$ and $p_{1,-t}=0$. If $i>1$ by induction hypothesis we have that

$$\begin{aligned} x_t^{-1}x_i = x_t^{-1}(x_ix_t)x_t^{-1} &= x_t^{-1}\left(q_{ti}x_tx_i + a_{ti}^{(0)} + \sum_{l=1}^{i-1} a_{ti}^{(l)}x_l\right)x_t^{-1} \\ &= q_{ti}x_ix_t^{-1} + a_{ti}^{(0)}x_t^{-2} + \sum_{l=1}^{i-1} a_{ti}^{(l)}x_t^{-1}x_lx_t^{-1} \\ &= q_{it}^{-1}x_ix_t^{-1} + a_{ti}^{(0)}x_t^{-2} + \sum_{l=1}^{i-1} a_{ti}^{(l)}q_{lt}^{-1}x_lx_t^{-2} + \sum_{l=1}^{i-1} a_{ti}^{(l)}p_{l,-t}x_t^{-1} \end{aligned}$$

since $p_{l,-t} = \sum_{h=1}^{l-1}\sum_{k=1}^{s(l,-1)} r_{h,k}x_hx_t^{-k}$ with $\deg(p_{l,-t}) < 0$ for all $l = 1,\dots,i-1$ then $\deg(p_{l,-t})x_t^{-1} < -1$ for all l and

$$p_{l,-t}x_t^{-1} = \sum_{h=1}^{l-1}\sum_{k=1}^{s(l,-1)} r_{h,k}x_hx_t^{-k-1}$$

so we have the claimed.

(d) ($-(b+1)$, $b>0$ if $t\leqslant r$) The same way, put $i<t$, by induction hypothesis there exist $p_{i,-1}, p_{i,-b}$ as in (c) such that $x_t^{-1}x_i = q_{it}^{-1}x_ix_t^{-1} + p_{i,-1}$ and $x_t^{-b}x_i = q_{i,-t}^{-b}x_ix_t^{-b} + p_{i,-b}$, with $\deg(p_{i,-1})<0$ and $\deg(p_{i,-b}) < -b+1$ then

$$\begin{aligned}
x_t^{-b-1}x_i &= x_t^{-1}(x_t^{-b}x_i) = x_t^{-1}(q_{it}^{-b}x_ix_t^{-b} + p_{i,-b})\\
&= q_{it}^{-b}x_t^{-1}x_ix_t^{-b} + x_t^{-1}p_{i,-b} = q_{it}^{-b}(q_{it}^{-1}x_ix_t^{-1} + b_{i,-1})x_t^{-b} + x_t^{-1}p_{i,-b}\\
&= q_{it}^{-b-1}x_ix_t^{-b-1} + q_{it}^{-b}b_{i,-1}x_t^{-b} + x_t^{-1}\sum_{h=1}^{i-1}\sum_{k=1}^{s(i,-b)} r_{h,k}x_hx_t^{-k}\\
&= q_{it}^{-b-1}x_ix_t^{-b-1} + q_{it}^{-b}b_{i,-1}x_t^{-b} + \sum_{h=1}^{i-1}\sum_{k=1}^{s(i,-b)} r_{h,k}x_t^{-1}x_hx_t^{-k}\\
&= q_{it}^{-b-1}x_ix_t^{-b-1} + q_{it}^{-b}b_{i,-1}x_t^{-b} + \sum_{h=1}^{i-1}\sum_{k=1}^{s(i,-b)} r_{h,k}q_{ht}^{-1}x_hx_t^{-b-1}\\
&\qquad + \sum_{h=1}^{i-1}\sum_{k=1}^{s(i,-b)} r_{h,k}p_{h,-1}x_t^{-b}
\end{aligned}$$

Note that $x_tr_{h,k} = r_{h,k}x_t$ for all h,k because for all t and $g\in G$, $\delta_t(g)=0$ so

$$p_{i,-b-1} = b_{i,-1}x_t^{-b} + \sum_{h=1}^{i-1}\sum_{k=1}^{s(i,-b)} r_{h,k}q_{ht}^{-1}x_hx_t^{-b-1} + \sum_{h=1}^{i-1}\sum_{k=1}^{s(i,-b)} r_{h,k}p_{h,-1}x_t^{-b}$$

where we have the claimed.

2) ($j+1$) We can suppose that $t_i = i$ and $i<t_{j+1}$ because in other way we can put $x_i^{a_i}$ with $a_i=0$. By induction hypothesis on $j=1$ there exist $p_{i,t_{j+1}}$ such that $\deg(p_{i,t_{j+1}}) < a_{j+1}+1$ and $x_{t_{j+1}}^{a_{j+1}}x_i = q_{it_{j+1}}^{a_{j+1}}x_ix_{t_{j+1}}^{a_{j+1}} + p_{i,t_{j+1}}$, again by induction hypothesis there exist $p_{i,t'}$, $t'=(t_1,\dots,t_j)$, such that $\deg(p_{i,t'}) < a_1+\cdots+a_i+1$ and $x_{t_1}^{a_1}\cdots x_{t_i}^{a_i}\cdots x_{t_j}^{a_j}x_i = \left(\prod_{l=i+1}^{j} q_{it_l}^{a_l}\right)x_{t_1}^{a_1}\cdots x_{t_i}^{a_i+1}\cdots x_{t_j}^{a_j} + p_{i,t'}$ then

$$\begin{aligned}
x_{t_1}^{a_1}\cdots x_{t_j}^{a_j}x_{t_{j+1}}^{a_{j+1}}x_i &= x_{t_1}^{a_1}\cdots x_{t_j}^{a_j}\left(q_{it_{j+1}}^{a_{j+1}}x_ix_{t_{j+1}}^{a_{j+1}} + p_{i,t_{j+1}}\right)\\
&= \left(x_{t_1}^{a_1}\cdots x_{t_j}^{a_j}q_{it_{j+1}}^{a_{j+1}}x_ix_{t_{j+1}}^{a_{j+1}}\right) + \left(x_{t_1}^{a_1}\cdots x_{t_j}^{a_j}p_{i,t_{j+1}}\right)\\
&= q_{it_{j+1}}^{a_{j+1}}\left(x_{t_1}^{a_1}\cdots x_{t_j}^{a_j}x_ix_{t_{j+1}}^{a_{j+1}}\right) + \left(x_{t_1}^{a_1}\cdots x_{t_j}^{a_j}p_{i,t_{j+1}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= q_{it_{j+1}}^{a_{j+1}}\left(\left(\prod_{l=i+1}^{j} q_{it_l}^{a_l}\right)x_{t_1}^{a_1}\cdots x_{t_i}^{a_i+1}\cdots x_{t_j}^{a_j}+p_{i,t'}\right)x_{t_{j+1}}^{a_{j+1}}\\
&\quad+\left(x_{t_1}^{a_1}\cdots x_{t_j}^{a_j}p_{i,t_{j+1}}\right)\\
&=\left(\prod_{l=i+1}^{j+1} q_{it_l}^{a_l}\right)x_{t_1}^{a_1}\cdots x_{t_i}^{a_i+1}\cdots x_{t_{j+1}}^{a_{j+1}}+q_{it_{j+1}}^{a_{j+1}}p_{i,t'}x_{t_{j+1}}^{a_{j+1}}+x_{t_1}^{a_1}\cdots x_{t_j}^{a_j}p_{i,t_{j+1}}\\
&=\left(\prod_{l=i+1}^{j+1} q_{it_l}^{a_l}\right)x_{t_1}^{a_1}\cdots x_{t_i}^{a_i+1}\cdots x_{t_{j+1}}^{a_{j+1}}+p_{i,t}
\end{aligned}
$$

where $\deg(q_{it_{j+1}}^{a_{j+1}}p_{i,t'}x_{t_{j+1}}^{a_{j+1}}) \leqslant \deg(p_{i,t'})+\deg(x_{t_{j+1}}^{a_{j+1}}) < a_1+\cdots+a_j+1+a_{j+1}$ and $\deg(x_{t_1}^{a_1}\cdots x_{t_j}^{a_j}p_{i,t_{j+1}}) \leqslant \deg(x_{t_1}^{a_1}\cdots x_{t_j}^{a_j}) + \deg(p_{i,t_{j+1}}) < a_1+\cdots+a_j+a_{j+1}+1$ later $\deg(p_{it}) = \deg(q_{it_{j+1}}^{a_{j+1}}p_{i,t'}x_{t_{j+1}}^{a_{j+1}} + x_{t_1}^{a_1}\cdots x_{t_j}^{a_j}p_{i,t_{j+1}}) < a_1+\cdots+a_{j+1}+1$. The second claimed is proved the same way we proved the first claimed. □

2. Derivations over $\mathcal{O}_{q,\delta}^{r,n}$

Definition 2.1. Let d be a linear operator over $\mathcal{O}_{q,\delta}^{r,n}$ such that

$$d(ab) = d(a)b + \gamma(a)d(b) + \sum_s \theta_s\alpha_s(a)\beta_s(b)$$

for all $a, b \in \mathcal{O}_{q,\delta}^{r,n}$, where $\theta_s \in A$ and $\gamma, \alpha_s, \beta_s$ are toric automorphisms of A, i.e. $\gamma(x_i) = \gamma_i x_i$, $\alpha_s(x_i) = a_{s,i}x_i$, and $\beta_s(x_i) = b_{s,i}x_i$ for all $i = 1, \dots, n$ where $\gamma_i, a_{s,i}, b_{s,i} \in A$.

1) If $\theta_s = 0$ for all s then d is a γ-derivation.
2) If $\theta_s = 0$ and γ is the identity of A then d is a derivation.
3) A inner γ-derivation $[\mathrm{ad}_\gamma u]a$ is defined by

$$[\mathrm{ad}_\gamma u]a := ua - \gamma(a)u$$

when γ is the identity we denote $[\mathrm{ad}_\gamma u]a = [\mathrm{ad}\, u]a$.

Lemma 2.2. Let $u_i = d(x_i)$ then they are a solution of

$$
\begin{aligned}
u_jx_i + \gamma(x_j)u_i - q_{ij}u_ix_j - q_{ij}\gamma(x_i)u_j + \theta_{ij}x_ix_j + Kx_i\\
+ K'x_j + \hat{\theta}_{ij}p_{ij}(x_1,\dots,x_n) - \bar{p}_{ij}(u_1,\dots,u_n) + a_{ij}^{(0)}\theta = 0
\end{aligned}
\tag{1}
$$

where $\hat{\theta}_{ij}, \theta_{ij}, K, K' \in A$, $K = 0$ if $\delta_j = 0$, and $K' = 0$ if $\delta_i = 0$.

Proof. Note that $d(1) = d(1 \cdot 1) = d(1) + d(1) + \sum_s \theta_s = 2d(1) + \theta$, so we have that

$$\begin{aligned} d(x_jx_i) &= d(q_{ij}x_ix_j + p_{ij}(x_1, \dots, x_n)) = q_{ij}d(x_i)x_j + q_{ij}\gamma(x_i)d(x_j) \\ &\quad + q_{ij}\sum_s \theta_s a_{s,i}x_ib_{s,j}x_j + d(a_{ij}^{(0)}) + d(a_{ij}^{(1)}x_1) + \cdots + d(a_{ij}^{(n)}x_n) \\ &= q_{ij}d(x_i)x_j + q_{ij}\gamma(x_i)d(x_j) + K_1x_ix_j \\ &\quad + \sum_s \theta_s a_{s,i}\delta_i(b_{s,i})x_j + \bar{p}_{ij}(u_1, \dots, u_n) + a_{ij}^{(0)}d(1) \\ &= q_{ij}u_ix_j + q_{ij}\gamma(x_i)u_j + K_1x_ix_j + K_2x_j + \bar{p}_{ij}(u_1, \dots, u_n) - a_{ij}^{(0)}\theta. \end{aligned}$$

On the other hand,

$$\begin{aligned} d(x_jx_i) &= u_jx_i + \gamma(x_j)u_i + \sum_s \theta_s a_{s,j}x_jb_{s,i}x_i \\ &= u_jx_i + \gamma(x_j)u_i + \sum_s \theta_s a_{s,j}b_{s,i}x_jx_i + \sum_s \theta_s a_{s,j}\delta_j(b_{s,i})x_i \\ &= u_jx_i + \gamma(x_j)u_i + K_3x_i + K_4(q_{ij}x_ix_j + p_{ij}(x_1, \dots, x_n)) \end{aligned}$$

so that (1) holds. □

Lemma 2.3. Let $z \in \mathcal{O}_{q,\delta}^{r,n}$ then $u_i - [\mathrm{ad}_\gamma z]x_i$ are solutions of (1).

Proof.

$$\begin{aligned} &(u_j - [\mathrm{ad}_\gamma z]x_j)x_i + \gamma(x_j)(u_i - [\mathrm{ad}_\gamma z]x_i) - q_{ij}(u_i - [\mathrm{ad}_\gamma z]x_i)x_j \\ &\quad - q_{ij}\gamma(x_i)(u_j - [\mathrm{ad}_\gamma z]x_j) + \theta_{ij}x_ix_j + Kx_i + K'x_j \\ &\quad + \hat{\theta}_{ij}p_{ij}(x_1, \dots, x_n) - \bar{p}_{ij}((u_1 - [\mathrm{ad}_\gamma z]x_1), \dots, (u_n - [\mathrm{ad}_\gamma z]x_n)) \\ &\quad + a_{ij}^{(0)}\theta \\ &= \big(u_jx_i + \gamma(x_j)u_i - q_{ij}u_ix_j - q_{ij}\gamma(x_i)u_j + \theta_{ij}x_ix_j + Kx_i + K'x_j \\ &\quad + \hat{\theta}_{ij}p_{ij}(x_1, \dots, x_n) + a_{ij}^{(0)}\theta\big) + (-[\mathrm{ad}_\gamma z]x_j)x_i - \gamma(x_j)[\mathrm{ad}_\gamma z]x_i \\ &\quad + q_{ij}([\mathrm{ad}_\gamma z]x_i)x_j + q_{ij}\gamma(x_i)[\mathrm{ad}_\gamma z]x_j - \sum_{t=1}^{n} a_{ij}^{(t)}(u_t - [\mathrm{ad}_\gamma z]x_t) \\ &= (u_jx_i + \gamma(x_j)u_i - q_{ij}u_ix_j - q_{ij}\gamma(x_i)u_j + \theta_{ij}x_ix_j + Kx_i + K'x_j \\ &\quad + \hat{\theta}_{ij}p_{ij}(x_1, \dots, x_n) - \bar{p}_{ij}(u_1, \dots, u_n) + a_{ij}^{(0)}\theta) + (-[\mathrm{ad}_\gamma z]x_j)x_i \\ &\quad - \gamma(x_j)[\mathrm{ad}_\gamma z]x_i + q_{ij}([\mathrm{ad}_\gamma z]x_i)x_j + q_{ij}\gamma(x_i)[\mathrm{ad}_\gamma z]x_j \end{aligned}$$

$$
\begin{aligned}
&+\sum_{t=1}^{n} a_{ij}^{(t)}([\operatorname{ad}_\gamma z]x_t)\\
&= -zx_jx_i + \gamma(x_j)zx_i - \gamma(x_j)zx_i + \gamma(x_j)\gamma(x_i)z + q_{ij}zx_ix_j\\
&\quad - q_{ij}\gamma(x_i)zx_j + q_{ij}\gamma(x_i)zx_j - q_{ij}\gamma(x_i)\gamma(x_j)z\\
&\quad + \sum_{t=1}^{n} a_{ij}^{(t)}(zx_t - \gamma(x_t)z)\\
&= [-zx_jx_i + q_{ij}zx_ix_j + \sum_{t=1}^{n} a_{ij}^{(t)} zx_t + a_{ij}^{(0)} z] + [\gamma(x_j)zx_i - \gamma(x_j)zx_i]\\
&\quad + [-q_{ij}\gamma(x_i)zx_j + q_{ij}\gamma(x_i)zx_j] + [\gamma(x_j)\gamma(x_i)z - \gamma(x_j)\gamma(x_i)z]\\
&\quad + [\gamma(x_j)\gamma(x_i)z - q_{ij}\gamma(x_ix_j)z - \sum_{t=1}^{n} a_{ij}^{(t)}\gamma(x_t)z - a_{ij}^{(0)} z]\\
&= [z\big(-x_jx_i + q_{ij}x_ix_j + \sum_{t=1}^{n} a_{ij}^{(t)} x_t + a_{ij}^{(0)}\big)]\\
&\quad + [\gamma\big(x_jx_i - q_{ij}x_ix_j - \sum_{t=1}^{n} a_{ij}^{(t)} x_i - a_{ij}^{(0)}\big)z] = 0. \qquad \square
\end{aligned}
$$

Proposition 2.4. Let $v = ax_1^{m_1}\cdots x_n^{m_n}$ with $a \in A$ and $m = (m_1, \ldots, m_n) \in Z$ such that either $\gamma_1 \neq 1$ and $\big(1 - \gamma_1 \prod_{j=2}^{n} q_{j1}^{m_j}\big) \in A^*$ or $\gamma_1 = 1$ and exists $m_j \neq 0$ for $j \neq 1$. Then there exists $w \in A^*$ with

$$[\operatorname{ad}_\gamma wvx_1^{-1}]x_1 = v$$

Proof. Take $w = \big(1 - \gamma_1 \prod_{j=2}^{n} q_{j1}^{m_j}\big)^{-1}$ that exists by hypothesis. Note that $\gamma_1 w = w\gamma_1$ so

$$
\begin{aligned}
[\operatorname{ad}_\gamma wvx_1^{-1}]x_1 &= wvx_1^{-1}x_1 - \gamma_1 x_1 wvx_1^{-1}\\
&= w(v - \gamma_1 a x_1 x_1^{m_1}\cdots x_n^{m_n} x_1^{-1})\\
&= w(v - \gamma_1 a x_1^{m_1} x_1 x_2^{m_2}\cdots x_n^{m_n} x_1^{-1})\\
&= w(v - \gamma_1 a x_1^{m_1} q_{21}^{m_2} x_2^{m_2} x_1 \cdots x_n^{m_n} x_1^{-1})\\
&= w(v - \gamma_1 a x_1^{m_1} q_{21}^{m_2} x_2^{m_2} q_{31}^{m_3} x_3 \cdots q_{n1}^{m_n} x_n^{m_n} x_1 x_1^{-1})\\
&= w\left(v - \gamma_1 \prod_{j=2}^{n} q_{j1}^{m_j} a x_1^{m_1}\cdots x_n^{m_n}\right) = w\left(1 - \gamma_1 \prod_{j=2}^{n} q_{j1}^{m_j}\right)v = v. \square
\end{aligned}
$$

Corollary 2.5. Let $v = \sum_t \rho_t X_t \in \mathcal{O}_{q,\delta}^{r,n}$ such that $\rho_t X_t$ satisfies the hypothesis of the above proposition for all t, where $X_t \in \operatorname{Mon}(\mathcal{O}_{q,\delta}^{r,n})$. Then there exists $w \in \mathcal{O}_{q,\delta}^{r,n}$ such that

$$[\mathrm{ad}_\gamma w]x_1 = v.$$

Proof. We have that $[\mathrm{ad}_\gamma w + w']z = (w + w')z - \gamma(z)(w + w') = (wz - \gamma(z)w) + (w'z - \gamma(z)w) = [\mathrm{ad}_\gamma w]z + [\mathrm{ad}_\gamma w']z$ for all $w, w', z \in \mathcal{O}_{q,\delta}^{r,n}$ so for each $\rho_t X_t$ there exists $w_t \in A$ such that

$$[\mathrm{ad}_\gamma w_t \rho_t X_t x_1^{-1}]x_1 = \rho_t X_t$$

then

$$\begin{aligned} v &= \sum_t \rho_t X_t = \sum_t [\mathrm{ad}_\gamma w_t \rho_t X_t x_1^{-1}]x_1 = [\mathrm{ad}_\gamma \sum_t w_t \rho_t X_t x_1^{-1}]x_1 \\ &= [\mathrm{ad}_\gamma w]x_1. \end{aligned}$$

□

Theorem 2.6. Let $u_i = d(x_i) \in \mathcal{O}_{q,\delta}^{r,n}$ for $i = 1, \dots, n$.

1) If $\gamma_1 \neq 1$ and $\left(\prod_{j=2}^n q_{j1}^{m_j} - \gamma_1 q_{1t}\right) \in A^*$ for all $(0, m_2, \dots, m_n) \in Z$. Then there exists $w \in \mathcal{O}_{q,\delta}^{r,n}$ and $\rho_j \in A$ such that

$$u_1 = [\mathrm{ad}_\gamma w]x_1 \quad \text{and} \quad u_j = \rho_j x_j + [\mathrm{ad}_\gamma w]x_j$$

for all $j \neq 1$ where $(q_{1t} - q_{1t}\gamma_1)\rho_t = -\theta_{1t}$.

2) If $\gamma_j = 1$ for all $i = 1, \dots, n$. Then there exists $w \in \mathcal{O}_{q,\delta}^{r,n}$ such that

$$u_j = \lambda_j x_j + [\mathrm{ad}\, w]x_j$$

for all $j = 1, \dots, n$ and some $\lambda_j = \lambda(x_j) \in A$.

Proof. 1) By the above corollary there exists $w \in \mathcal{O}_{q,\delta}^{r,n}$ such that $u_1 = [\mathrm{ad}_\gamma w]x_1$ so we $u_1 = [\mathrm{ad}_\gamma w]x_1$ so we define $\bar{u}_t = u_t - [\mathrm{ad}_\gamma w]x_t$ where they hold the equation (1). In

$$0 = \bar{u}_t x_1 - q_{1t}\gamma_1 x_1 \bar{u}_t + \theta_{1t} x_1 x_t + K_t x_1$$

If $\bar{u}_t = \sum_{m \in Z} \eta_m x_1^{m_1} \cdots x_n^{m_n} \in \mathcal{O}_{q,\delta}^{r,n}$ we have

$$\begin{aligned} 0 &= \left(\sum_{m \in Z} \eta_m x_1^{m_1} \cdots x_n^{m_n} x_1 - \gamma_1 q_{1t} \sum_{m \in Z} \eta_m x_1 x_1^{m_1} \cdots x_n^{m_n} \right) \\ &\quad + \theta_{1t} x_1 x_t + K x_t \\ &= \sum_{m \in Z} \eta_m \prod_{j=2}^n q_{1j}^{m_j} x_1 x_1^{m_1} \cdots x_n^{m_n} - \gamma_1 q_{1t} \sum_{m \in Z} \eta_m x_1 x_1^{m_1} \cdots x_n^{m_n} \end{aligned}$$

$$+ \theta_{1t}x_1x_t + Kx_t$$
$$= \left(\sum_{m\in Z}\eta_m\left(\prod_{j=2}^{n} q_{1j}^{m_j} - \gamma_1 q_{1t}\right)x_1^{m_1+1}\cdots x_n^{m_n}\right)$$
$$+ \theta_{1t}x_1x_t + Kx_t.$$

As $\mathrm{Mon}(\mathcal{O}_{q,\delta}^{r,n})$ is a basis of $\mathcal{O}_{q,\delta}^{r,n}$ and $\prod_{j=2}^{n} q_{j1}^{m_j} - \gamma_1 q_{1t} \in A^*$ we have that $\eta_m = 0$ if $m \neq (0,\ldots,1,\ldots,0)$ where 1 is in the tth position, so $\bar{u}_t = \eta_t x_t$ and $K = 0$. In the other hand

$$0 = \eta_t x_t x_1 - q_{1t}\gamma_1 x_1 \eta_t x_t + \theta_{1t}x_1x_t = ((q_{1t} - q_{1t}\gamma_1)\eta_t + \theta_{1t})x_1x_t$$

then $(q_{1t} - q_{1t}\gamma_1)\eta_t + \theta_{1t} = 0$.

2) Put $u_1 = u_1' + v_1$ with $v_1 \in \mathcal{O}_{q,\delta}^{r,n} \setminus A((x_1))$. By the corollary (2.5) there exists $w \in \mathcal{O}_{q,\delta}^{r,n}$ with $v_1 = [\mathrm{ad}\, w]x_1$, we will denote $\bar{u}_i = u_i - [\mathrm{ad}\, w]x_i$ so we have $\bar{u}_1 \in A((x_1))$. Let $t \neq 1$

$$\bar{u}_t x_1 - q_{1t}x_1\bar{u}_t = -x_t\bar{u}_1 + q_{1t}\bar{u}_1x_t - \theta_{1t}x_1x_t - Kx_t \in A((x_1))x_t$$

because if $\bar{u}_1 = \sum_t \rho_t x_1^t$ then $x_t\bar{u}_1 = \sum_t \rho_t q_{1t}^t x_1^t x_t$. If

$$\bar{u}_t = \sum_{m\in Z}\eta_m x_1^{m_1}\cdots x_n^{m_n}$$

then

$$\bar{u}_t x_1 - x_1 q_{1t}\bar{u}_t = \sum_{m\in Z}\eta_m x_1^{m_1}\cdots x_n^{m_n}x_1 - x_1 q_{1t}\sum_{m\in Z}\eta_m x_1^{m_1}\cdots x_n^{m_n}$$
$$= \sum_{m\in Z}\eta_m\left(\prod_{j=2}^{n} q_{1j}^{m_j} - q_{1t}\right)x_1x_1^{m_1}\cdots x_n^{m_n}$$
$$= x_1\sum_{m\in Z}\eta_m\left(\prod_{j=2}^{n} q_{1j}^{m_j} - q_{1t}\right)x_1^{m_1}\cdots x_n^{m_n}$$

multiplying by x_1^{-1} we have

$$\sum_{m\in Z}\eta_m\left(\prod_{j=2}^{n} q_{1j}^{m_j} - q_{1t}\right)x_1^{m_1}\cdots x_n^{m_n} \in A((x_1))x_t.$$

Since $\prod_{j=2}^{n} q_{1j}^{m_j} - q_{1t} \in A^*$ we have $\eta_m = 0$ if $m_i \neq 0$ for $i \neq 1, t$ or if $m_t \neq 1$. Then $\bar{u}_t = f_t(x_1)x_t$ where $f_t(x_1) \in A((x_1))$. Take $t = 2$, if

$f_2(x_1) = \lambda_2 + \sum_{t\neq 0} \rho_t x_1^t$ then for all t put $w_t = (1 - q_{12}^t)^{-1}$, note that $w_t x_i = x_i w_t$ for all $i = 1, \ldots, n$ so

$$\begin{aligned}[\operatorname{ad} w_t \rho_t x_1^t]x_2 &= w_t\rho_t x_1^t x_2 - x_2 w_t \rho_t x_1^t \\ &= w_t(\rho_t x_1^t x_2 - x_2\rho_t x_1^t) = w_t(\rho_t x_1^t x_2 - \rho_t x_2 x_1^t - \delta_2(\rho_t)x_1^t) \\ &= w_t(1-q_{12}^t)\rho_t x_1^t x_2 - w_t\delta_2(\rho_t)x_1^t = \rho_t x_1^t x_2 - w_t\delta_2(\rho_t)x_1^t.\end{aligned}$$

Then there exists $w' \in A((x_1)) \setminus A$ or $w' = 0$ with $\bar{u}_2 - [\operatorname{ad} w']x_2 = \lambda_2 x_2 + f'(x_1)$ and $f'(x_1) \in A((x_1)) \setminus A$ or $f'(x_1) = 0$.

Take $\tilde{u}_l = \bar{u}_l - [\operatorname{ad} w']x_l = g_l(x_1)x_l + g_l'(x_1)$, ($g_l'(x_1) \in A((x_1)) \setminus A$ or $g_l'(x_1) = 0$), because if $w' = \sum_t a_t x_1^t$ then

$$\begin{aligned}[\operatorname{ad} w']x_l &= \sum_t a_t x_1^t x_l - \sum_t x_l a_t x_1^t \\ &= \sum_t a_t x_1^t x_l - \sum_t a_t q_{1t}^t x_1^t x_l - \sum_t \delta_l(a_t)x_1^t\end{aligned}$$

and $\tilde{u}_2 = \lambda_2 x_2 + g_2'(x_1)$.

By the equation (1) for ($t = 1$) if $\tilde{u}_1 = \sum_t a_t x_1^t$ then

$$\begin{aligned}0 &= \lambda_2 x_2 x_1 + g_2'(x_1)x_1 + x_2\tilde{u}_1 - q_{12}\tilde{u}_1 x_2 - q_{12}x_1\lambda_2 x_2 - q_{12}x_1 g_2'(x_1) \\ &\quad + \theta_{12}x_1x_2 + Kx_1 \\ &= [\lambda_2 q_{12}x_1x_2 + \sum_t x_2 a_t x_1^t - \sum_t q_{12}a_t x_1^t x_2 - q_{12}\lambda_2 x_1 x_2 + \theta_{12}x_1x_2] \\ &\quad + [g_2'(x_1)x_1 + q_{12}x_1 g_2'(x_1) + Kx_1] \\ &= \left(\lambda_2 q_{12}x_1x_2 + \sum_t a_t x_2 x_1^t - \sum_t q_{12}a_t x_1^t x_2 - q_{12}\lambda_2 x_1x_2 + \theta_{12}x_1x_2\right) \\ &\quad + \left(g_2'(x_1)x_1 - q_{12}x_1 g_2'(x_1) + Kx_1 + \sum_t \delta_2(a_t)x_1^t\right) \\ &= \left(\sum_t a_t(q_{12}^t - q_{12})x_1^t x_2 + \theta_{12}x_1x_2\right) \\ &\quad + \left(g_2'(x_1)x_1 - q_{12}x_1 g_2'(x_1) + Kx_1 + \sum_t \delta_2(a_t)x_1^t\right).\end{aligned}$$

Then $t = 1$, $\theta_{12} = 0$, and $\tilde{u}_1 = ax_1$. For this equations if $g_2'(x_1) \neq 0$ put $it(g_2'(x_1)) = b_v x_1^v$ with $v \neq 0$ then

$$0 = b_v x_1^{v+1} - q_{12}b_v x_1^{v+1} = b_v(1-q_{12})x_1^{v+1}$$

so $b_v = 0$ but this is not possible. Thus $g_2'(x_1) = 0$.

Take $3 \leqslant t$ and $\tilde{u}_1 = a_1 x_1$.

$$\begin{aligned}
0 &= \big(g_t(x_1)x_t + g_t'(x_1)\big)x_1 + x_t a_1 x_1 - q_{1t} a_1 x_1 x_t \\
&\qquad - q_{1t} x_1 \big(g_t(x_1)x_t + g_t'(x_1)\big) + \theta_{1t} x_1 x_t + K x_1 \\
&= \big(g_t(x_1)x_t x_1 + a_1 x_t x_1 - q_{1t} a_1 x_1 x_t - q_{1t} x_1 g_t(x_1) x_t + \theta_{1t} x_1 x_t\big) \\
&\qquad + \big(g_t'(x_1)x_1 + \delta_t(a_1)x_1 - q_{1t} x_1 g_t'(x_1) + K x_1\big) \\
&= \big(g_t(x_1)x_t x_1 - q_{1t} x_1 g_t(x_1) x_t + \theta_{1t} x_1 x_t\big) \\
&\qquad + \big(g_t'(x_1)x_1 + \delta_t(a_1)x_1 - q_{1t} x_1 g_t'(x_1) + K x_1\big) \\
&= \theta_{1t} x_1 x_t + \big(g_t'(x_1)x_1 + \delta_t(a_1)x_1 - q_{1t} x_1 g_t'(x_1) + K x_1\big).
\end{aligned}$$

Then $\theta_{1t} = 0$ and if $g_t'(x_1) \neq 0$ we can take $it(g_t'(x_1)) = b_v x_1^v$ with $b_v \neq 0$. Of this equation we can claim that $b_v(1 - q_{1t}) = 0$ then $b_v = 0$ but this is not possible so $g_t'(x_1) = 0$.

In the other hand, if $g_t(x_1) = \sum_t c_t x_1^t$

$$\begin{aligned}
0 &= g_t(x_1)x_t x_2 + x_t \lambda_2 x_2 - q_{2t} \lambda_2 x_2 x_t - q_{2t} x_2 g_t(x_1) x_t + \theta_{2t} x_2 x_t \\
&\qquad + K x_2 + K' x_t + \hat{\theta}_{2t} p_{2t}(x_1, \ldots, x_n) - \bar{p}_{2t}(\tilde{u}_1, \ldots, \tilde{u}_n) + a_{2t}^{(0)} \theta \\
&= g_t(x_1)x_t x_2 + (\lambda_2 x_t x_2 - \lambda_2 q_{2t} x_2 x_t) - q_{2t} x_2 g_t(x_1) x_t + \theta_{2t} x_2 x_t + K x_2 \\
&\qquad + K' x_t + \hat{\theta}_{2t} p_{2t}(x_1, \ldots, x_n) - \bar{p}_{2t}(\tilde{u}_1, \ldots, \tilde{u}_n) + \delta_t(\lambda_2) x_2 + a_{2t}^{(0)} \theta \\
&= g_t(x_1)\big(q_{2t} x_2 x_t + p_{2t}(x_1, \ldots, x_n)\big) - q_{2t} x_2 \sum_t c_t x_1^t x_t + \theta_{2t} x_2 x_t \\
&\qquad + \lambda_2 p_{2t}(x_1, \ldots, x_n) + K x_2 + K' x_t + \hat{\theta}_{2t} p_{2t}(x_1, \ldots, x_n) \\
&\qquad - \bar{p}_{2t}(\tilde{u}_1, \ldots, \tilde{u}_n) + \delta_t(\lambda_2) x_2 + a_{2t}^{(0)} \theta \\
&= q_{2t}\left(\sum_t c_t x_1^t x_2 x_t - \sum_t c_t x_2 x_1^t x_t\right) + \theta_{2t} x_2 x_t + g_t(x_1) p_{2t}(x_1, \ldots, x_n) \\
&\qquad + \sum_t \delta_2(c_t) x_1^t x_t + K x_2 + K' x_t + (\hat{\theta}_{2t} + \lambda_2) p_{2t}(x_1, \ldots, x_n) \\
&\qquad - \bar{p}_{2t}(\tilde{u}_1, \ldots, \tilde{u}_n) + \delta_t(\lambda_2) x_2 + a_{2t}^{(0)} \theta \\
&= q_{2t}\left(\sum_t c_t (1 - q_{12}^t) x_1^t x_2 x_t\right) + \theta_{2t} x_2 x_t + g_t(x_1) p_{2t}(x_1, \ldots, x_n) \\
&\qquad + \sum_t \delta_2(c_t) x_1^t x_t + K x_2 + K' x_t + (\hat{\theta}_{2t} + \lambda_2) p_{2t}(x_1, \ldots, x_n) \\
&\qquad - \bar{p}_{2t}(\tilde{u}_1, \ldots, \tilde{u}_n) + \delta_t(\lambda_2) x_2 + a_{2t}^{(0)} \theta.
\end{aligned}$$

Of this last equation we have that $c_t(1-q_{12}^t)=0$ for $t\neq 0$ so $c_t=0$ since $(1-q_{12}^t)\in A^*$, thus $g_t(x_1)=\lambda_t$ of this way $u_t-[\operatorname{ad} w+w']x_t=\tilde{u}_t=\lambda_t x_t$. □

3. Poisson brackets on $\mathcal{O}_{q,\delta}^{r,n}$

Definition 3.1. A Poisson bracket $\{\cdot,\cdot\}$ is a $\mathbb{Z}$-bilinear function on $\mathcal{O}_{q,\delta}^{r,n}$ such that

1) $\{\cdot,\cdot\}$ is a Lie bracket.
2) $\{ab,c\}=\{a,c\}b+a\{b,c\}$ for all $a,b,c\in\mathcal{O}_{q,\delta}^{r,n}$.
3) $0=\{a,b\}+\{b,a\}$ for all $a,b\in\mathcal{O}_{q,\delta}^{r,n}$.

Remark 3.2. If ∂ is a derivation on $\mathcal{O}_{q,\delta}^{r,n}$ then for all $i=1,\ldots,r$ we have $0=\partial(1)=\partial(x_ix_i^{-1})=\partial(x_i)x_i^{-1}+x_i\partial(x_i^{-1})$ then $\partial(x_i^{-1})=-x_i^{-1}\partial(x_i)x_i^{-1}$.

Proposition 3.3. Let be given a Poisson bracket $\{a,b\}$ on an extension $\mathcal{O}_{q,\delta}^{r,n}$ of A. Then there exists $\xi,\xi_j\in A$ such that $\{x_i,x_j\}=\xi[x_i,x_j]+\delta_i(\xi)x_i-\delta_i(\xi)x_j-\delta_i(\xi_j)$ with $\delta_j(\xi_j)=0$ for all $j,i=1,\ldots,n$, where $[a,b]:=ab-ba$.

Proof. By theorem (2.6) for all $a\in\mathcal{O}_{q,\delta}^{r,n}$ there exists $\lambda_a(x_i)\in A$ and $w(a)\in\mathcal{O}_{q,\delta}^{r,n}$ with

$$\{x_i,a\}=\lambda_a(x_i)x_i+[\operatorname{ad} w(a)]x_i.$$

So we have

$$0=\{x_i,x_i\}=\lambda_i(x_i)x_i+[\operatorname{ad} w(x_i)]x_i.$$

Put $w(x_i)=\sum_{m=0}^{k}\eta_{v_m}x_1^{v_{m_1}}\cdots x_n^{v_{m_n}}$, we will see that $w(x_i)\in A((x_i))$ for $i=1,\ldots,r$ or $w(x_i)\in A[x_i]$ for $i=r+1,\ldots,n$. If $w(x_i)=0$ we have the claimed, in the other hand,

$$\begin{aligned}0&=\lambda_i(x_i)x_i-\sum_{m=0}^{k}x_i\eta_{v_m}x_1^{v_{m_1}}\cdots x_n^{v_{m_n}}+\sum_{m=0}^{k}\eta_{v_m}x_1^{v_{m_1}}\cdots x_n^{v_{m_n}}x_i\\&=\lambda_i(x_i)x_i-\sum_{m=0}^{k}\delta_i(\eta_{v_m})x_1^{v_{m_1}}\cdots x_n^{v_{m_n}}-\sum_{m=0}^{k}\eta_{v_m}x_ix_1^{v_{m_1}}\cdots x_n^{v_{m_n}}\\&\quad+\sum_{m=0}^{k}\eta_{v_m}x_1^{v_{m_1}}\cdots x_n^{v_{m_n}}x_i\end{aligned}$$

$$
\begin{aligned}
&= \lambda_i(x_i)x_i - \sum_{m=0}^{k} \delta_i(\eta_{v_m}) x_1^{v_{m_1}} \cdots x_n^{v_{m_n}} \\
&\quad - \sum_{m=0}^{k} \Big(\eta_{v_m} \prod_{s<i} q_{si}^{v_{m_s}} x_1^{v_{m_1}} \cdots x_i^{v_{m_i}+1} \cdots x_n^{v_{m_n}} + p_{v_m,i}(x_1,\dots,x_n) \Big) \\
&\quad + \sum_{m=0}^{k} \Big(\eta_{v_m} \prod_{s>i} q_{is}^{v_{m_s}} x_1^{v_{m_1}} \cdots x_i^{v_{m_i}+1} \cdots x_n^{v_{m_n}} + p_{i,v_m}(x_1,\dots,x_n) \Big) \\
&= \sum_{m=0}^{k} \eta_{v_m} \Big(\prod_{s>i} q_{is}^{v_{m_s}} - \prod_{s<i} q_{si}^{v_{m_s}} \Big) x_1^{v_{m_1}} \cdots x_i^{v_{m_i}+1} \cdots x_n^{v_{m_n}} \\
&\quad + \lambda_i(x_i)x_i - \sum_{m=0}^{k} \delta_i(\eta_{v_m}) x_1^{v_{m_1}} \cdots x_n^{v_{m_n}} + \sum_{m=0}^{k} p_{i,v_m}(x_1,\dots,x_n) \\
&\quad - p_{v_m,i}(x_1,\dots,x_n),
\end{aligned}
$$

where $\deg(p_{v_m,i}), \deg(p_{i,v_m}) < v_{m_1} + \cdots + v_{m_n} + 1 < |v_k| + 1$ for all $m = 0,\dots,k$. By the last equation we have $\big(\prod_{s<i} q_{si}^{v_{k_s}} - \prod_{s>i} q_{is}^{v_{k_s}}\big)\eta_{v_k} = 0$ and since $\eta_{v_k} \neq 0$ then $v_{k_l} = 0$ for $l \neq i$, because if there exists $v_{k_l} \neq 0$ for some $l \neq i$ then $\prod_{s<i} q_{si}^{v_{m_s}} - \prod_{s>i} q_{is}^{v_{m_s}} \in A^*$ by definition of $\mathcal{O}_{q,\delta}^{r,n}$, so this implies that $\eta_{v_k} = 0$, but this is not possible. Later $p_{i,v_k} = p_{v_k,i} = 0$ and

$$
\begin{aligned}
0 &= \sum_{m=0}^{k-1} \eta_{v_m} \Big(\prod_{s>i} q_{is}^{v_{m_s}} - \prod_{s<i} q_{si}^{v_{m_s}} \Big) x_1^{v_{m_1}} \cdots x_i^{v_{m_i}+1} \cdots x_n^{v_{m_n}} + \lambda_i(x_i)x_i \\
&\quad - \sum_{m=0}^{k-1} \delta_i(\eta_{v_m}) x_1^{v_{m_1}} \cdots x_n^{v_{m_n}} + \sum_{m=0}^{k-1} p_{i,v_m}(x_1,\dots,x_n) \\
&\quad - p_{v_m,i}(x_1,\dots,x_n).
\end{aligned}
$$

So $\eta_{v_{k-1}} = 0$, if $\eta_{v_{k-1}} \neq 0$ then $v_{k-1_l} = 0$ for $l \neq i$ because $\Big(\prod_{s<i} q_{si}^{v_{k-1_s}} - \prod_{s>i} q_{is}^{v_{k-1_s}}\Big) \in A^*$ if there exist $v_{k-1_l} \neq 0$ for some $l \neq i$. By recurrently way we can claim that $\eta_{v_m} = 0$ if there exists $v_{m_l} \neq 0$ for some $l \neq i$, then we have the claimed. Since $w(x_i) = \sum_{t\in\mathbb{Z}} \xi_{it} x_i^t$, where $\xi_{it} = 0$ if $t < 0$ and $i > r$, then

$$
0 = \lambda_i(x_i)x_i - x_i \sum_{t\in Z} \xi_{it} x_i^t + \sum_{t\in\mathbb{Z}} \xi_{it} x_i^{t+1} = \lambda_i(x_i)x_i - \sum_{t\in\mathbb{Z}} \delta_i(\xi_{it}) x_i^t
$$

then $\delta_i(\xi_{it}) = 0$ for $t \neq 1$ and $\lambda_i(x_i) = \delta_i(\xi_{i1})$. Put $i < j$, $w(x_i) = \sum_{l=0}^{k} \xi_{iv_{i_l}} x_i^{v_{i_l}}$, and $w(x_j) = \sum_{l=0}^{r} \xi_{jv_{j_l}} x_j^{v_{j_l}}$, so

$$\begin{aligned}
0 &= \{x_j, x_i\} + \{x_i, x_j\} = \lambda_i(x_j)x_j + [\operatorname{ad} w(x_i)]x_j + \lambda_j(x_i)x_i + [\operatorname{ad} w(x_j)]x_i \\
&= \lambda_i(x_j)x_j + \lambda_j(x_i)x_i + \sum_{l=0}^{k} \xi_{iv_{i_l}} x_i^{v_{i_l}} x_j - x_j \sum_{l=0}^{k} \xi_{iv_{i_l}} x_i^{v_{i_l}} + \sum_{l=0}^{r} \xi_{jv_{j_l}} x_j^{v_{j_l}} x_i \\
&\quad - x_i \sum_{l=0}^{r} \xi_{jv_{j_l}} x_j^{v_{j_l}} \\
&= \lambda_i(x_j)x_j + \lambda_j(x_i)x_i + \xi_{iv_{i_k}} x_i^{v_{i_k}} x_j - \xi_{iv_{i_k}} x_j x_i^{v_{i_k}} + \xi_{jv_{j_r}} x_j^{v_{j_r}} x_i \\
&\quad - \xi_{jv_{j_r}} x_i x_j^{v_{j_r}} + \sum_{l=0}^{k-1} \xi_{iv_{i_l}} x_i^{v_{i_l}} x_j - \sum_{l=0}^{k-1} \xi_{iv_{i_l}} x_j x_i^{v_{i_l}} + \sum_{l=0}^{r-1} \xi_{jv_{j_l}} x_j^{v_{j_l}} x_i \\
&\quad - \sum_{l=0}^{r-1} \xi_{jv_{j_l}} x_i x_j^{v_{j_l}} - \sum_{l=0}^{k-1} \delta_j(\xi_{iv_{i_l}}) x_i^{v_{i_l}} - \sum_{l=0}^{r-1} \delta_i(\xi_{jv_{j_l}}) x_j^{v_{j_l}} - \delta_j(\xi_{iv_{i_k}}) x_i^{v_{i_k}} \\
&\quad - \delta_i(\xi_{jv_{j_r}}) x_j^{v_{j_r}} \\
&= \lambda_i(x_j)x_j + \lambda_j(x_i)x_i + \xi_{iv_{i_k}} x_i^{v_{i_k}} x_j - \xi_{iv_{i_k}} q_{ij}^{v_{i_k}} x_i^{v_{i_k}} x_j + \xi_{jv_{j_r}} q_{ij}^{v_{j_r}} x_i x_j^{v_{j_r}} \\
&\quad - \xi_{jv_{j_r}} x_i x_j^{v_{j_r}} + \sum_{l=0}^{k-1} \xi_{iv_{i_l}} x_i^{v_{i_l}} x_j - \sum_{l=0}^{k-1} \xi_{iv_{i_l}} q_{ij}^{v_{i_l}} x_i^{v_{i_l}} x_j + \sum_{l=0}^{r-1} \xi_{jv_{j_l}} q_{ij}^{v_{j_l}} x_i x_j^{v_{j_l}} \\
&\quad - \sum_{l=0}^{r-1} \xi_{jv_{j_l}} x_i x_j^{v_{j_l}} - \sum_{l=0}^{k} \delta_j(\xi_{iv_{i_l}}) x_i^{v_{i_l}} - \sum_{l=0}^{r} \delta_i(\xi_{jv_{j_l}}) x_j^{v_{j_l}} - \xi_{iv_{i_k}} p_{v_{i_k},j} \\
&\quad + \xi_{jv_{j_r}} p_{i,v_{j_r}} - \sum_{l=0}^{k-1} \xi_{iv_{i_l}} p_{v_{i_l},j} + \sum_{l=0}^{r-1} \xi_{jv_{j_l}} p_{i,v_{j_l}} \\
&= \lambda_i(x_j)x_j + \lambda_j(x_i)x_i + \xi_{iv_{i_k}} (1 - q_{ij}^{v_{i_k}}) x_i^{v_{i_k}} x_j + \xi_{jv_{j_r}} (q_{ij}^{v_{j_r}} - 1) x_i x_j^{v_{j_r}} \\
&\quad + \sum_{l=0}^{k-1} \xi_{iv_{i_l}} (1 - q_{ij}^{v_{i_l}}) x_i^{v_{i_l}} x_j + \sum_{l=0}^{r-1} \xi_{jv_{j_l}} (q_{ij}^{v_{j_l}} - 1) x_i x_j^{v_{j_l}} \\
&\quad - \sum_{l=0}^{k} \delta_j(\xi_{iv_{i_l}}) x_i^{v_{i_l}} - \sum_{l=0}^{r} \delta_i(\xi_{jv_{j_l}}) x_j^{v_{j_l}} - \sum_{l=0}^{k} \xi_{iv_{i_l}} p_{v_{i_l},j} + \sum_{l=0}^{r} \xi_{jv_{j_l}} p_{i,v_{j_l}}
\end{aligned}$$

where $\deg(p_{v_{i_l},j}) < v_{i_l} + 1 < v_{i_k} + 1$ and $\deg(p_{i,v_{j_l}}) < v_{j_l} + 1 < v_{j_r} + 1$ for all l. Since $0 = \xi_{iv_{i_k}} (1 - q_{ij}^{v_{i_k}}) x_i^{v_{i_k}} x_j + \xi_{jv_{j_r}} (q_{ij}^{v_{j_r}} - 1) x_i x_j^{v_{j_r}}$, we have $v_{j_r} = v_{i_k}$ and $v_{j_r} \in \{0, 1\}$, later $p_{v_{i_k},j},\ p_{i,v_{j_r}} \in A + Ax_1 + \cdots + Ax_n$. The same way

$$
\begin{aligned}
0 &= \lambda_i(x_j)x_j + \lambda_j(x_i)x_i + \xi_{iv_{i_{k-1}}}(1 - q_{ij}^{v_{i_{k-1}}})x_i^{v_{i_{k-1}}}x_j \\
&\quad + \xi_{jv_{j_{r-1}}}(q_{ij}^{v_{j_{r-1}}} - 1)x_ix_j^{v_{j_{r-1}}} + \sum_{l=0}^{k-2}\xi_{iv_{i_l}}(1 - q_{ij}^{v_{i_l}})x_i^{v_{i_l}}x_j \\
&\quad + \sum_{l=0}^{r-2}\xi_{jv_{j_l}}(q_{ij}^{v_{j_l}} - 1)x_ix_j^{v_{j_l}} - \sum_{l=0}^{k}\delta_j(\xi_{iv_{i_l}})x_i^{v_{i_l}} - \sum_{l=0}^{r}\delta_i(\xi_{jv_{j_l}})x_j^{v_{j_l}} \\
&\quad - \sum_{l=0}^{k}\xi_{iv_{i_l}}p_{v_{i_l},j} + \sum_{l=0}^{r}\xi_{jv_{j_l}}p_{i,v_{j_l}}.
\end{aligned}
$$

Then $0 = \xi_{iv_{i_{k-1}}}(1 - q_{ij}^{v_{i_{k-1}}})x_i^{v_{i_{k-1}}}x_j + \xi_{jv_{j_{r-1}}}(q_{ij}^{v_{j_{r-1}}} - 1)x_ix_j^{v_{j_{r-1}}}$, since $v_{i_{k-1}},\ v_{j_{r-1}} \leqslant 0$, $\xi_{jv_{j_{k-1}}},\ \xi_{iv_{i_{r-1}}} \neq 0$, and $(1 - q_{ij}^{v_{i_k}}), (q_{ij}^{v_{j_k}} - 1) \in A^*$ if $v_{i_{k-1}},\ v_{j_{r-1}} \neq 0$ then we can claim that $v_{i_{k-1}},\ v_{j_{r-1}} = 0$. Of recurrently form we can claim that $w(x_t) = \xi_{t1}x_t + \xi_{t0}$ with $\xi_{t1}, \xi_{t0} \in A$ and $\delta_t(\xi_{t0}) = 0$ for all $t = 1, \ldots, n$. Again put $i < j$

$$
\begin{aligned}
0 &= \{x_j, x_i\} + \{x_i, x_j\} = \lambda_i(x_j)x_j + [\operatorname{ad} w(x_i)]x_j + \lambda_j(x_i)x_i + [\operatorname{ad} w(x_j)]x_i \\
&= \lambda_i(x_j)x_j + \lambda_j(x_i)x_i + \xi_{i1}x_ix_j + \xi_{i0}x_j - x_j\xi_{i1}x_i - x_j\xi_{i0} \\
&\qquad + \xi_{j1}x_jx_i + \xi_{j0}x_i - x_i\xi_{j1}x_j - x_i\xi_{j0} \\
&= \lambda_i(x_j)x_j + \lambda_j(x_i)x_i + \xi_{i1}x_ix_j + \xi_{i0}x_j \\
&\qquad - \big(\delta_j(\xi_{i1})x_i + \xi_{i1}q_{ij}x_ix_j + \xi_{i1}p_{ij}(x_1, \ldots, x_n)\big) \\
&\qquad - \big(\xi_{i0}x_j + \delta_j(\xi_{i0})\big) + \big(\xi_{j1}q_{ij}x_ix_j + \xi_{j1}p_{ij}(x_1, \ldots, x_n)\big) \\
&\qquad + \xi_{j0}x_i - \big(\delta_i(\xi_{j1})x_j + \xi_{j1}x_ix_j\big) - \big(\delta_i(\xi_{j0}) + \xi_{j0}x_i\big) \\
&= \xi_{i1}x_ix_j - \xi_{i1}q_{ij}x_ix_j + \xi_{j1}q_{ij}x_ix_j - \xi_{j1}x_ix_j \\
&\qquad + \lambda_i(x_j)x_j + \lambda_j(x_i)x_i - \delta_j(\xi_{i1})x_i - \delta_j(\xi_{i0}) - \xi_{i1}p_{ij}(x_1, \ldots, x_n) \\
&\qquad + \xi_{j1}p_{ij}(x_1, \ldots, x_n) - \delta_i(\xi_{j1})x_j - \delta_i(\xi_{j0}).
\end{aligned}
$$

So $(\xi_{i1} - \xi_{j1})(1 - q_{ij}) = 0$ as $1 - q_{ij} \in A^*$ then $\xi_{1j} = \xi_{1i} = \xi$. Later

$$
0 = (\lambda_i(x_j) - \delta_j(\xi))x_j + \big(\lambda_j(x_i) - \delta_i(\xi)\big)(x_i)x_i - \delta_j(\xi_{i0}) - \delta_i(\xi_{j0})
$$

and $\lambda_i(x_j) = \delta_j(\xi)$, $\lambda_j(x_i) = \delta_i(\xi)$, $\delta_j(\xi_{i0}) = -\delta_i(\xi_{j0})$. Note that

$$
\begin{aligned}
\{x_i, x_j\} &= \lambda_j(x_i)x_i + [\operatorname{ad}\xi x_j + \xi_{j0}]x_i = \lambda_j(x_i)x_i + \xi x_jx_i + \xi_{j0}x_i \\
&\qquad - x_i\xi x_j - x_i\xi_{j0} \\
&= \delta_i(\xi)x_i + \xi x_jx_i + \xi_{j0}x_i - \xi x_ix_j + \delta_i(\xi)x_j - \xi_{j0}x_i - \delta_i(\xi_{j0}) \\
&= \xi[x_i, x_j] + \delta_i(\xi)x_i - \delta_i(\xi)x_j - \delta_i(\xi_{j0}).
\end{aligned}
$$

□

Lemma 3.4. Let $\{\cdot,\cdot\}$ be a Poisson bracket on $\mathcal{O}^{r,n}_{q,\delta}$ where $\delta_i = 0$ for all $i = 1,\dots,n$. Then there exists $\xi \in A$ such that $\{x_i, a\} = \xi[x_i, a]$ for all $i = 1,\dots,n$ and $a = x_{t_1}^{b_1}\cdots x_{t_n}^{b_n} \in \mathcal{O}^{r,n}_{q,\delta}$ monomial.

Proof. Let $\xi \in A$ such that $\{x_i, x_j\} = \xi[x_i, x_j]$ for all $i, j = 1,\dots,n$ which is given by proposition (3.3) and $a = x_{t_1}^{b_1}\cdots x_{t_l}^{b_l}$ with $b_r \neq 0$ for all $r = 1,\dots,l$. Fix $i = 1,\dots,n$, we will show, by induction on l, that $\{a, x_i\} = \xi[a, x_i]$.

1) $(a = x_t^n)$ For this case we will do induction on n.

(a) $(a = x_t)$ we have $\{x_t, x_i\} = \xi[x_t, x_i]$ by the above proposition.

(b) $(a = x_t^{n+1})$

$$\begin{aligned}\{x_t^{n+1}, x_i\} &= \{x_t^n, x_i\}x_t + x_t^n\{x_t, x_i\} = \big(\xi[x_t^n, x_i]\big)x_t + x_t^n\big(\xi[x_t, x_i]\big)\\ &= \xi\big(-x_ix_t^{n+1} + x_t^nx_ix_t - x_t^nx_ix_t + x_t^{n+1}x_i\big) = \xi[x_t^{n+1}, x_i]\end{aligned}$$

(c) $(a = x_t^{-1}$, if $t \leqslant r)$

$$\begin{aligned}\{x_t^{-1}, x_i\} &= -x_t^{-1}\{x_t, x_i\}x_t^{-1} = -\xi x_t^{-1}\big([x_t, x_i]\big)x_t^{-1}\\ &= \xi\big(-x_t^{-1}(x_tx_i - x_ix_t)x_t^{-1}\big) = \xi(x_t^{-1}x_i - x_ix_t^{-1}) = \xi[x_t^{-1}, x_i]\end{aligned}$$

(d) $(a = x_t^{-n-1}$, if $t \leqslant r)$

$$\begin{aligned}\{x_t^{-1-n}, x_i\} &= \{x_t^{-n}, x_i\}x_t^{-1} + x_t^{-n}\{x_t^{-1}, x_i\}\\ &= \xi\big(([x_t^{-n}, x_i])x_t^{-1} + x_t^{-n}([x_t^{-1}, x_i])\big)\\ &= \xi\big(-x_ix_t^{-n-1} + x_t^{-n}x_ix_t^{-1} - x_t^{-n}x_ix_t^{-1} + x_t^{-1-n}x_i\big)\\ &= \xi[x_t^{-n-1}, x_i].\end{aligned}$$

2) $(a = x_{t_1}^{b_1}\cdots x_{t_l}^{b_l})$

$$\begin{aligned}\{a, x_i\} &= \{x_{t_1}^{b_1}\cdots x_{t_l}^{b_l}, x_i\} = \{x_{t_1}^{b_1}, x_i\}x_{t_2}^{b_2}\cdots x_{t_l}^{b_l} + x_{t_1}^{b_1}\{x_{t_2}^{b_2}\cdots x_{t_l}^{b_l}, x_i\}\\ &= \xi\big(([x_{t_1}^{b_1}, x_i])x_{t_2}^{b_2}\cdots x_{t_l}^{b_l} + x_{t_1}^{b_1}([x_{t_2}^{b_2}\cdots x_{t_l}^{b_l}, x_i])\big)\\ &= \xi\big(-x_ia + x_{t_1}^{b_1}x_ix_{t_2}^{b_2}\cdots x_{t_l}^{b_l} - x_{t_1}^{b_1}x_ix_{t_2}^{b_2}\cdots x_{t_l}^{b_l} + ax_i\big) = \xi[a, x_i].\end{aligned}$$

So we have the claimed. By theorem (2.6)

$$\begin{aligned}0 &= \{x_i, a\} + \{a, x_i\} = \lambda_a(x_i)x_i + [\operatorname{ad} w(a)]x_i + \{a, x_i\}\\ &= \lambda_a(x_i)x_i + [\operatorname{ad} w(a)]x_i - \xi[\operatorname{ad} a]x_i = \lambda_a(x_i)x_i + [\operatorname{ad} w(a) - \xi a]x_i\end{aligned}$$

but if we take $w(a) - \xi a$ as $w(x_i)$ in the proof of the above proposition, we can show that $w(a) - \xi a \in A((x_i))$ (if $i = 1,\dots,r$) or $w(a) - \xi a \in A[x_i]$ (if

$i = r+1, \dots, n$) for all $i = 1, \dots, n$ then $w(a) - \xi a = l$ with $l \in A$. Then $[\operatorname{ad} w(a)]x_i = [\operatorname{ad} \xi a + l]x_i = [\operatorname{ad} \xi a]x_i + [\operatorname{ad} l]x_i = (\xi aa) + (lx_i - x_i l) = \xi[x_i, a]$ and $\lambda_a(x_i) = 0$ for all $i = 1, \cdots, n$. □

Theorem 3.5. Let $\{\cdot,\cdot\}$ be a Poisson bracket on $\mathcal{O}^{r,n}_{q,\delta}$. If A is a commutative ring, $\delta_i = 0$ for all $i = 1, \dots, n$, and $\{\cdot,\cdot\}$ is a A-bilinear function, then there exists $\xi \in A$ such that $\{a, b\} = \xi[a, b]$ for all $a, b \in \mathcal{O}^{r,n}_{q,\delta}$.

Proof. Let $a, b \in \mathcal{O}^{r,n}_{q,\delta}$ and $\xi \in A$ such that $\{x_i, X\} = \xi[x_i, X]$ for all $i = 1, \dots, n$, where $X \in \operatorname{Mon}\{\mathcal{O}^{r,n}_{q,\delta}\}$, ξ is given by lemma (3.4). If $b = \sum_{r=0}^{k} \eta_{v_r} x_1^{v_{r_1}} \cdots x_n^{v_{r_n}}$ since $\{\cdot,\cdot\}$ is a A-bilinear function and $x_i c = cx_i + \delta_i(c) = cx_i$ for all $c \in A$ and $i = 1, \dots, n$, it is enough to see this when $a = x_{t_1}^{a_1} \cdots x_{t_l}^{a_l}$ with $a_t \neq 0$. We will show this by induction on l.

1) ($a = x_t^n$) For this case we will do induction on n.

(a) ($a = x_t$) By the above lemma we have that

$$\begin{aligned}\{x_t, b\} &= \{x_t, \sum_{r=0}^{k} \eta_{v_r} x_1^{v_{r_1}} \cdots x_n^{v_{r_n}}\} = \sum_{r=0}^{k} \eta_{v_r} \{x_t, x_1^{v_{r_1}} \cdots x_n^{v_{r_n}}\} \\ &= \sum_{r=0}^{k} \eta_{v_r} \xi [x_t, x_1^{v_{r_1}} \cdots x_n^{v_{r_n}}] = \xi \sum_{r=0}^{k} \eta_{v_r} [x_t, x_1^{v_{r_1}} \cdots x_n^{v_{r_n}}] \\ &= \xi[x_t, b].\end{aligned}$$

(b) ($a = x_t^{n+1}$)

$$\begin{aligned}\{x_t^{n+1}, b\} &= \{x_t^n, b\}x_t + x_t^n\{x_t, b\} = \xi\big([x_t^n, b]x_t + x_t^n[x_t, b]\big) \\ &= \xi\big(x_t^n b x_t - b x_t^{n+1} + x_t^{n+1} b - x_t^n b x_t\big) = \xi[x_t^{n+1}, b].\end{aligned}$$

(c) ($a = x_t^{-1}$)

$$\{x_t^{-1}, b\} = -x_t^{-1}\{x_t, b\}x_t^{-1} = \xi\big(-x_t^{-1}[x_t, b]x_t^{-1}\big) = \xi[x_t^{-1}, b].$$

(d) ($a = x_t^{-1-n}$)

$$\begin{aligned}\{x_t^{-1-n}, b\} &= \{x_t^{-n}, b\}x_t^{-1} + x_t^{-n}\{x_t^{-1}, b\} \\ &= \xi\big([x_t^{-n}, b]x_t^{-1} + x_t^{-n}[x_t^{-1}, b]\big) \\ &= \xi\big(x_t^{-n} b x_t^{-1} - b x_t^{-n-1} + x_t^{-n-1} b - x_t^{-n} b x_t^{-1}\big) \\ &= \xi[x_t^{-n-1}, b].\end{aligned}$$

2) $(a = x_{t_1}^{a_1} \cdots x_{t_l}^{a_l})$

$$\begin{aligned}\{a,b\} &= \{x_{t_1}^{a_1} \cdots x_{t_l}^{a_l}, b\} = \{x_{t_1}^{a_1}, b\}x_{t_2}^{a_2} \cdots x_{t_l}^{a_l} + x_{t_1}^{a_1}\{x_{t_2}^{a_2} \cdots x_{t_l}^{a_l}, b\} \\ &= \xi\left([x_{t_1}^{a_1}, b]x_{t_2}^{a_2} \cdots x_{t_l}^{a_l} + x_{t_1}^{a_1}[x_{t_2}^{a_2} \cdots x_{t_l}^{a_l}, b]\right) \\ &= \xi\left(x_{t_1}^{a_1} b x_{t_2}^{a_2} \cdots x_{t_l}^{a_l} - ba + ab - x_{t_1}^{a_1} b x_{t_2}^{a_2} \cdots x_{t_l}^{a_l}\right) = \xi[a,b]. \quad \square\end{aligned}$$

Lemma 3.6. Let $B = \sigma(A)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of A with

1) σ_i is the identity of A for all $i = 1, \ldots, n$.
2) $q_{ij}, a_{ij}^{(t)} \in Z(A)$ for all $1 \leqslant i, j \leqslant n$ and $t = 0, \ldots, n$.
3) $\delta_1 = 0$
4) $\delta_t(q_{ij}) = \delta_t(a_{ij}^{(m)}) = 0$ for all $m = 0, \cdots, n$ and $i, j, t = 1, \cdots, n$.
5) $p_{1j} = 0$ for all $j = 1, \cdots, n$.

then there exists $\mathcal{O}_{q,\delta}^{1,n}$ with the same properties.

Proof. We will see that BS^{-1} exists, showing the Ore conditions on S, and it is an extension of A of type $\mathcal{O}_{q,\delta}^{1,n}$ for some set S. Let $S := \{x_1^m | m \in \mathbb{N}\}$, S is a multiplicative set of B. We will see that S is a right (left) Ore set.

(a) (Right) Take $P \in B$ and $s \in S$ with $sP = 0$, if $P = \sum_{t \in Z} b_t x_1^{t_1} \cdots x_n^{t_n}$ and $s = x_1^l$ we have $0 = \sum_{t \in Z} b_t x_1^l x_1^{t_1} \cdots x_n^{t_n}$ then $b_t = 0$ for all $t \in Z$ then $P = 0$ and $Ps = 0$.

(b) (Left) If $Ps = 0$,

$$0 = \sum_{t \in Z} b_t x_1^{t_1} \cdots x_n^{t_n} x_1^l = \sum_{t \in Z} b_t \prod_{j=2}^{n} q_{1j}^{t_j} x_1^{t_1 + l} x_2^{t_2} \cdots x_n^{t_n}$$

since $q_{ij} \in A^*$ then $b_t = 0$ for all $t \in Z$ and $sP = 0$.

(c) (Right) Let $P = \sum_{t \in Z} b_t x_1^{t_1} \cdots x_n^{t_n} \in B$ and $x_1^l \in S$ then

$$Px_1^l = \sum_{t \in Z} b_t \prod_{j=2}^{n} q_{1j}^{t_j} x_1^{t_1 + l} x_2^{t_2} \cdots x_n^{t_n} = x_1^l \left(\sum_{t \in Z} b_t \prod_{j=2}^{n} q_{1j}^{t_j} x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}\right).$$

(d) (Left) Let $P = \sum_{t \in Z} b_t x_1^{t_1} \cdots x_n^{t_n} \in B$ and $x_1^l \in S$ then

$$x_1^l P = \sum_{t \in Z} b_t x_1^l x_1^{t_1 + l} x_2^{t_2} \cdots x_n^{t_n} = \left(\sum_{t \in Z} b_t \prod_{j=2}^{n} q_{j1}^{t_j} x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}\right) x_1^l.$$

Then BS^{-1} and $S^{-1}B$ exist, so $BS^{-1} \cong S^{-1}B$. We will see that $D = BS^{-1}$ is an extension of A of type $\mathcal{O}_{q,\delta}^{1,n}$ and holds the conditions.

1) $A \hookrightarrow B \hookrightarrow D$.
2) $\frac{x_i}{1}\frac{a}{1} = \frac{x_i a}{1} = \frac{a x_i + \delta_i(a)}{1} = \frac{a x_i}{1} + \frac{\delta_i(a)}{1} = \frac{a}{1}\frac{x_i}{1} + \frac{\delta_i(a)}{1}$ for $a \in A$.
3) Let $i < j$, then

$$\begin{aligned}\frac{x_j}{1}\frac{x_i}{1} = \frac{x_j x_i}{1} &= \frac{q_{ij}x_ix_j + a_{ij}^{(0)} + a_{ij}^{(1)}x_1 + \cdots + a_{ij}^{(n)}x_n}{1} \\ &= \frac{q_{ij}}{1}\frac{x_i}{1}\frac{x_j}{1} + \frac{a_{ij}^{(0)}}{1} + \frac{a_{ij}^{(1)}}{1}\frac{x_1}{1} + \cdots + \frac{a_{ij}^{(n)}}{1}\frac{x_n}{1}.\end{aligned}$$

4) Let $\frac{\sum_{t\in Z} b_t x_1^{t_1}\cdots x_n^{t_n}}{x_1^l} \in B$, note that

$$\frac{1}{x_1}\frac{a}{1} = \frac{a}{1}\frac{1}{x_1} \qquad \left(\frac{1}{x_1}\left(\frac{x_1}{1}\frac{a}{1}\right)\frac{1}{x_1} = \frac{1}{x_1}\left(\frac{a}{1}\frac{x_1}{1}\right)\frac{1}{x_1}\right)$$

then

$$\begin{aligned}\frac{\sum_{t\in Z} b_t x_1^{t_1}\cdots x_n^{t_n}}{x_1^l} &= \left(\frac{x_1}{1}\right)^{-l}\frac{\sum_{t\in Z} b_t x_1^{t_1}\cdots x_n^{t_n}}{1} \\ &= \sum_{t\in Z}\frac{b_t}{1}\left(\frac{x_1}{1}\right)^{t_1-l}\cdots\left(\frac{x_n}{1}\right)^{t_n}.\end{aligned}$$

5) If $P = \sum_{t\in Z}\frac{b_t}{1}\left(\frac{x_1}{1}\right)^{t_1-l}\cdots\left(\frac{x_n}{1}\right)^{t_n} = \frac{0}{1}$ then $\frac{0}{1} = \left(\frac{x_1}{1}\right)^{l} P = \frac{\sum_{t\in Z} b_t x_1^{t_1}\cdots x_n^{t_n}}{1}$ then there exists $x_1^l \in S$ with

$$0 = \sum_{t\in Z} b_t x_1^{t_1}\cdots x_n^{t_n} x_1^l = \sum_{t\in Z} b_t \prod_{j=2}^{n} q_{1j}^{t_j} x_1^{t_1+l}\cdots x_n^{t_n}.$$

As $\prod_{j=2}^n q_{1j}^{t_j} \in A^*$ then $b_t = 0$ for all $t \in Z$ so if we denote $\frac{x_i}{1} := x_i$, then $\mathrm{Mon}\{x_1^{\pm 1}, x_2, \ldots, x_n\}$ is a A-basis of D. Note that if we put $\bar{\delta}_i(\frac{a}{1}) = \frac{\delta_i(a)}{1}$ then $\bar{\delta}_i\left(\frac{a_{ij}^{(t)}}{1}\right)$, $\bar{\delta}_i\left(\frac{q_{ij}}{1}\right) = 0$ for all $i, j = 1, \ldots, n$ and $t = 0, \ldots, n$, and $\bar{\delta}_1 = 0$. □

Lemma 3.7. Let $\{\cdot,\cdot\}$ be a Poisson bracket over a ring A then $\{a^l, a^r\} = 0$ for all $a \in A$ and $l, r \in \mathbb{N}$.

Proof. Take $l = 1$ and $r = 1$ then $\{a, a\} = 0$, now $\{a, a^{l+1}\} = \{a, a\}a^l + a\{a, a^l\} = 0$ by induction, later $\{a^r, a^{l+1}\} = \{a^r, a^l\}a + a^l\{a^r, a\} = -a^l\{a, a^r\} = 0$. □

Proposition 3.8. Let $B = \sigma(A)\langle x_1, \ldots, x_n\rangle$ be a skew PBW extension of a commutative ring A such that

1) σ_i is the identity of A for all $i = 1, \ldots, n$.
2) $\delta_i = 0$ for all $i = 1, \ldots, n$.
3) For every i fixed, with $i = 0, \ldots, n$ and $(m_1, \ldots, m_n) \in Z\setminus\{(0, \ldots, 0\}$, $\big(1 - \prod_{j=1, j\neq i}^{n} q_{ij}^{m_j}\big) \in A^*$.
4) $p_{1j} = 0$ for all $j = 1, \cdots, n$.

and $\{\cdot, \cdot\}$ a Poisson bracket on B then there exists a Poisson bracket $\{\cdot, \cdot\}_0$ on $\mathcal{O}_{q,\delta}^{1,n}$ such that $\{\cdot, \cdot\}_0|_B = \{\cdot, \cdot\}$.

Proof. Let $D = \mathcal{O}_{q,\delta}^{1,n}$, which is given by lemma (3.6). Put $s \in B$ and define $g_s(x_t) = \{x_t, s\}$ for $t = 1, \ldots, n$, $g_s(x_1^{-1}) = -x_1^{-1}\{x_1, s\}x_1^{-1}$, and

$$g_s(x_{t_1}^{a_1} \cdots x_{t_n}^{a_n}) = g_s(x_{t_1}^{a_1})x_{t_2}^{a_2} \cdots x_{t_n}^{a_n} + x_{t_1}^{a_1} g_s(x_{t_2}^{a_2} \cdots x_{t_n}^{a_n}).$$

of recurrently form.

By the universal property of basis there exists an A-homomorphism on D to itself, g_s with $g_s|_B = \{\ , s\}$. We will prove that this is a derivation. Since a g_s is an A-homomorphism then it is enough to see this to products of monomials. Let $p = x_{t_1}^{a_1} \cdots x_{t_s}^{a_s}$ and $q = x_{l_1}^{b_1} \cdots x_{l_r}^{b_r}$, we see the claimed by induction on s.

1) Let $s = 1$, we will do induction on r.

(a) $(r = 1)$ We will denote $p = x_i^a$ and $q = x_j^b$.

i. $(i < j)$ We have

$$g_s(x_i^a x_j^b) := g_s(x_i^a)x_j^b + x_i^a g_s(x_j^b).$$

ii. $(x_i^a, x_j^b \in B)$ Then

$$g_s(x_i^a x_j^b) := \{x_i^a x_j^b, s\} = \{x_i^a, s\}x_j^b + x_i^a\{x_j^b, s\} = g_s(x_i^a)x_j^b + x_i^a g_s(x_j^b).$$

iii. $(x_1^b \notin B,\ x_j^a \in B,\ 1 \leqslant j)$ Note that for all $r \in A$ we have

$$x_1^{-1}r = x_1^{-1}(rx_1)x_1^{-1} = x_1^{-1}(x_1 r)x_1^{-1} = rx_1^{-1}$$

and

$$x_j x_1^{-1} = x_1^{-1}(x_1 x_j)x_1^{-1} = x_1^{-1}(q_{j1}x_j x_1)x_1^{-1} = q_{j1}x_j x_1^{-1}$$

then for all $b \in \mathbb{Z}^-$ and $a \in \mathbb{N}$, $x_j^a x_1^b = q_{j1}^{-ab} x_1^b x_j^a = c^{-1} x_1^b x_j^a$ where $c \in Z(B)$,

$$\begin{aligned} c^{-1} g_s(x_j^a) x_1^{-b} + c^{-1} x_j^a g_s(x_1^{-b}) &= g_s(c^{-1} x_j^a x_1^{-b}) = g_s(x_1^{-b} x_j^a) \\ &= g_s(x_1^{-b}) x_j^a + x_1^{-b} g_s(x_j^a) \end{aligned}$$

and

$$\begin{aligned} &g_s(x_j^a x_1^b) = g_s(c^{-1} x_1^b x_j^a) = c^{-1}\big(g_s(x_1^b) x_j^a + x_1^b g_s(x_j^a)\big) \\ &= c^{-1}\big(-x_1^b g_s(x_1^{-b}) x_1^b x_j^a + x_1^b g_s(x_j^a)\big) = -x_1^b g_s(x_1^{-b}) c^{-1} x_1^b x_j^a + c^{-1} x_1^b g_s(x_j^a) \\ &= -x_1^b g_s(x_1^{-b}) x_j^a x_1^b + c^{-1} x_1^b g_s(x_j^a) = x_1^b\big(-g_s(x_1^{-b}) x_j^a + c^{-1} g_s(x_j^a) x_1^{-b}\big) x_1^b \\ &= x_1^b\big(x_1^{-b} g_s(x_j^a) - c^{-1} x_j^a g_s(x_1^{-b})\big) x_1^b = g_s(x_j^a) x_1^b - c^{-1} x_1^b x_j^a g_s(x_1^{-b}) x_1^b \\ &= g_s(x_j^a) x_1^b - x_j^a x_1^b g_s(x_1^{-b}) x_1^b = g_s(x_j^a) x_1^b + x_j^a g_s(x_1^b). \end{aligned}$$

(b) Let $q = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$ and $p = x_1^b$ then

$$\begin{aligned} g_s(pq) = g_s(x_1^{l_1+b} x_2^{l_2} \cdots x_n^{l_n}) &:= g_s(x_1^{l_1+b}) x_2^{l_2} \cdots x_n^{l_n} + x_1^{l_1+b} g_s(x_2^{l_2} \cdots x_n^{l_n}) \\ &= \big(g_s(x_1^b) x_1^{l_1} + x_1^b g_s(x_1^{l_1})\big) x_2^{l_2} \cdots x_n^{l_n} + x_1^{l_1+b} g_s(x_2^{l_2} \cdots x_n^{l_n}) \\ &= g_s(p) q + x_1^b\big(g_s(x_1^{l_1}) x_2^{l_2} \cdots x_n^{l_n} + x_1^{l_1} g_s(x_2^{l_2} \cdots x_n^{l_n})\big) \\ &= g_s(p) q + p g_s(q). \end{aligned}$$

2) ($q = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$, $p = x_{t_1}^{a_1} \cdots x_{t_s}^{a_s}$). Let $T \in D$ such that $x_2^{t_2} \cdots x_n^{t_n} q = T$, later

$$\begin{aligned} g_s(pq) &= g_s(x_{t_1}^{a_1} x_{t_2}^{a_2} \cdots x_{t_s}^{a_s} q) = g_s(x_{t_1}^{a_1} T) = g_s(x_{t_1}^{a_1}) T + x_{t_1}^{a_1} g_s(T) \\ &= g_s(x_{t_1}^{a_1}) x_{t_2}^{a_2} \cdots x_{t_s}^{a_s} q + x_{t_1}^{a_1} g_s(x_{t_2}^{a_2} \cdots x_{t_s}^{a_s} q) \\ &= g_s(x_{t_1}^{a_1}) x_{t_2}^{a_2} \cdots x_{t_s}^{a_s} q + x_{t_1}^{a_1}\big(g_s(x_{t_2}^{a_2} \cdots x_{t_s}^{a_s}) q + x_{t_2}^{a_2} \cdots x_{t_s}^{a_s} g_s(q)\big) \\ &= \big(g_s(x_{t_1}^{a_1}) x_{t_2}^{a_2} \cdots x_{t_s}^{a_s} + x_{t_1}^{a_1} g_s(x_{t_2}^{a_2} \cdots x_{t_s}^{a_s})\big) q + p g_s(q) \\ &= g_s(p) q + p g_s(q). \end{aligned}$$

We will define $\{\cdot,\cdot\}_0$ of recurrently form on the basis of D. Put $p \in D$ and define

1) $f_p(X) = -g_X(p)$ for $X \in \mathrm{Mon}\{x_1, \ldots, x_n\}$.
2) $f_p(x_1^{-a}) = -x_1^{-a} f_p(x_1^a) x_1^{-a}$, $a > 0$.

And for $x_{l_1}^{b_1} \cdots x_{l_r}^{b_r} \in \mathrm{Mon}\{x_1^{\pm}, x_2, \ldots, x_n\}$

$$f_p(x_{t_1}^{a_1} \cdots x_{t_n}^{a_n}) = f_p(x_{t_1}^{a_1}) x_{t_2}^{a_2} \cdots x_{t_n}^{a_n} + x_{t_1}^{a_1} f_p(x_{t_2}^{a_2} \cdots x_{t_n}^{a_n}).$$

By the universal property of the basis f has a extension to an A-homomorphism f in all D, now we will take $\{a,b\}_0 := f_a(b)$ and we will prove it is a bracket Poisson on D.

1) ($\{a,b\}_0 + \{b,a\}_0 = 0$) Since $\{a,b\}_0$ is a A-bilinear function, it is enough to see this when $a,b \in D$ are monomials, put $a = x_1^{-l}a'$ and $b = x_1^{-r}b'$ where $0 \leqslant l, r$, $a' = x_1^{a_1}\cdots x_n^{a_n} \in B$, $b' = x_1^{b_1}\cdots x_n^{b_n} \in B$. We have

$$\begin{aligned}
\{a,b\}_0 + \{b,a\}_0 &:= f_a(b) + f_b(a) = f_a(x_1^{-r}b') + f_b(x_1^{-l}a') \\
&:= \big(f_a(x_1^{-r})b' + x_1^{-r}f_a(b')\big) + \big(f_b(x_1^{-l})a' + x_1^{-l}f_b(a')\big) \\
&:= \big(-x_1^{-r}f_a(x_1^{r})b + x_1^{-r}f_a(b')\big) + \big(-x_1^{-l}f_b(x_1^{l})a + x_1^{-l}f_b(a')\big) \\
&:= \big(x_1^{-r}g_{x_1^r}(a)b - x_1^{-r}g_{b'}(a)\big) + \big(x_1^{-l}g_{x_1^l}(b)a - x_1^{-l}g_{a'}(b)\big) \\
&= \big(x_1^{-r}g_{x_1^r}(x_1^{-l}a')b - x_1^{-r}g_{b'}(x_1^{-l}a')\big) \\
&\quad + \big(x_1^{-l}g_{x_1^l}(x_1^{-r}b')a - x_1^{-l}g_{a'}(x_1^{-r}b')\big) \\
&= x_1^{-r}\big(g_{x_1^r}(x_1^{-l})a' + x_1^{-l}g_{x_1^r}(a')\big)b - x_1^{-r}\big(g_{b'}(x_1^{-l})a' + x_1^{-l}g_{b'}(a')\big) \\
&\quad + x_1^{-l}\big(g_{x_1^l}(x_1^{-r})b' + x_1^{-r}g_{x_1^l}(b')\big)a - x_1^{-l}\big(g_{a'}(x_1^{-r})b' + x_1^{-r}g_{a'}(b')\big) \\
&= x_1^{-r}\big(-x_1^{-l}g_{x_1^r}(x_1^{l})x_1^{-l}a' + x_1^{-l}g_{x_1^r}(a')\big)b \\
&\quad - x_1^{-r}\big(-x_1^{-l}g_{b'}(x_1^{l})x_1^{-l}a' + x_1^{-l}g_{b'}(a')\big) \\
&\quad + x_1^{-l}\big(-x_1^{-r}g_{x_1^l}(x_1^{r})x_1^{-r}b' + x_1^{-r}g_{x_1^l}(b')\big)a \\
&\quad - x_1^{-l}\big(-x_1^{-r}g_{a'}(x_1^{r})x_1^{-r}b' + x_1^{-r}g_{a'}(b')\big) \\
&:= \big(-x_1^{-l-r}\{x_1^l, x_1^r\}ab + x_1^{-l-r}\{a', x_1^r\}b\big) \\
&\quad - \big(-x_1^{-l-r}\{x_1^l, b'\}a + x_1^{-l-r}\{a', b'\}\big) \\
&\quad + \big(-x_1^{-r-l}\{x_1^r, x_1^l\}ba + x_1^{-r-l}\{b', x_1^l\}a\big) \\
&\quad - \big(-x_1^{-r-l}\{x_1^r, a'\}b + x_1^{-r-l}\{b', a'\}\big) \\
&= \big(-x_1^{-r-l}\{x_1^l, x_1^r\}ab - x_1^{-l-r}\{x_1^r, x_1^l\}ba\big) \\
&\quad + \big(x_1^{-r-l}\{a', x_1^r\}b + x_1^{-r-l}\{x_1^r, a'\}b\big) \\
&\quad + \big(x_1^{-l-r}\{x_1^l, b'\}a + x_1^{-r-l}\{b', x_1^l\}a\big) \\
&\quad - \big(x_1^{-r-l}\{a', b'\} + x_1^{-r-l}\{b', a'\}\big) \\
&= \big(-x_1^{-r-l}\{x_1^l, x_1^r\}ab + x_1^{-l-r}\{x_1^r, x_1^l\}ba\big) = 0.
\end{aligned}$$

2) ($\{a,a\}_0 = 0$) Let $a = x_1^l a'$ where $a' = \sum_t \eta_t x_1^{t_1}\cdots x_n^{t_n} \in B$ and $l \leqslant 0$ then

$$\{a,a\}_0 := f_a(x_1^l a') = \sum_t \eta_t f_a(x_1^l(x_1^{t_1}\cdots x_n^{t_n}))$$

$$
\begin{aligned}
&:= \sum_t \eta_t \big(f_a(x_1^l)x_1^{t_1}\cdots x_n^{t_n} + x_1^l f_a(x_1^{t_1}\cdots x_n^{t_n})\big)\\
&= f_a(x_1^l)a' + x_1^l f_a(a') := -x_1^l f_a(x_1^{-l})x_1^l a' + x_1^l f_a(a')\\
&= x_1^l g_{x_1^{-l}}(x_1^l a')x_1^l a' - x_1^l g_{a'}(x_1^l a')\\
&= x_1^l \big(g_{x_1^{-l}}(x_1^l)a' + x_1^l g_{x_1^{-l}}(a')\big)x_1^l a' - x_1^l \big(g_{a'}(x_1^l)a' + x_1^l g_{a'}(a')\big)\\
&= x_1^l g_{x_1^{-l}}(x_1^l)a'a + x_1^{2l} g_{x_1^{-l}}(a')x_1^l a' - x_1^l g_{a'}(x_1^l)a' - x_1^{2l} g_{a'}(a')\\
&= \big(x_1^l g_{x_1^{-l}}(x_1^l)a'a - x_1^{2l} g_{a'}(a')\big) + \big(x_1^{2l} g_{x_1^{-l}}(a')x_1^l a' - x_1^l g_{a'}(x_1^l)a'\big)\\
&= \big(x_1^{2l} g_{x_1^{-l}}(x_1^{-l})x_1^l a'a - x_1^{2l} g_{a'}(a')\big) + \big(x_1^{2l} g_{x_1^{-l}}(a')x_1^l a' + x_1^{2l} g_{a'}(x_1^{-l})x_1^l a'\big)\\
&:= x_1^{2l}\big(\{x_1^{-l}, x_1^{-l}\}aa - \{a', a'\}\big) + x_1^{2l}\big(\{a', x_1^{-l}\} + \{x_1^{-l}, a'\}\big)a = 0.
\end{aligned}
$$

3) ($\{ab, c\}_0 = \{a, c\}_0 b + a\{b, c\}_0$) It can be proved the same way we proved that $g_X(\cdot)$ is a derivations for all $X \in \operatorname{Mon}\{x_1, \dots, x_n\}$. □

Theorem 3.9. Let $B = \sigma(A)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of A as above and $\{\cdot,\cdot\}$ a poisson bracket on B, then there exists $\xi \in A$ such that $\{a, b\} = \xi[a, b]$ for all $a, b \in B$.

Proof. Since $\{\cdot,\cdot\}$ is a Poisson bracket on B then there exists a poisson bracket $\{\cdot,\cdot\}_0$ on $\mathcal{O}_{q,\delta}^{1,n}$ with $\{\cdot,\cdot\}_0|_B = \{\cdot,\cdot\}$ which is A-bilinear function, by theorem (3.5) there exists $\xi \in A$ such that $\{a, b\}_0 = \xi[a, b]$ for all $a, b \in \mathcal{O}_{q,\delta}^{1,n}$ so if $a, b \in B$ then $\{a, b\} = \{a, b\}_0 = \xi[a, b]$. □

4. Some examples

In this section we will show some algebras where we can give a characterization of the poisson brackets.

1) *The algebra of q-differentiable operators.* $D_{q,h}[x, y]$. Let $q, h \in \mathbb{K}$, $q \neq 0$ consider $\mathbb{K}[y][x; \sigma, \delta]$, $\sigma(y) := qy$ and $\delta(y) := h$. By definitions of skew polynomial ring, it is the $\mathbb{K}$-algebra defined by the relation $xy = qtx + h$. If we put $h = 0$ and $q^l - 1 \in \mathbb{K}^*$ for all $l \in \mathbb{N}$ we have that $xr = rx$ and $yr = ry$ for all $r \in \mathbb{K}$.

2) *The algebra of linear partial q-dilatation operators.* For a fixed $q \in \mathbb{K} - \{0\}$, the $\mathbb{K}$-algebra of linear partial q-dilatation operators with polynomial coefficients, respectively, with rational coefficients, is $\mathbb{K}[t_1, \dots, t_n][H_1^{(q)}, \dots, H_m^{(q)}]$, respectively $\mathbb{K}(t_1, \dots, t_n)[H_1^{(q)}, \dots, H_m^{(q)}]$ $n \leqslant m$, subject to the relations:

$$t_j t_i = t_i t_j, \qquad 1 \leqslant i < j \leqslant n,$$

$$H_i^{(q)} t_i = q t_i H_i^{(q)}, \qquad 1 \leqslant i \leqslant n,$$
$$H_j^{(q)} t_i = t_i H_j^{(q)}, \qquad i \neq j,$$
$$H_i^{(q)} H_j^{(q)} = H_j^{(q)} H_i^{(q)}, \qquad 1 \leqslant i \leqslant n.$$

If we take $n = m = 1$ and $q^l - 1 \in \mathbb{K}^*$ for all $l \in \mathbb{N}$.

3) *Multiplicative analogue of the Weyl algebra.* The $\mathbb{K}$-algebra $O_n(\lambda_{ij})$ is generated by $x_1, \ldots, x_n$ subject to the relations:

$$x_j x_i = \lambda_{ij} x_i x_j, \qquad 1 \leqslant i < j \leqslant n,$$

where $\lambda_{ij} \in \mathbb{K} - \{0\}$. If we take $n > 1$ and λ_{ij} as $\mathbb{N}$ independent, i.e. for all $i = 1, \ldots, n$ and $m \in \mathbb{N}^n - \{(0, \ldots, 0)\ ,\ 1 - \prod_{j \neq i} \lambda_{ij}^{m_j} \in \mathbb{K}^*$.

4) *3-dimensional skew polynomial algebra A.* It is given by the relations

$$yz - \alpha zy = \lambda, \qquad zx - \beta xz = \mu, \qquad xy - \gamma yx = v$$

such that $\lambda, \mu, v \in \mathbb{K} + \mathbb{K}x + \mathbb{K}y + \mathbb{K}z$, and $\alpha, \beta, \gamma \in \mathbb{K} - \{0\}$. If we take $\lambda, \mu = 0$ and α, β, γ, $\mathbb{N}$-independent.

5) *Quantum Space S_q.* Let $\mathbb{K}$ be a commutative ring and let $\mathbf{q} = [q_{ij}]$ be a matrix with entries in $\mathbb{K}^*$ such that $q_{ii} = 1 = q_{ij} q_{ji}$ for all $1 \leqslant i, j \leqslant n$. The $\mathbb{K}$-algebra S_q is generated by $x_1, \ldots, x_n$, subject to the relations

$$x_i x_j = q_{ij} x_j x_i.$$

If we take $n > 1$ and q_{ij} $\mathbb{N}$-independent.

6) *Witten's deformation of $\mathcal{U}(\mathcal{SL}(2, \mathbb{K}))$.* Let $\underline{\xi} = (\xi_1, \ldots, \xi_7)$ a 7-tuple of parameters, it is generated by x, y, z subject to relations

$$xz - \xi_1 zx = \xi_2 x, \qquad zy - \xi_3 yz = \xi_4 y, \qquad yx - \xi_5 xy = \xi_6 z^2 + \xi_7 z.$$

If we take $\xi_7, \xi_6, \xi_2 = 0$ and $\xi_1, \xi_3, \xi_5 \in \mathbb{K}^*$ and $\mathbb{N}$-independent.

7) *Quantum symplectic space.* $O_q(\mathcal{SP}(\mathbb{K}^{2n}))$. For every nonzero element $q \in \mathbb{K}$, one defines this quantum algebra $O_q(\mathcal{SP}(\mathbb{K}^{2n}))$ to be the algebra generated by $\mathbb{K}$ and the variables $y_1 \cdots, y_n, x_1, \ldots, x_n$ subject to the relations

$$y_j x_i = q^{-1} x_i y_j, \qquad y_i y_j = y_i y_j, \qquad 1 \leqslant i < j \leqslant n,$$
$$x_j x_i = q^{-1} x_i x j, \qquad x_j y_i = q y_i x_j, \qquad 1 \leqslant i < j \leqslant n,$$
$$x_i y_i - q^2 y_i x_i = (q^2 - 1) \sum_{l=1}^{i-1} q^{i-l} y_l x_l, \qquad 1 \leqslant i \leqslant n.$$

If we take $n = 1$ and q $\mathbb{N}$-independent.

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References

[1] V.A Artamonov. Action of Hopf algebras on General Quantum Malcev power series and Quantum planes, J. Math. Sci., 2006, 134, N 1, pp. 1773–1798.

[2] Gallego, C. and Lezama, O., Gröbner bases for ideals of sigma-PBW extensions, Communications in Algebra, 39 (1), 2011, 50–75.

[3] Reyes, M. A., Ring and Module Theoretic Properties of -PBW Extensions, Ph.D. Thesis, Universidad Nacional de Colombia, 2013.

[4] O. Lezama, J. P. Acosta and A. Reyes, Prime ideals of skew PBW extensions, Rev. Un. Mat. Argentina, vol. 56, no. 2, pp. 39–55, 2015.

[5] O. Lezama and A. Reyes, Some Homological Properties of Skew PBW Extensions, Comm. Algebra, vol. 42, pp. 1200–1230, 2014.

[6] A. Reyes and H. Suárez, Armendariz property for skew PBW extensions and their classical ring of quotients, Rev. Integr. Temas Mat., vol. 34, no. 2, pp. 147–168, 2016.

[7] Suarez,H, Kozulity for graded skew PBW extensions, Comm. in Algebra, 45(10), 2017, 4569–4580.

[8] Acosta, J.P., Chaparro, C., Lezama, O., Ojeda, I., and Venegas, C., Ore and Goldie theorems for skew PBW extensions, Asian-European Journal of Mathematics, 6(4), 2013, 1350061-1–1350061-20.

Contact information

Brian Andres Zambrano Luna

Seminario de Álgebra Constructiva - SAC^2
Departamento de Matemáticas, Universidad Nacional de Colombia, Sede Bogotá
E-Mail(s): bazambranol@unal.edu.co

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